数理解析研究所講究録
1134巻 2000年 136-151

Mathematical Analysis of a Model in Relativistic Quantum Electrodynamics

Asao Arai (新井朝雄)*
Department of Mathematics, Hokkaido University
Sapporo 060-0810, Japan
E-mail: arai@math.sci.hokudai.ac.jp

Abstract

Rigorous results are reported on a model of a Dirac particle — a relativistic charged particle with spin 1/2 — minimally coupled to the quantized radiation field.

1991 MSC: 81Q10, 81T08, 81V10

1 Introduction

We consider mathematically a model in relativistic quantum electrodynamics, which describes a Dirac particle — a relativistic charged particle with spin 1/2 — coupled to the quantized radiation field. The Hamiltonian of the model is given by the sum of the Dirac operator with the minimal coupling to the quantized radiation field and the free Hamiltonian of the quantized radiation field. An approximate version of this model was discussed by Bloch and Nordsieck[5] in view of the infrared problem of quantum electrodynamics. The Hamiltonian they treated is the one obtained by replacing the anticommuting matrices contained in the Dirac operator by c-number constants and is much easier to analyze than the original one.

Discussions using informal perturbation methods[7] suggest that the model may have a physical meaning in a range of quantum electrodynamic phenomena such as the Lamb shift of a hydrogen-like atom and the Compton scattering of the electron where the effects of the quantized radiation field play essential roles. Besides this point, we think that mathematical analysis of the model is interesting also in its own right, because the Hamiltonian of the model belongs to a new class of Hamiltonians on a Hilbert space of Fock type. Moreover the model may be regarded as a model

*Supported by the Grant-in-Aid No.11440036 for Scientific Research from the Ministry of Education, Science, Sports and Culture, JAPAN.

for a quantum mechanical system unstable under the influence of the quantized radiation field. To our best knowledge, no mathematically rigorous analysis has been made on the model so far[1]. In the present note we report fundamental results on the model concerning (essential) self-adjointness, spectral properties of the Hamiltonian and existence of ground states with a fixed (deformed) total momentum. Proofs of these results are given in [1, 2].

2 Description of the Model

For a linear operator T on $\mathcal{H}$, we denote its domain by $D(T)$ and by $\sigma(T)$ the spectrum of T. For two objects $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ such that products $a_j b_j$ $(j = 1, 2, 3)$ and their sum can be defined, we set $\mathbf{a} \cdot \mathbf{b} := \sum_{j=1}^{3} a_j b_j$.

The free Dirac particle of mass $m \geq 0$ is described by the free Dirac operator

$$H_D := \boldsymbol{\alpha} \cdot (-i\nabla) + m\beta \tag{2.1}$$

acting in the Hilbert space

$$\mathcal{H}_{\mathrm{D}} := \oplus^4 L^2(\mathbf{R}^3) \tag{2.2}$$

with domain $D(H_D) := \oplus^4 H^1(\mathbf{R}^3)$ ($H^1(\mathbf{R}^3)$ is the Sobolev space of order 1), where α_j $(j = 1, 2, 3)$ and β are 4×4 Hermitian matrices satisfying

$$\{\alpha_j, \alpha_k\} = 2\delta_{jk}, \quad j, k = 1, 2, 3, \tag{2.3}$$

$$\{\alpha_j, \beta\} = 0, \quad \beta^2 = 1, \quad j = 1, 2, 3, \tag{2.4}$$

$\{A, B\} := AB + BA$, and $\nabla := (D_1, D_2, D_3)$, D_j being the generalized partial differential operator in the variable x_j [$\mathbf{x} = (x_1, x_2, x_3) \in \mathbf{R}^3$]. The operator H_D is self-adjoint and essentially self-adjoint on $\oplus^4 C_0^\infty(\mathbf{R}^3 \setminus \{0\})$ ([11, p.11, Theorem 1.1]). Moreover, the spectrum $\sigma(H_D)$ of H_D is purely absolutely continuous and

$$\sigma(H_D) = (-\infty, -m] \cup [m, \infty). \tag{2.5}$$

As for the radiation field, we use the Coulomb gauge in quantizing it. In general, given a Hilbert space $\mathcal{H}$, we have the Boson Fock space

$$\mathcal{F}_{\mathrm{b}}(\mathcal{H}) := \oplus_{n=0}^{\infty} (\otimes_{\mathrm{s}}^n \mathcal{H}) \tag{2.6}$$

over $\mathcal{H}$, where $\otimes_{\mathrm{s}}^n \mathcal{H}$ denotes the n-fold symmetric tensor product Hilbert space of $\mathcal{H}$ with convention $\otimes_{\mathrm{s}}^0 \mathcal{H} := \mathbf{C}$. For basic facts on the theory of the Boson Fock space, we refer the reader to [9, §X.7].

The Hilbert space of one-photon states in momentum representation is given by

$$\mathcal{H}_{\mathrm{ph}} := L^2(\mathbf{R}^3) \oplus L^2(\mathbf{R}^3), \tag{2.7}$$

[1] Prosser [8] discusses a *modified* version of the model with relatively much mathematical rigor.

where $\mathbf{R}^3 := \{\mathrm{k} = (k_1, k_2, k_3) | k_j \in \mathbf{R},\ j = 1,2,3\}$ physically means the momentum space of photons. The Boson Fock space

$$\mathcal{F}_{\mathrm{rad}} := \mathcal{F}_{\mathrm{b}}(\mathcal{H}_{\mathrm{ph}}) \tag{2.8}$$

over $\mathcal{H}_{\mathrm{ph}}$ serves as a Hilbert space for the quantized radiation field in the Coulomb gauge.

We denote by $a(F)$ $(F \in \mathcal{H}_{\mathrm{ph}})$ the annihilation operator with test vector F on $\mathcal{F}_{\mathrm{rad}}$. By definition, $a(F)$ is a densely defined closed linear operator and antilinear in F. The Segal field operator

$$\Phi_{\mathrm{S}}(F) := \frac{\overline{a(F) + a(F)^*}}{\sqrt{2}} \tag{2.9}$$

is self-adjoint, where, for a closable operator T, $\overline{T}$ denotes its closure.

We take a nonnegative Borel measurable function ω on $\mathbf{R}^3$ to denote the one free photon energy. We assume that, for almost everywhere (a.e.) $\mathrm{k} \in \mathbf{R}^3$ with respect to the Lebesgue measure on $\mathbf{R}^3$, $0 < \omega(\mathrm{k}) < \infty$. Then the function ω defines uniquely a multiplication operator on $\mathcal{H}_{\mathrm{ph}}$ which is nonnegative, self-adjoint and injective. We denote it by the same symbol ω also. The free Hamiltonian of the quantized radiation field is then defined by

$$H_{\mathrm{rad}} := d\Gamma(\omega), \tag{2.10}$$

the second quantization of ω. The operator H_{rad} is a nonnegative self-adjoint operator.

Remark 2.1 Usually ω is taken to be of the form

$$\omega_{\mathrm{phys}}(\mathrm{k}) := |\mathrm{k}|, \quad \mathrm{k} \in \mathbf{R}^3, \tag{2.11}$$

but, in this note, for mathematical generality, we do not restrict ourselves to this case.

There exist an $\mathbf{R}^3$-valued continuous function $\mathbf{e}^{(r)}$ $(r = 1,2)$ on the non-simply connected space

$$\mathbf{M}_0 := \mathbf{R}^3 \setminus \{(0,0,k_3) | k_3 \in \mathbf{R}\}. \tag{2.12}$$

such that, for all $\mathrm{k} \in \mathbf{M}_0$,

$$\mathbf{e}^{(r)}(\mathrm{k}) \cdot \mathbf{e}^{(s)}(\mathrm{k}) = \delta_{rs}, \quad \mathbf{e}^{(r)}(\mathrm{k}) \cdot \mathrm{k} = 0, \quad r,s = 1,2. \tag{2.13}$$

These vector-valued functions $\mathbf{e}^{(r)}$ are called the polarization vectors of one photon.

Let $g \in L^2(\mathbf{R}^3)$. Then, each $\mathrm{x} \in \mathbf{R}^3$ and $j = 1,2,3$, we can define an element g_j^{x} of $\mathcal{H}_{\mathrm{ph}}$ by

$$g_j^{\mathrm{x}}(\mathrm{k}) := \left(g(\mathrm{k}) e_j^{(1)}(\mathrm{k}) e^{-i\mathrm{k}\cdot\mathrm{x}}, g(\mathrm{k}) e_j^{(2)}(\mathrm{k}) e^{-i\mathrm{k}\cdot\mathrm{x}}\right) \in \mathbf{C}^2.$$

Then the quantized radiation field $\mathbf{A}^g(\mathrm{x}) := (A_1^g(\mathrm{x}), A_2^g(\mathrm{x}), A_3^g(\mathrm{x}))$ with momentum cutoff function g is defined by

$$A_j^g(\mathrm{x}) := \Phi_{\mathrm{S}}\left(g_j^{\mathrm{x}}\right), \quad j = 1,2,3. \tag{2.14}$$

Remark 2.2 The case $g = 1/\sqrt{(2\pi)^3\omega}$ corresponds to the case without momentum cutoff.

We now move to the Hilbert space

$$\mathcal{F} := \mathcal{H}_{\mathrm{D}} \otimes \mathcal{F}_{\mathrm{rad}} \tag{2.15}$$

of state vectors for the coupled system of the Dirac particle and the quantized radiation field. This Hilbert space can be identified as

$$\mathcal{F} = L^2(\mathbf{R}^3; \oplus^4\mathcal{F}_{\mathrm{rad}}) = \int_{\mathbf{R}^3}^{\oplus} \oplus^4\mathcal{F}_{\mathrm{rad}} d\mathbf{x} \tag{2.16}$$

the Hilbert space of $\oplus^4\mathcal{F}_{\mathrm{rad}}$-valued Lebesgue square integrable functions on $\mathbf{R}^3$ [the constant fibre direct integral with base space $(\mathbf{R}^3, dx)$ and fibre $\oplus^4\mathcal{F}_{\mathrm{rad}}$ [10, §XIII.6]. We freely use this identification.

Let $\tau \in \mathbf{R}$ be a constant. Since the mapping:$\mathbf{x} \to g_j^{\mathbf{x}}$ from $\mathbf{R}^3$ to $\mathcal{H}_{\mathrm{ph}}$ is strongly continuous, we can show that the decomposable operator

$$A_j^{g,\tau} := \int_{\mathbf{R}^3}^{\oplus} A_j^g(\tau\mathbf{x}) d\mathbf{x} \tag{2.17}$$

acting on $\mathcal{F}$ is self-adjoint [10, Theorem XIII.85].

We denote by $q \in \mathbf{R} \setminus \{0\}$ the charge of the Dirac particle. We consider the situation where the Dirac particle is in an external field described by a 4×4 Hermitian matrix-valued Borel measurable function $V = (V_{ab})_{a,b=1,\cdots,4}$ such that each V_{ab} is in

$$L^2(\mathbf{R}^3)_{\mathrm{loc}} := \{f : \mathbf{R}^3 \to \mathbf{C}; \text{Borel measurable} | \textstyle\int_{|\mathbf{x}|\leq R} |f(\mathbf{x})|^2 d\mathbf{x} < \infty \text{ for all } R > 0\}.$$

Then the Hamiltonian of the Dirac particle is given by

$$H_D(V) := H_D + V \tag{2.18}$$

The minimal interaction between the Dirac particle and the quantized radiation field with momentum cutoff g is given by

$$H_{I,\tau}(g) := -q\boldsymbol{\alpha} \cdot \mathbf{A}^{g,\tau}. \tag{2.19}$$

Thus the total Hamiltonian of the coupled system is defined by

$$H_\tau(V, g) := H_D(V) + H_{\mathrm{rad}} + H_{I,\tau}(g). \tag{2.20}$$

Remark 2.3 The orignal Hamiltonian of the model is $H_1(V, g)$ (the case $\tau = 1$). On the other hand, $H_0(V, g)$ (the case $\tau = 0$) is the Hamiltonian with the "dipole approximation".

Remark 2.4 For a class of V, the essential spectrum $\sigma_{\rm ess}(H_D(V))$ of $H_D(V)$ coincides with that of H_D:

$$\sigma_{\rm ess}(H_D(V)) = (-\infty, -m] \cup [m, \infty), \tag{2.21}$$

so that the discrete spectrum $\sigma_{\rm d}(H_D(V))$ of $H_D(V)$ is a subset of the interval $(-m, m)$ if m is positive [11, p.116, Theorem 4.7]. Suppose that (2.21) holds with $\sigma_{\rm d}(H_D(V)) = \{E_n\}_{n=1}^N$ ($N < \infty$ or N is countably infinite) and that $\{\omega(\mathbf{k})|\mathbf{k} \in \mathbf{R}^3\} = [\nu, \infty)$ with a constant $\nu \geq 0$. Then we have

$$\sigma_{\rm ess}(H_D(V) + H_{\rm rad}) = \mathbf{R}$$

and each E_n is an eigenvalue of $H_D(V) + H_{\rm rad}$ *embedded in its continuous spectrum.* Hence the spectral analysis of $H_\tau(V, g)$ includes a perturbation problem of embedded eigenvalues.

Remark 2.5 We can not expect that $H_\tau(V, g)$ is bounded below. Hence the model may be unphysical *in view of stability of matter.* From this point of view, we can consider a modified version of the model: Let E_D be the spectral measure of H_D and $\Lambda_+ := E_D((0, \infty))$, the projection of $\mathcal{H}_D$ onto the positive spectral subspace of the free Dirac operator H_D. Then the operator

$$H_\tau^{\rm BR}(V, g) := \Lambda_+ H_\tau(V, g) \Lambda_+ \tag{2.22}$$

may be a Hamiltonian for a quantum system of of a Dirac particle interacting with the quantized radiation field. This operator is an extended version of the Brown-Ravenhall Hamiltonian $\Lambda_+ H_D(V) \Lambda_+$ [6]. As for certain aspects (e.g., self-adjointness, boundedness from below), the operator $H_\tau^{\rm BR}(V, g)$ is more tractable than $H_\tau(V, g)$. The model discussed in [8] is in fact the one described by $H_\tau^{\rm BR}(V, g)$.

3 Self-Ajointness of the Total Hamiltonian

In what follows we fix $\tau \in \mathbf{R}$, unless otherwise stated.

3.1 Numerical range and a self-adjoint extension

For a linear operator T on a Hilbert space $\mathcal{X}$, its numerical range is defined by

$$\Theta(T) := \{(u, Tu)_{\mathcal{X}} | u \in D(T),\ \|u\|_{\mathcal{X}} = 1\}, \tag{3.1}$$

where $(\cdot, \cdot)_{\mathcal{X}}$ (resp. $\| \cdot \|_{\mathcal{X}}$) denotes the inner product (resp. norm) of $\mathcal{X}$.

Proposition 3.1 *Suppose that*

$$g,\ \frac{g}{\sqrt{\omega}} \in L^2(\mathbf{R}^3). \tag{3.2}$$

Then $H_\tau(V,g)$ is a symmetric operator with $D(H_\tau(V,g) = D(H_D) \cap D(V) \cap D(H_{\rm rad})$. Moreover

$$\Theta(H_D(V)) \subset \Theta(H_\tau(V,g)). \tag{3.3}$$

Remark 3.1 It is well known that, for a wide class of V, $H_D(V)$ is not semibounded (i.e., neither bounded from below nor above) [11, Chapter 4, §4.3]. Hence, for such a function V, (3.3) implies that $H_\tau(V,g)$ is not semibounded. In particular, in the case of the Coulomb potential

$$V(\mathbf{x}) = V_{\rm Cl}(\mathbf{x}) := -\frac{Z}{|\mathbf{x}|} \quad (Z > 0 : \text{a constant}),$$

which is a physically important case, one can show that $H_\tau(V_{\rm Cl},g)$ is not semibounded.

By Pauli's lemma [11, p.14 and p.74], there exists a 4×4 unitary matrix $U_{\rm C}$ such that

$$U_{\rm C}^{-1}\alpha_j U_{\rm C} = \bar{\alpha}_j, \quad j = 1,2,3, \quad U_{\rm C}^{-1}\beta U_{\rm C} = -\bar{\beta}, \tag{3.4}$$

where, for a matrix M, $\overline{M}$ denotes its complex conjugate.

Theorem 3.2 *Assume (3.2). Suppose that g is real-valued and that*

$$U_{\rm C}^{-1}V(\mathbf{x})U_{\rm C} = \overline{V(-\mathbf{x})}. \tag{3.5}$$

for a.e. $\mathbf{x}$. Then $H_\tau(V,g)$ has a self-adjoint extension.

Remark 3.2 The Coulomb potential $V = V_{\rm Cl}$ (Remark 3.1) satisfies condition (3.5).

3.2 Essential self-adjointness

We define

$$\Delta := \sum_{j=1}^{3} D_j^2 \tag{3.6}$$

the Laplacian acting in $\mathcal{H}_{\rm D}$.

For a subspace $\mathcal{D}$ of $\mathcal{H}_{\rm ph}$, we define $\mathcal{F}_{\rm rad}^{\rm fin}(\mathcal{D}) \subset \mathcal{F}_{\rm rad}$ to be the subspace algebraically spanned by Ω_0 and all the vectors of the form

$$a(F_1)^* \cdots a(F_n)^* \Omega_0, \quad n \geq 1, \ F_j \in \mathcal{D}, \ j = 1, \cdots, n.$$

If $\mathcal{D}$ is dense in $\mathcal{H}_{\rm ph}$, then $\mathcal{F}_{\rm rad}^{\rm fin}(\mathcal{D})$ is dense in $\mathcal{F}_{\rm rad}$.

Theorem 3.3 *Suppose that*

$$g, \ \frac{g}{\sqrt{\omega}}, \ \omega g, \ |\mathbf{k}|g, \ \frac{|\mathbf{k}|g}{\sqrt{\omega}} \in L^2(\mathbf{R}^3). \tag{3.7}$$

Assume the following (V.1) and (V.2):

(V.1) *V is $-\Delta$-bounded.*

(V.2) *For each $j = 1,2,3$ and $a,b = 1,\cdots,4$, the distribution D_jV_{ab} is in $L^2(\mathbf{R}^3)_{\rm loc}$ and there exists a constant $c > 0$ such that, for all $f \in \oplus^4 C_0^\infty(\mathbf{R}^3)$,*

$$\|(D_jV)f\| \leq c\|(-\Delta+1)^{1/2}f\|, \quad j = 1,2,3.$$

Let $\mathcal{D} \subset \mathcal{H}_{\rm ph}$ be a core of the self-adjoint operator ω. Then $H_\tau(V,g)$ is essentially self-adjoint on $[\oplus^4 C_0^\infty(\mathbf{R}^3)] \otimes_{\rm alg} \mathcal{F}_{\rm rad}^{\rm fin}(\mathcal{D})$ ($\otimes_{\rm alg}$ means algebraic tensor product) and its closure is essentially self-adjoint on every core of $-\Delta + H_{\rm rad}$.

Remark 3.3 Theorem 3.3 excludes the Coulomb potential case $V = V_{\rm cl}$.

As a corollary to Theorem 3.3, we have the following.

Corollary 3.4 *Let V be bounded. Assume (3.7). Let $\mathcal{D}$ be as in Theorem 3.3. Then $H_\tau(V,g)$ is essnetially self-adjoint on $[\oplus^4 C_0^\infty(\mathbf{R}^3)] \otimes_{\rm alg} \mathcal{F}_{\rm rad}^{\rm fin}(\mathcal{D})$.*

4 Direct Integral Decomposition

We consider the total Hamiltonain without the external field V

$$H_\tau := H_\tau(0,g) = H_D + H_{\rm rad} + H_{I,\tau}(g). \tag{4.1}$$

This is a Hamiltonian of a *relativistic polaron with spin 1/2.*

The momentum operator $\mathbf{P}^{\rm rad} := (P_1^{\rm rad}, P_2^{\rm rad}, P_3^{\rm rad})$ of the quantized radiation field is defined by

$$P_j^{\rm rad} := d\Gamma(k_j), \tag{4.2}$$

the second quantization of the multiplication operator k_j on $\mathcal{H}_{\rm ph}$, while the momentum operator of the Dirac particle is $-i\nabla$. We define a deformed total momentum operator $\mathbf{P}(\tau) := (P_1(\tau), P_2(\tau), P_3(\tau))$ with parameter $\tau \in \mathbf{R}$ is given by

$$P_j(\tau) := \overline{-iD_j + \tau P_j^{\rm rad}} \tag{4.3}$$

on $\mathcal{F}$ ($j = 1,2,3$). Each $P_j(\tau)$ is self-adjoint and its spectrum is purely absolutely continuous with

$$\sigma(P_j(\tau)) = \mathbf{R}. \tag{4.4}$$

Physically $P_j(\tau)$ is interpreted as the generator of a unitary representation of a (deformed) translation to the j-th direction. It is not difficult to see that, for all $t \in \mathbf{R}$ and $j = 1,2,3$,

$$e^{itP_j(\tau)} H_\tau \subset H_\tau e^{itP_j(\tau)}.$$

This shows a translation invariance of H_τ.

For all $\mathbf{x} \in \mathbf{R}^3$, the operator

$$Q(\mathbf{x}) := \overline{\sum_{j=1}^{3} x_j P_j^{\mathrm{rad}}} \tag{4.5}$$

acting in $\mathcal{F}_{\mathrm{rad}}$ is self-adjoint. Since the mapping: $\mathbf{x} \to e^{iQ(\mathbf{x})}$ is strongly continuous, we can define a decomposable operator

$$W_\tau := \int_{\mathbf{R}^3}^{\oplus} e^{i\tau Q(\mathbf{x})} d\mathbf{x} \tag{4.6}$$

on $\mathcal{F} = \int_{\mathbf{R}^3}^{\oplus} \oplus^4 \mathcal{F}_{\mathrm{rad}} d\mathbf{x}$. It follows that W_τ is unitary.

The Fourier transform on $\mathcal{H}_D = \oplus^4 L^2(\mathbf{R}^3)$ can be naturally extended to a unitary operator on $\mathcal{F}$ by

$$(U_F \Psi)(\mathbf{p}) := \frac{1}{\sqrt{(2\pi)^3}} \int_{\mathbf{R}^3} e^{-i\mathbf{p}\cdot\mathbf{x}} \Psi(\mathbf{x}) d\mathbf{x}, \quad \text{a.e.} \mathbf{p} \in \mathbf{R}^3, \ \Psi \in \mathcal{F}. \tag{4.7}$$

We define a unitary operator on $\mathcal{F}$ by

$$U_\tau := U_F W_\tau. \tag{4.8}$$

Then we have a direct integral decomposition

$$U_\tau \mathcal{F} = \int_{\mathbf{R}^3}^{\oplus} \oplus^4 \mathcal{F}_{\mathrm{rad}} d\mathbf{p}. \tag{4.9}$$

We can show that, for $j = 1, 2, 3$,

$$U_\tau P_j(\tau) U_\tau^{-1} = \int_{\mathbf{R}^3}^{\oplus} p_j d\mathbf{p}. \tag{4.10}$$

Thus the Hilbert space $U_\tau \mathcal{F}$ carries a spectral representation of $\mathbf{P}(\tau)$ and the index parameter $\mathbf{p}$ in the decomposition (4.9) physically means an observed value of the deformed total momentum $\mathbf{P}(\tau)$.

Let

$$H_I := -q \sum_{j=1}^{3} \alpha_j \Phi_{\mathrm{S}}(g_j) \tag{4.11}$$

and, for each $\mathbf{p} \in \mathbf{R}^3$ and $\tau \in \mathbf{R}$,

$$h_D(\mathbf{p}) := \boldsymbol{\alpha} \cdot \mathbf{p} + m\beta, \tag{4.12}$$

$$L(\tau) := \overline{H_{\mathrm{rad}} - \tau \boldsymbol{\alpha} \cdot \mathbf{P}^{\mathrm{rad}}}. \tag{4.13}$$

In terms of these operators, we define

$$H_\tau(\mathbf{p}) := h_D(\mathbf{p}) + L(\tau) + H_I \tag{4.14}$$

acting on $\oplus^4\mathcal{F}_{\rm rad}$. Physically $H_\tau(\mathbf{p})$ is the polaron Hamiltonian of the Dirac particle with a deformed total momentum $\mathbf{p}$.

It should be noted that $H_\tau(\mathbf{p})$ is not in the class of the generalized spin-boson model [3, 4] except for the case $\tau = 0$

We introduce a subspace of $\mathcal{F}_{\rm rad}$:

$$\mathcal{F}_{\rm rad,0}^{\infty} := \mathcal{F}_{\rm rad}^{\rm fin}\left(C_0^\infty(\mathbf{R}^3) \oplus C_0^\infty(\mathbf{R}^3)\right). \tag{4.15}$$

Theorem 4.1 *Assume (3.7). Suppose that $\omega \in L^2(\mathbf{R}^3)_{\rm loc}$. Then, for all $\mathbf{p} \in \mathbf{R}^3$, $H_\tau(\mathbf{p})$ is essentially self-adjoint on $\oplus^4\mathcal{F}_{\rm rad,0}^{\infty}$.*

Theorem 4.2 *Under the same assumption as in Theorem 4.1, H_τ is essentially self-adjoint and*

$$U_\tau \overline{H_\tau} U_\tau^{-1} = \int_{\mathbf{R}^3}^{\oplus} \overline{H_\tau(\mathbf{p})} d\mathbf{p}. \tag{4.16}$$

Remark 4.1 Let

$$\omega_D(\mathbf{p}) := \sqrt{\mathbf{p}^2 + m^2}, \tag{4.17}$$

the energy of the free Dirac particle with momentum $\mathbf{p}$. It is well known (or easy to see) that

$$\sigma(h_D(\mathbf{p}) = \sigma_{\rm d}(h_D(\mathbf{p})) = \{\pm\omega_D(\mathbf{p})\}, \tag{4.18}$$

the multiplicity of each eigenvalue being two. Suppose that $\{\omega(\mathbf{k}) - |\tau|\,|\mathbf{k}|\,|\mathbf{k} \in \mathbf{R}^3\} = [M_\tau, \infty)$ with some constant $M_\tau \geq 0$. Then $\sigma_{\rm ess}(h_D(\mathbf{p}) + L(\tau)) = [-\omega_D(\mathbf{p}) + M_\tau, \infty)$. Hence, if $2\omega_D(\mathbf{p}) \geq M_\tau$, then the eigenvalue $\omega_D(\mathbf{p})$ of $h_D(\mathbf{p}) + L(\tau)$ is embedded in its continuous spectrum. Thus $H_\tau(\mathbf{p})$ gives rise to a preturbation problem of embedded (*degenerate*) eigenvalues. This problem concerns the instability of the Dirac particle with a positive energy under the influence of the quantized radiation field.

5 The Ground-State Energy of the Polaron with a Fixed Deformed Total Momentum

In this section we describe fundamental properties of the ground-state energy of $\overline{H_\tau(\mathbf{p})}$ defined by

$$E_\tau(\mathbf{p}) := \inf\sigma\left(\overline{H(\mathbf{p})}\right), \tag{5.1}$$

provided that $H_\tau(\mathbf{p})$ is essentially self-adjoint. At this stage, however, $\overline{H_\tau(\mathbf{p})}$ is not necessarily bounded below: It may happen that $E_\tau(\mathbf{p}) = -\infty$.

5.1 Self-adjointness and boundedness from below of $H_\tau(\mathbf{p})$

Let

$$\mu_\tau(\mathbf{k}) := \omega(\mathbf{k}) - |\tau||\mathbf{k}|, \quad \mathbf{k} \in \mathbf{R}^3. \tag{5.2}$$

We assume the following:

Hypothesis (H.1)$_\tau$

(i) $\mu_\tau(\mathbf{k}) > 0$ for a.e.$\mathbf{k}$.

(ii) $g,\ g/\sqrt{\mu_\tau} \in L^2(\mathbf{R}^3)$.

Remark 5.1 Hypothesis (H.1)$_\tau$ implies (3.2).

Remark 5.2 The physical case $\omega = \omega_{\rm phys}$ (Remark 2.1), which gives $\mu_1(\mathbf{k}) = 0$ for all $\mathbf{k} \in \mathbf{R}^3$, does not satisfy (H.1)$_1$-(i). On the other hand, if $|\tau| < 1$, then $\omega = \omega_{\rm phys}$ satisfies (H.1)$_\tau$-(i).

Hypothesis (H.2)$_\tau$-(i) may be regarded as a spectral condition for the photon energy-momentum operator $(\omega(\mathbf{k}), \mathbf{k})$, implying that, for a.e. $\mathbf{k} \in \mathbf{R}^3$, $\mu_\tau(\mathbf{k})^{-1}$ exists and the Hermitian matrix

$$\nu_\tau(\mathbf{k}) := \omega(\mathbf{k}) - \tau\boldsymbol{\alpha} \cdot \mathbf{k} \tag{5.3}$$

is nonnegative, invertible with

$$\nu_\tau(\mathbf{k})^{-1} = \omega(\mathbf{k})^{-1} \sum_{n=0}^{\infty} \frac{\tau^n (\boldsymbol{\alpha} \cdot \mathbf{k})^n}{\omega(\mathbf{k})^n}. \tag{5.4}$$

It is easy to see that $H_{\rm rad}$ and $\boldsymbol{\alpha} \cdot \mathbf{P}^{\rm rad}$ are strongly commuting[2]. Hence $L(\tau)$ is self-adjoint. It follows from (H.1)$_\tau$ that, for a.e.$\mathbf{k}$, the matrix $\nu_\tau(\mathbf{k})$ is positive definite, which implies that $L(\tau)$ is nonnegative.

Theorem 5.1 *Assume (H.1)$_\tau$. Then, for all $\mathbf{p} \in \mathbf{R}^3$, $H_\tau(\mathbf{p})$ is self-adjoint with $D(H_\tau(\mathbf{p})) = D(L(\tau))$ and essentially self-adjoint on every core of $L(\tau)$. Moreover, $H_\tau(\mathbf{p})$ is bounded from below.*

[2]Two self-adjoint operators on a Hilbert space are said to *strongly commute* if their spectral measures commute.

5.2 Bounds of the ground-state energy of $H_\tau(\mathbf{p})$

Assume $(\mathrm{H.1})_\tau$. Then, by Theorem 5.1, the ground-state energy $E_\tau(\mathbf{p})$ is finite. We introduce a 4×4 Hermitian matrix:

$$R_\tau(g) := \sum_{r=1}^{2} \frac{1}{2} \int_{\mathbf{R}^3} d\mathbf{k}\, \boldsymbol{\alpha} \cdot \mathbf{e}^{(r)}(\mathbf{k}) \nu_\tau(\mathbf{k})^{-1} \boldsymbol{\alpha} \cdot \mathbf{e}^{(r)}(\mathbf{k}) |g(\mathbf{k})|^2, \tag{5.5}$$

which is positive semi-definite. We have

$$\|R_\tau(g)\| \leq \int_{\mathbf{R}^3} \frac{|g(\mathbf{k})|^2}{\mu_\tau(\mathbf{k})} d\mathbf{k}. \tag{5.6}$$

Proposition 5.2 *Assume $(H.1)_\tau$. Suppose that ω is in $L^2(\mathbf{R}^3)_{\mathrm{loc}}$. Then, for all $\mathbf{p} \in \mathbf{R}^3$,*

$$H_\tau(\mathbf{p}) \geq h_D(\mathbf{p}) - q^2 R_\tau(g) \tag{5.7}$$

In particular,

$$E_\tau(\mathbf{p}) \geq -\omega_D(\mathbf{p}) - q^2 \|R_\tau(\mathbf{g})\|, \tag{5.8}$$

where ω_D is defined by (4.17).

We next estimate $E_\tau(\mathbf{p})$ from above. For $z \in \mathbf{C}^4$ with $\|z\| = 1$, we define

$$\xi_{z,\tau}(\mathbf{k}) := \omega(\mathbf{k}) - \tau \mathbf{u}(z) \cdot \mathbf{k}, \tag{5.9}$$

where

$$\mathbf{u}(z) := ((z, \alpha_1 z), (z, \alpha_2 z), (z, \alpha_3 z)) \in \mathbf{R}^3.$$

It is easy to see that

$$\xi_{z,\tau}(\mathbf{k}) \geq \mu_\tau(\mathbf{k}). \tag{5.10}$$

By this fact, we can define

$$C_\tau(z) := \frac{1}{2} \int_{\mathbf{R}^3} \frac{|g(\mathbf{k})|^2}{\xi_{z,\tau}(\mathbf{k})} \left(|\mathbf{u}(z)|^2 - \frac{|\mathbf{u}(z) \cdot \mathbf{k}|^2}{|\mathbf{k}|^2} \right) d\mathbf{k} \geq 0. \tag{5.11}$$

We set

$$\beta(z) := (z, \beta z)_{\mathbf{C}^4}, \quad z \in \mathbf{C}^4. \tag{5.12}$$

Proposition 5.3 *Assume $(H.1)_\tau$. Then, for all $\mathbf{p} \in \mathbf{R}^3$,*

$$E_\tau(\mathbf{p}) \leq \inf_{z \in \mathbf{C}^4; \|z\|=1} \left\{ \mathbf{u}(z) \cdot \mathbf{p} + m\beta(z) - q^2 C_\tau(z) \right\} \tag{5.13}$$

Remark 5.3 Let $g \neq 0$ as an element of $L^2(\mathbf{R}^3)$. Then (5.13) implies that, for all $\mathbf{p} \in \mathbf{R}^3$, $\lim_{|q| \to \infty} E_\tau(\mathbf{p}) = -\infty$.

Remark 5.4 Estimates (5.8) and (5.13) give an order of ultraviolet divergence of the ground-state energy. To be concrete, consider the case $\omega = \omega_{\rm phys}$, $0 \leq |\tau| < 1$, and $g = \chi_\Lambda / \sqrt{(2\pi)^3 \omega_{\rm phy}}$, where χ_Λ is the characteristic function of the set $\{\mathbf{k} \in \mathbf{R}^3 | \, |\mathbf{k}| \leq \Lambda\}$ ($\Lambda > 0$ is a momentum cutoff parameter). We denote the ground-state energy in this case by $E_\tau^\Lambda(\mathbf{p})$. Applying (5.8) and (5.13) to the present case, we have for all $\mathbf{p} \in \mathbf{R}^3$ and $z \in \mathbf{C}^4$ with $\|z\| = 1$

$$-\frac{q^2}{2\pi^2(1-|\tau|)}\Lambda - \omega_D(\mathbf{p}) \leq E_\tau^\Lambda(\mathbf{p}) \leq \mathbf{u}(z)\cdot\mathbf{p} + m\beta(z) - q^2 G_\tau(z)\Lambda,$$

where

$$G_\tau(z) := \frac{1}{8\pi^2}|\mathbf{u}(z)|^2 \int_{-1}^{1} \frac{1-t^2}{1-\tau|\mathbf{u}(z)|t} dt.$$

In particular, $\lim_{\Lambda\to\infty} E_\tau^\Lambda(\mathbf{p}) = -\infty$.

Let

$$F_\tau(\mathbf{p}) := \frac{1}{2}\int_{\mathbf{R}^3} d\mathbf{k} \frac{|g(\mathbf{k})|^2}{\omega(\mathbf{k}) + \tau \frac{\mathbf{p}\cdot\mathbf{k}}{\omega_D(\mathbf{p})}} \left(\mathbf{p}^2 - \frac{(\mathbf{p}\cdot\mathbf{k})^2}{|\mathbf{k}|^2}\right) \frac{1}{\omega_D(\mathbf{p})^2}. \tag{5.14}$$

Proposition 5.4 *Assume (H.1)$_\tau$. Then, for all $\mathbf{p} \in \mathbf{R}^3$,*

$$E_\tau(\mathbf{p}) \leq -\omega_D(\mathbf{p}) - q^2 F_\tau(\mathbf{p}). \tag{5.15}$$

Proposition 5.5 *Assume (H.1)$_\tau$. Suppose that $\omega \in L^2(\mathbf{R}^3)_{\rm loc}$. Then:*

(i)

$$\lim_{q\to 0} E_\tau(\mathbf{p}) = -\omega_D(\mathbf{p}). \tag{5.16}$$

(ii)

$$\lim_{|\mathbf{p}|\to\infty} \frac{E_\tau(\mathbf{p})}{\omega_D(\mathbf{p})} = -1. \tag{5.17}$$

5.3 Physical mass of the polaron

The physical mass of the polaron may be defined by

$$m_\tau^*(q) := -E_\tau(\mathbf{0}) \tag{5.18}$$

Assume (H.1)$_\tau$ and suppose that $\omega \in L^2(\mathbf{R}^3)_{\rm loc}$. Then it follows from Propositions 5.2 and 5.4 that

$$\sup_{z\in\mathbf{C}^4;\|z\|=1} \left\{q^2 C_\tau(z) - m\beta(z)\right\} \leq m_\tau^*(q) \leq m + q^2\|R_\tau(g)\|. \tag{5.19}$$

In particular,

$$\lim_{q\to 0} m_\tau^*(q) = m. \tag{5.20}$$

If $g \neq 0$ as an element of $L^2(\mathbf{R}^3)$, then

$$\lim_{|q|\to\infty} m_\tau^*(q) = \infty. \tag{5.21}$$

5.4 Properties of $E_\tau(\mathbf{p})$ as a function of $\mathbf{p}$

Proposition 5.6 *Assume (H.1)$_\tau$. Then, for all* $\mathbf{p}, \mathbf{p}' \in \mathbf{R}^3$,

$$|E_\tau(\mathbf{p}) - E_\tau(\mathbf{p}')| \leq |\mathbf{p} - \mathbf{p}'|. \tag{5.22}$$

Proposition 5.7 *Assume (H.1)$_\tau$. Suppose that g is rotation invariant. Then the function:*$\mathbf{p} \to E_\tau(\mathbf{p})$ *is rotation invariant.*

Proposition 5.8 *Assume (H.1)$_\tau$.*

(i) *(concavity) For all* $\mathbf{p}, \mathbf{p}' \in \mathbf{R}^3$ *and* $\lambda \in [0,1]$,

$$\lambda E_\tau(\mathbf{p}) + (1-\lambda)E_\tau(\mathbf{p}') \leq E_\tau(\lambda\mathbf{p} + (1-\lambda)\mathbf{p}'). \tag{5.23}$$

(ii) *For all* $\mathbf{p}, \mathbf{p}' \in \mathbf{R}^3$ *and* $\varepsilon, \lambda \in [0,1]$,

$$E_\tau(\lambda\mathbf{p} + (1-\lambda)\mathbf{p}') \leq \varepsilon E_\tau(\mathbf{p}) + (1-\varepsilon)E_\tau(\mathbf{p}') + (\varepsilon + \lambda - 2\varepsilon\lambda)|\mathbf{p} - \mathbf{p}'|. \tag{5.24}$$

6 Existence of a Ground State of $H_\tau(\mathbf{p})$

A ground state of $H_\tau(\mathbf{p})$ is, by definition, a non-zero vector of $\ker(H_\tau(\mathbf{p}) - E_\tau(\mathbf{p}))$.

6.1 The Massive Case

We define

$$M_\tau := \text{ess.inf}_{\mathbf{k}\in\mathbf{R}^3}\mu_\tau(\mathbf{k}), \tag{6.1}$$

where ess.inf means essential infimum. We assume the following two conditions (H.2)$_\tau$ and (H.3).

Hypothesis (H.2)$_\tau$ $M_\tau > 0$.

Hypothesis (H.3)

(i) $g \in L^2(\mathbf{R}^3)$

(ii) The function ω is uniformly continuous on $\mathbf{R}^3$.

Note that (H.2)$_\tau$ and (H.3)-(i) imply (H.1)$_\tau$ with

$$\omega(\mathbf{k}) \geq M_\tau, \tag{6.2}$$

which physically means that the photon is "massive" or has a "low energy cutoff".

We introduce

$$\Delta_\tau(\mathbf{p}) := \inf_{n\geq 1}\inf_{\mathbf{k}_1,\cdots,\mathbf{k}_n\in\mathbf{R}^3}\left\{E_\tau\left(\mathbf{p} - \sum_{j=1}^n \tau\mathbf{k}_n\right) + \sum_{j=1}^n \omega(\mathbf{k}_j)\right\} - E_\tau(\mathbf{p}). \tag{6.3}$$

Using Proposition 5.6, we see that

$$M_\tau \leq \Delta_\tau(\mathbf{p}) \leq \omega(\mathbf{0}). \tag{6.4}$$

Theorem 6.1 *Assume $(H.2)_\tau$ and (H.3). Suppose that*

$$\lim_{|\mathbf{k}|\to\infty} \mu_\tau(\mathbf{k}) = \infty. \tag{6.5}$$

Then, for all $\mathbf{p} \in \mathbf{R}^3$, $H_\tau(\mathbf{p})$ has purely discrete spectrum in $[E_\tau(\mathbf{p}), E_\tau(\mathbf{p})+\Delta_\tau(\mathbf{p}))$. In particular, $H_\tau(\mathbf{p})$ has a ground state.

6.2 The Massless Case

We next consider the case where Hypothesis $(H.2)_\tau$ does not necessarily hold. We define

$$\left\| \frac{g}{\mu_\tau} \right\| := \sum_{j=1}^{3} \sqrt{\int_{\mathbf{R}^3} \frac{|g(\mathbf{k})|^2}{\mu_\tau(\mathbf{k})^2} \left(1 - \frac{k_j^2}{|\mathbf{k}|^2}\right) d\mathbf{k}} \tag{6.6}$$

Theorem 6.2 *Assume $(H.1)_\tau$, (H.3)-(ii) and (6.5). Suppose that $g/\mu_\tau \in L^2(\mathbf{R}^3)$ with*

$$|q| \left\| \frac{g}{\mu_\tau} \right\| < \sqrt{2}. \tag{6.7}$$

Then, for all $\mathbf{p} \in \mathbf{R}^3$, $H_\tau(\mathbf{p})$ has a ground state $\Psi_\tau(\mathbf{p})$ with $\|\Psi_\tau(\mathbf{p})\| = 1$. Moreover, $\Psi_\tau(\mathbf{p}) \in D(N^{1/2})$ and

$$\|N^{1/2}\Psi_\tau(\mathbf{p})\| \leq \frac{|q|}{\sqrt{2}} \left\| \frac{g}{\mu_\tau} \right\| \tag{6.8}$$

Remark 6.1 Theorem 6.2 does not cover the original physical case: $\omega = \omega_{\text{phys}}$ and $\tau = 1$. But, for $|\tau| < 1$, Theorem 6.1 can be applied to the case $\omega = \omega_{\text{phys}}$.

7 Spectral Properties

7.1 Essential spectrum of $H_\tau(\mathbf{p})$

Theorem 7.1 *Assume $(H.1)_\tau$. Suppose that ω is continuous on $\mathbf{R}^3$. Then, for all $\mathbf{p} \in \mathbf{R}^3$,*

$$\overline{\{E_\tau(\mathbf{p} - \tau\mathbf{k}) + \omega(\mathbf{k}) | \mathbf{k} \in \mathbf{R}^3\}} \subset \sigma_{\text{ess}}(H_\tau(\mathbf{p})). \tag{7.1}$$

We define

$$\delta_\tau(\mathbf{p}) := \inf_{\mathbf{k}\in\mathbf{R}^3} \{E_\tau(\mathbf{p} - \tau\mathbf{k}) + \omega(\mathbf{k})\} - E_\tau(\mathbf{p}). \tag{7.2}$$

It follows that

$$M_\tau \leq \delta_\tau(\mathbf{p}) \leq \Delta_\tau(\mathbf{p}) \leq \omega(\mathbf{0}). \tag{7.3}$$

Corollary 7.2 *Let the same assumption as in Theorem 7.1 be satisfied. Assume (6.5). Then:*

(i) *For all* $\mathbf{p} \in \mathbf{R}^3$,

$$[E_\tau(\mathbf{p}) + \delta_\tau(\mathbf{p}), \infty) \subset \sigma_{\rm ess}(H_\tau(\mathbf{p})). \tag{7.4}$$

(ii) *If* $\omega(\mathbf{0}) = 0$, *then*

$$\sigma(H_\tau(\mathbf{p})) = [E_\tau(\mathbf{p}), \infty). \tag{7.5}$$

Corollary 7.2(ii) shows that the (essential) spectrum of $H_\tau(\mathbf{p})$ in the *massless case* is completely located under a weaker condition than in Theorem 6.1. If we impose stronger conditions than in Theorem 7.1, then we can completely locate the essential spectrum of $H_\tau(\mathbf{p})$ in the massive case too:

Theorem 7.3 *Let the same assumption as in Theorem 6.1 be satisfied. Suppose that, for all* $\mathbf{k}, \mathbf{k}' \in \mathbf{R}^3$,

$$\omega(\mathbf{k} + \mathbf{k}') \leq \omega(\mathbf{k}) + \omega(\mathbf{k}'). \tag{7.6}$$

Then

$$\sigma_{\rm ess}(H_\tau(\mathbf{p})) = [E_\tau(\mathbf{p}) + \delta_\tau(\mathbf{p}), \infty). \tag{7.7}$$

7.2 Spectrum of $\overline{H_\tau}$

Theorem 7.4 *Assume (3.7),* $(H.1)_\tau$ *and (6.5). Suppose that* ω *is continuous on* $\mathbf{R}^3$. *Then*

$$\sigma(\overline{H_\tau}) = \mathbf{R}. \tag{7.8}$$

References

[1] A. Arai, Fundamental properties of the Hamiltonian of a Dirac particle coupled to the quantized radiation field, *Hokkaido University Preprint Series* #447, February 1999.

[2] A. Arai, Spectral analysis of the Hamiltonian of a Dirac particle coupled to the quantized radiation field, preprint, 1999.

[3] A. Arai and M. Hirokawa, On the existence and uniqueness of ground states of a generalized spin-boson model, *J. Funct. Anal.* **151** (1997), 455–503.

[4] A. Arai and M. Hirokawa, Ground states of a general class of quantum field Hamiltonians, to be published in *Rev. Math. Phys.*

[5] F. Bloch and A. Nordsieck, Notes on the radiation field of the electron, *Phys. Rev.* **52** (1937), 54–59.

[6] G. E. Brown and D. G. Ravenhall, On the interaction of two electrons, *Proc. Roy. Soc. London A* **208** (1952), 552–559.

[7] K. Nishijima, *Relativistic Quantum Mechanics*(in Japanese), Baihu-kan, Tokyo, 1973.

[8] R. T. Prosser, On the energy spectrum of the hydrogen atom in a photon field. I, *J. Math. Phys.* **39** (1998), 229–277.

[9] M. Reed and B. Simon, *Methods of Modern Mathematical Physics II: Fourier Analysis, Self-adjointness*, Academic Press, New York, 1975.

[10] M. Reed and B. Simon, *Methods of Modern Mathematical Physics IV: Analysis of Operators*, Academic Press, New York, 1978.

[11] B. Thaller, *The Dirac Equation*, Springer-Verlag, Berlin, Heidelberg, 1992.