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Smooth solutions for degenerate parabolic equations

早稲田大学 理工学部 杉山由恵 (Yoshie SUGIYAMA)
Department of Mathematics,
School of Science and Engineering, Waseda University,
3-4-1 Okubo Tokyo, JAPAN 169-8555

1 Introduction

Many models have been proposed to describe the diffusion process. One for the linear Newtonian diffusion process is the well-known heat equation ; (HE)$u_t = u_{xx}$. A typical example for the nonlinear diffusion process for the Newtonian filtration problem is known as the porous medium equation ; (PE) $u_t = (u^\ell u_x)_x$. Here we note that (PE) is equivalent to (HE) as $\ell = 0$. Concerning the non-Newtonian polytropic filtration problem, the following doubly nonlinear equation is proposed. ; (see [?, Kalashnikov])(DE) $u_t = (u^\ell |u_x|^{p-2} u_x)_x$. In this note, we consider the following more generalized equation which covers all these examples.

$$(E)\begin{cases} \frac{\partial u}{\partial t} - (\varphi(u, u_x)u_x)_x = 0, & (x,t) \in \mathbb{R} \times [0,T], \\ u(x,0) = u_0(x), & x \in \mathbb{R}. \end{cases}$$

These equations have been studied by so many people and many interesting results are obtained so far. Among them, concerning the regularity of solutions for nonlinear problems, the Hölder continuity of u has been well known. However, as for the estimate for the derivative of u itself, little is known. In our recent works [13], [14], [15] and [16], we constructed a time local solution which is smooth with respect to time and space variables. Our aim here is to investigate the structure condition on $\varphi(u, u_x)$ which assures the existence of time local smooth solutions of (E). Our main result is given in the next section, and the sketch of its proof is given in §3. This work is a collaboration with Prof. M. Ôtani (Waseda Univ.).

2 Main Result

Throughout of this note, we always assume the following conditions.

(A.1) $\varphi(u,z) \in C^\infty(\mathbb{R}\times\mathbb{R})$, $\varphi(u,z)$, $\varphi_z(u,z)z \geq 0$, $\forall(u,z)\in\mathbb{R}\times\mathbb{R}$, *and* $\varphi(0,0)=0$.

(A.2) There exist functions $g_1(u,z), g_2(u,z) \in L^\infty_{\mathrm{loc}}(\mathbb{R}\times\mathbb{R})$ such that

$$\left|\frac{(\varphi_z(u,z))^2}{\varphi(u,z)}\right| \leq g_1(u,z), \quad \left|\frac{(\varphi_{zz}(u,z)z)^2}{\varphi_z(u,z)z}\right| \leq g_2(u,z).$$

(A.3) $u_0(x) \in \bigcap_{m=0}^{\infty} H^m(\mathbb{R})$

Then main result is stated as follows.

Theorem (Local existence for (E)) *Assume* (A.1) *and* (A.3), *then there exists a number* $T_2 \in (0,T]$ *depending only on* $|u_0|_{L^\infty}$, $|u_{0x}|_{L^\infty}$, $|D_x^2 u_0|_{L^\infty}$ *and* $|D_x^3 u_0|_{L^\infty}$ *such that* (E) *has a solution* u *belonging to* $C^\infty([0,T_2]\times\mathbb{R})$. Furthermore if (A.2) is satisfied, then there exists a number $T_1 \in (0,T]$ depending only on $|u_0|_{L^\infty}$, $|u_{0x}|_{L^\infty}$ and $|D_x^2 u_0|_{L^\infty}$ such that (E) has a solution u belonging to $C^\infty([0,T_1]\times\mathbb{R})$.

Remark Note that (A.2) is not so restrictive because of the existence of the square power raised to the numerators in (A.2). In fact there are many functions satisfying (A.1) and (A.2) such as $\varphi(u,z) = u^{2\ell}z^{2p}$ with $\ell, p \in \mathbb{N}$.

3 Proof of main theorem

We here give the sketch of a proof of our main theorem. We first prepare the following approximate equations for (E):

$$(\mathrm{E})^\varepsilon \begin{cases} \frac{\partial u^\varepsilon}{\partial t} - \varepsilon u^\varepsilon_{xx} - \left((\varphi(u^\varepsilon, u^\varepsilon_x)u^\varepsilon_x\right)_x = 0, & (x,t)\in\Omega\times[0,T], \\ u^\varepsilon(x,0) = u_0(x), & x\in\Omega, \end{cases} \tag{1}$$

$$(\mathrm{E})^\varepsilon \begin{cases} \frac{\partial u^\varepsilon}{\partial t} - \varepsilon u^\varepsilon_{xx} - \left((\varphi(u^\varepsilon, u^\varepsilon_x)u^\varepsilon_x\right)_x = 0, & (x,t) \in \Omega \times [0,T], \\ u^\varepsilon(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1}$$

where ε is a positive parameter. In order to see that $(\mathrm{E})^\varepsilon$ have C^∞-solutions, we have only to show the following proposition.

Proposition 3.1 *For any $T > 0$, $n \in \mathbb{N}$ and $\varepsilon \in (0,1]$, $(\mathrm{E})^\varepsilon$ has a unique solution u^ε belonging to $H^k(0,T;H^{2(n-k+1)}(\mathbb{R}))$ for all $k = 0,1,\ldots,n$.*

To prove Proposition 3.1, we reduce $(\mathrm{E})^\varepsilon$ to the following evolution equations in $H_k = H^{2k}(\mathbb{R})$:

$$(\mathrm{E})^\varepsilon \begin{cases} u_{\varepsilon t} + \varepsilon A u_\varepsilon + \varphi(u_\varepsilon, u_{\varepsilon x})Au + \varphi_z(u_\varepsilon, u_{\varepsilon x})u_{\varepsilon x}Au_\varepsilon \\ \qquad\qquad = \varphi_u(u_\varepsilon, u_{\varepsilon x})(u_{\varepsilon x})^2, \quad t \in [0,T], \\ u_\varepsilon(0) = u_0. \end{cases}$$

Here A is an operator in H_k defined by $A = -\frac{d^2}{dx^2}$ and $D(A) = H^{2k+2}(\mathbb{R})$, and the inner product of H_k is given by $(u,v)_{H_k} = (u,v)_{L^2(\mathbb{R})} + (A^k u, A^k v)_{L^2(\mathbb{R})}$. In what follows we always assume $u_0 \in D(A^{1/2}) = H^{2k+1}(\mathbb{R})$ and for the sake of simplicity we denote u_ε by u. In solving $(\mathrm{E})^\varepsilon$, we regard the terms $\varphi(u,u_x)Au$, $\varphi_z(u,u_x)u_xAu$ and $\varphi_u(u,u_x)(u_{\varepsilon x})^2$ as perturbations for εAu. We first solve the following equation with the perturbation $\varphi(u,u_x)Au$.

$$(\mathrm{E})^\varepsilon_0 \begin{cases} u_t + \varepsilon Au + \varphi(u,u_x)Au = f, \quad t \in [0,T], \\ u(0) = u_0, \end{cases}$$

where f is a given function in $L^2(0,T;H_k)$.

The difficulty in solving this equation lies in the fact that $\varphi(u,u_x)Au$ is not a small perturbation nor monotone perturbation for εAu. However, roughly speaking, $\varphi(u,u_x)Au$ can be decomposed into the monotone perturbation part and the small perturbation part. In fact, we get

$$\begin{aligned} &(\varphi(u,u_x)Au,\ Au)_{H_k} \\ &= (\varphi(u,u_x)Au,\ Au)_{L^2(\mathbb{R})} + (A^k(\varphi(u,u_x)Au),\ A^{k+1}u)_{L^2(\mathbb{R})} \\ &= \int_{\mathbb{R}} \varphi(u,u_x)(Au)^2\,dx + \int_{\mathbb{R}} \varphi(u,u_x)(A^{k+1}u)^2\,dx + R_k \\ &\geq R_k, \end{aligned}$$

$$R_k = \sum_{j=1}^{2k} {}_{2k}\mathrm{C}_j \int_{\mathrm{R}} D_x^j(\varphi(u,u_x)) \cdot D_x^{2k-j} Au \cdot A^{k+1} u dx, \quad D_x = \frac{\partial}{\partial x}.$$

Since the most singular part of the integrand of R_k is $D_x(\varphi(u,u_x)) \cdot D_x^{2k-1} Au \cdot A^{k+1}u$, R_k can be handled as a small perturbation for $\varepsilon \|A^{k+1}u\|_{L^2}^2$.

To solve $(\mathrm{E})_0^\varepsilon$, we introduce another auxiliary equation:

$$^{\lambda}(\mathrm{E})_0^\varepsilon \begin{cases} u_t + \varepsilon Au + \lambda\varphi(u,u_x)Au = h + f, & t \in [0,T], \ \lambda \in [0,1], \\ u(0) = u_0. \end{cases}$$

If $^{\lambda}(\mathrm{E})_0^\varepsilon$ has a unique solution u^h, we define the operator $^{\lambda}\mathcal{F}_{\eta_0}$ by the following correspondence:

$$^{\lambda}\mathcal{F}_{\eta_0} \ : \ h \mapsto u^h \mapsto -\eta_0\varphi(u^h, u_x^h)Au^h, \qquad \eta_0 \in \mathbb{R}.$$

By making use of the property of $\varphi(u,u_x)Au$ observed above, we can show the following lemma.

Lemma 3.2 *There exist a (sufficiently small) positive number η_0 and a positive number R independent of λ such that $^{\lambda}\mathcal{F}_{\eta_0}$ becomes a contraction mapping from K_R^T into itself, where $K_R^T = \left\{v \in L^2(0,T;H_k); \|v\|_{L^2(0,T;H_k)} \le R\right\}$.*

It is clear that $^{\lambda}(\mathrm{E})_0^\varepsilon$ with $\lambda = 0$ has a unique solution, so $^{0}\mathcal{F}_{\eta_0}$ is well defined. Hence $^{0}\mathcal{F}_{\eta_0}$ has a fixed point by the contraction mapping principle, which implies that $^{\lambda}(\mathrm{E})_0^\varepsilon$ with $\lambda = \eta_0$ admits a unique solution, so $^{\eta_0}\mathcal{F}_{\eta_0}$ is well defined. Hence $^{\lambda}\mathcal{F}_{\eta_0}$ with $\lambda = 2\eta_0$ admits a unique solution. Thus repeating this procedure finite times, we can construct a unique solution of $^{1}(\mathrm{E})_0^\varepsilon$.

To solve the original approximate equation $(\mathrm{E})^\varepsilon$, we next introduce another mapping $^{\mu}\mathcal{S}_{\eta_1}$ defined by the following correspondence:

$$^{\mu}\mathcal{S}_{\eta_1} \ : \ h \mapsto u^h \mapsto -\eta_1\varphi_z(u,u_x)u_x Au,$$

where u^h is the unique solution of the following auxiliary equation.

$$^{1}_{\mu}(\mathrm{E})_0^\varepsilon \begin{cases} u_t + \varepsilon Au + \varphi(u,u_x)Au + \mu\varphi_z(u,u_x)u_x Au = f, \\ \qquad\qquad\qquad\qquad\qquad\qquad t \in [0,T], \ \lambda \in [0,1], \\ u(0) = u_0. \end{cases}$$

By using much the same arguments as for ${}^{\lambda}\mathcal{F}_{\eta_0}$, we can show the following Lemma 3.3.

Lemma 3.3 *There exist a (sufficiently small) positive number η_1 and a positive number R independent of μ such that ${}^{\mu}\mathcal{S}_{\eta_1}$ becomes a contraction mapping from K_R^T into itself, where $K_R^T = \left\{v \in L^2(0,T;H_k); \|v\|_{L^2(0,T;H_k)} \leq R\right\}$.*

Then we can assure the existence of solutions of ${}^{1}_{\mu}(E)^{\varepsilon}_{0}$ with $\mu = 1$. Finally we introduce the mapping W as follows.

$$\mathcal{W} \; : \; f \mapsto u^f \mapsto \varphi(u^f, u_x^f)(u_x^f)^2,$$

where u^f is the unique solution of ${}^{1}_{1}(\mathrm{E})^{\varepsilon}_{0}$. Since $\mathcal{W}$ does not involve any small parameter such as η_0 for ${}^{\lambda}\mathcal{F}_{\eta_0}$ or η_1 for ${}^{\mu}\mathcal{S}_{\eta_1}$ to control the size of the value of $\mathcal{W}$, we need the smallness of T_0. However, by the standard energy estimates for $(\mathrm{E})^{\varepsilon}$, we can establish a priori bound for $\|A^{1/2}u(t)\|_{H_k}$, which assures that the local solution on $[0, T_0]$ can be continued up to $[0, T]$. Thus the first step is completed.

Step2 (A priori estimates)

We apply the "L^∞-energy method" introduced in ([13],[14]). Since the presence of the term εAu does not disturb the following argument, it suffices to establish a priori estimates for the original equation (E).

(i) Estimate for $\|u(x,t)\|_{L^\infty(\mathrm{R}\times[0,T])}$:

Multiply $(\mathrm{E})^{\varepsilon}$ with $\varepsilon = 0$ by $|u|^{r-2}u$, then by the integration by parts, we get

$$\begin{aligned} \frac{1}{r}\frac{d}{dt}\|u(t)\|^r_{L^r(\mathrm{R})} &= -(r-1)\int \varphi(u,u_x)|u|^{r-2}|u_x|^2\,dx \\ &\leq 0. \end{aligned}$$

Hence $\|u(t)\|_{L^r(\mathrm{R})} \leq \|u_0\|_{L^r(\mathrm{R})}$ for all $r \in [2, \infty)$, then letting $r \to \infty$, we deduce

$$\|u(t)\|_{L^r(\mathrm{R})} \leq \|u_0\|_{L^r(\mathrm{R})} \quad \text{for all } r \in [0,\infty] \text{ and } t \in [0,T]. \qquad (2)$$

(ii) Estimate for $\|u_x(x,t)\|_{L^\infty(\mathbf{R}\times[0,T_0])}$:

Multiply $(\mathrm{E})^\varepsilon$ with $\varepsilon = 0$ by $-\Delta_r u = -(|u_x|^{r-2}u_x)_x$, then we have

$$\frac{1}{r}\frac{d}{dt}\|D_x u(t)\|^r_{L^r(\mathbf{R})} + J_0 + J_1 = J_2, \tag{3}$$
$$J_0 = (r-1)\int \varphi(u,u_x)|u_x|^{r-2}(D_x^2 u)^2\,dx \ \geq 0,$$
$$J_1 = (r-1)\int \varphi_z(u,u_x)u_x|u_x|^{r-2}(D_x^2 u)^2\,dx \ \geq 0,$$
$$J_2 = -\int \varphi_u(u,u_x)|u_x|^2(|u_x|^{r-2}u_x)_x\,dx.$$

Here

$$\begin{aligned} J_2 &= -\frac{r-1}{r+1}\int \varphi_u(u,u_x)(|u_x|^r u_x)_x\,dx \\ &= \frac{r-1}{r+1}\int \varphi_{uu}|u_x|^{r+2}\,dx + \frac{r-1}{r+1}\int \varphi_{uz}|u_x|^r u_x D_x^2 u\,dx \\ &= J_2^1 + J_2^2, \end{aligned}$$

where we get

$$\begin{aligned} J_2^1 &\leq C_0 \int |u_x|^{r+2}\,dx, \\ J_2^2 &= \frac{r-1}{r+1}\int \varphi_{uz}(u,u_x)|u_x|^r u_x D_x^2 u\,dx \\ &= \frac{r-1}{r+1}\int \varphi_{uz}(u,u_x)\Big(\frac{1}{r+2}|u_x|^{r+2}\Big)_x\,dx \\ &\leq \frac{1}{r+2}\int |\varphi_{uuz}(u,u_x)u_x + \varphi_{uzz}(u,u_x)D_x^2 u||u_x|^{r+2}\,dx \end{aligned}$$

There exists a constant C_ε depending on ε such that

$$|\varphi_{uuz}(u,u_x)u_x + \varphi_{uzz}(u,u_x)D_x^2 u| \ \leq\ C_\varepsilon.$$

Hence we find

$$J_2^2 \ \leq\ \frac{C_\varepsilon}{r+2}\int |u_x|^{r+2}\,dx. \tag{4}$$

Thus, in view of (2),(3) and (4), we derive

$$\|u_x(t)\|_{L^r(\mathrm{R})}^{r-1}\frac{d}{dt}\|u_x(t)\|_{L^r(\mathrm{R})} \ \leq\ \Big(C_0+\frac{C_\varepsilon}{r+2}\Big)\int |u_x|^{r+2}\ dx.$$

By dividing both sides by $\|u_x(t)\|_{L^r(\mathrm{R})}^{r-1}$, we get

$$\frac{d}{dt}\|u_x(t)\|_{L^r(\mathrm{R})} \ \leq\ \Big(C_0+\frac{C_\varepsilon}{r+2}\Big)\|u_x\|_{L^\infty(\mathrm{R})}^2\|u_x\|_{L^r(\mathrm{R})}.$$

Hence letting $r\to\infty$, we obtain

$$\|u_x(t)\|_{L^\infty(\mathrm{R})} \ \leq\ \|u_{0x}\|_{L^\infty(\mathrm{R})}+C_\varepsilon\int_0^t \|u_x(s)\|_{L^\infty(\mathrm{R})}^3 ds.$$

Then it is easy to show that there exists a positive number $T_0 = T_0(\|u_0\|_{L^\infty(\mathrm{R})},\|u_{0x}\|_{L^\infty(\mathrm{R})})\leq T$ independent of ε such that

$$\|u_x(t)\|_{L^\infty(\mathrm{R})} \ \leq\ 2\|u_{0x}\|_{L^\infty(\mathrm{R})} \quad \text{for all } t\in[0,T_0]. \tag{5}$$

(iii) Estimate for $\|D_x^2u(x,t)\|_{L^\infty(\mathrm{R}\times[0,T_1])}$:

We differentiate $(\mathrm{E})^\varepsilon$ once with respect to the x-variable to obtain $(\mathrm{E})_x^\varepsilon$, and multiply $(\mathrm{E})_x^\varepsilon$ by $-\Delta_r u_x = -(|D_x^2u|^{r-2}D_x^2u)_x$. Then we obtain

$$\frac{1}{r}\frac{d}{dt}\|D_x^2u(t)\|_{L^r(\mathrm{R})}^r \ =\ -\int D_x^2(\varphi(u,u_x)u_x)(|D_x^2u|^{r-2}D_x^2u)_x\ dx$$

$$\begin{aligned} D_x^2(\varphi(u,u_x)u_x) \ &=\ (\varphi+\varphi_z u_x)D_x^3u \\ &\quad +a_1(u,u_x)+a_2(u,u_x)D_x^2u+a_3(u,u_x)(D_x^2u)^2 \end{aligned}$$

So we have

$$\frac{d}{dt}\|D_x^2u(t)\|_{L^r(\mathrm{R})}^r+J_0 = J_1+J_2+J_3 \tag{6}$$

$$J_0 = (r-1)\int\Big(\varphi(u,u_x)+\varphi_z(u,u_x)u_x\Big)|D_x^2u|^{r-2}(D_x^3u)^2\ dx \geq 0,$$

$$J_1 = -\int a_1(u,u_x)(|D_x^2u|^{r-2}D_x^2u)_x\ dx,$$

$$J_2 = -\int a_2(u,u_x)D_x^2u(|D_x^2u|^{r-2}D_x^2u)_x\ dx,$$

$$J_3 = -\int a_3(u,u_x)(D_x^2u)^2(|D_x^2u|^{r-2}D_x^2u)_x\ dx.$$

By using integration by parts and (2) and (5), we get

$$\begin{aligned} J_1 &= -\int \Big(a_{1u}(u,u_x)u_x + a_{1z}(u,u_x)D_x^2u\Big)|D_x^2u|^{r-2}D_x^2u \\ &= C_1\int |D_x^2u|^{r-1}|u_x|\,dx + C_2\int |D_x^2u|^r\,dx, \end{aligned} \tag{7}$$

$$\begin{aligned} J_2 &= -\int a_2(u,u_x)D_x^2u\cdot(|D_x^2u|^{r-2}D_x^2u)_x\,dx \\ &= -\int a_2(u,u_x)\cdot\frac{r-1}{r}(|D_x^2u|^r)_x\,dx \\ &= \frac{r-1}{r}\int \Big(a_{1u}(u,u_x)u_x + a_{1z}(u,u_x)D_x^2u\Big)|D_x^2u|^r\,dx \\ &= C_3\int |D_x^2u|^r\,dx + C_4\int |D_x^2u|^{r+1}\,dx, \end{aligned} \tag{8}$$

$$\begin{aligned} J_3 &= -\int a_3(u,u_x)(D_x^2u)^2\cdot(|D_x^2u|^{r-2}D_x^2u)_x\,dx \\ &= -\int a_3(u,u_x)\cdot\frac{r-1}{r+1}(|D_x^2u|^rD_x^2u)_x\,dx \\ &= \frac{r-1}{r+1}\int \Big(a_{3u}(u,u_x)u_x + a_{3z}(u,u_x)D_x^2u\Big)|D_x^2u|^rD_x^2u\,dx \\ &= C_5\int |D_x^2u|^{r+1}\,dx + C_6\int |D_x^2u|^{r+2}\,dx. \end{aligned} \tag{9}$$

Then (6)–(9) give,

$$\begin{aligned} &\|D_x^2u(t)\|_{L^r(\mathrm{R})}^{r-1}\frac{d}{dt}\|D_x^2u(t)\|_{L^r(\mathrm{R})} \\ &\le C_1\|u_x\|_{L^r(\mathrm{R})}\|D_x^2u\|_{L^r(\mathrm{R})}^{r-1} \\ &\quad +C_7\Big(1+\|D_x^2u\|_{L^\infty(\mathrm{R}\times[0,T])}+\|D_x^2u\|_{L^\infty(\mathrm{R}\times[0,T])}^2\Big)\|D_x^2u\|_{L^r(\mathrm{R})}^r\,dx. \end{aligned}$$

Dividing both sides by $\|D_x^2u(t)\|_{L^r(\mathrm{R})}^{r-1}$ and letting $r\to\infty$, we obtain

$$\begin{aligned} &\|D_x^2u(t)\|_{L^\infty(\mathrm{R})} \\ &\le \|D_x^2u_0\|_{L^\infty(\mathrm{R})} + C_1\|u_x\|_{L^\infty(\mathrm{R})} \\ &\quad +C_7\int_0^t \Big(\|D_x^2(s)\|_{L^\infty(\mathrm{R})}^1 + \|D_x^2(s)\|_{L^\infty(\mathrm{R})}^2 + \|D_x^2(s)\|_{L^\infty(\mathrm{R})}^3\Big)\,ds. \end{aligned}$$

Then it is easy to show that there exists a positive number $T_1 = T_1(\|u_0\|_{L^\infty(\mathrm{R})}, \|u_{0x}\|_{L^\infty(\mathrm{R})}, \|D_x^2u_0\|_{L^\infty(\mathrm{R})}) \le T$ such that

$$\|D_x^2u(t)\|_{L^\infty(\mathrm{R})} \;\leq\; 2\|D_x^2u_0\|_{L^\infty(\mathrm{R})} \quad \text{for all } t\in[0,T_1]. \tag{10}$$

Here we note that the constants C_i $(i = 1,2,\cdots,7)$ depend on the initial data but not on ε.

(iv) Estimate for $\|D_x^3u(x,t)\|_{L^\infty(\mathrm{R}\times[0,T_1])}$:

In order to derive the a priori bound for $\|D_x^3u(x,t)\|_{L^\infty}$ under assumption (A.2), we rely on the following interpolation inequality:

$$\|D_x^3u(x,t)\|_{L^\infty} \;\leq\; \sqrt{2}\|D_x^3u(x,t)\|_{L^2}^{1/2}\cdot\|D_x^4u(x,t)\|_{L^2}^{1/2}. \tag{11}$$

That is to say, we have only to establish a priori bounds for $\|D_x^3u(x,t)\|_{L^2}$ and $\|D_x^4u(x,t)\|_{L^2}$.

(v) Estimate for $\sup_{0\leq t\leq T_1}\|D_x^3u(x,t)\|_{L^2(\mathrm{R})}$:

We differentiate $(\mathrm{E})^\varepsilon$ twice with respect to x-variable to obtain $D_x^2(\mathrm{E})^\varepsilon$, and multiply $D_x^2(\mathrm{E})^\varepsilon$ by $-D_x^4u$. Then we get

$$\frac{1}{2}\frac{d}{dt}\|D_x^3u(t)\|_{L^2(\mathrm{R})}^2 \;=\; -\int D_x^3(\varphi(u,u_x)u_x)D_x^4u\,dx,$$

$$\begin{aligned} D_x^3(\varphi(u,u_x)u_x) \;=\; & (\varphi+\varphi_z u_x)D_x^4u + \left(5\varphi_z + 3\varphi_{zz}u_x\right)D_x^2uD_x^3u \\ & + b_1(u,u_x,D_x^2u) + b_2(u,u_x)D_x^3u. \end{aligned}$$

So we have

$$\frac{1}{2}\frac{d}{dt}\|D_x^3u(t)\|_{L^2(\mathrm{R})}^2 + J_0 = J_1 + J_2 \tag{12}$$

$$J_0 = \int \left(\varphi(u,u_x)+\varphi_z(u,u_x)u_x\right)(D_x^4u)^2 \;\geq 0,$$

$$J_1 = -\int b_1(u,u_x,D_x^2u)D_x^4u\;dx,$$

$$J_2 = -\int b_2(u,u_x)D_x^3uD_x^4u\;dx,$$

$$J_3 = -5\int \varphi_z(u,u_x)D_x^2uD_x^3uD_x^4u\;dx,$$

$$J_4 = -3\int \varphi_{zz}(u,u_x)u_xD_x^2uD_x^3uD_x^4u\;dx.$$

It is easily follows from (2), (5) and (10) that

$$\begin{aligned} J_1 &= \int \Big(b_{1u}(u,u_x,D_x^2u)u_x + b_{1z}(u,u_x,D_x^2u)D_x^2u \\ &\qquad + b_{1v}(u,u_x,D_x^2u)D_x^3u\Big)D_x^3u\ dx \\ &\le C_8 \int (|u_x| + |D_x^2u| + |D_x^3u|)|D_x^3u|\ dx. \end{aligned}$$

Repeating the same arguments with a priori bounds for $\|u_x\|_{L^\infty(\mathrm{R})}$ and $\|D_x^2u\|_{L^\infty(\mathrm{R})}$, we easily obtain the following L^2-bound.

$$\sup_{t\in[0,T]} \|u_x(t)\|_{L^2(\mathrm{R})},\ \sup_{t\in[0,T]} \|D_x^2u(t)\|_{L^2(\mathrm{R})} \le C_9. \tag{13}$$

Therefore we get

$$J_1 \le 2C_8C_9\|D_x^3u\|_{L^2(\mathrm{R})} + C_8\|D_x^3u\|^2_{L^2(\mathrm{R})}. \tag{14}$$

As for J_2, (2) , (5) and (10) yield.

$$\begin{aligned} J_2 &= -\int b_2(u,u_x)D_x^3uD_x^4u\ dx \\ &= -\int b_2(u,u_x)\frac{1}{2}(D_x^3u)_x^2\ dx \\ &= \frac{1}{2}\int \Big(b_{2u}(u,u_x)u_x + b_{1z}(u,u_x)D_x^2u\Big)(D_x^3u)^2 \\ &= C_{10}\int (|u_x| + |D_x^2u|)|D_x^3u|^2\ dx. \end{aligned} \tag{15}$$

In order to estimate J_3, we note that $Z_z = \{(u,z) \in \mathbb{R}\times\mathbb{R};\ \varphi_z(u,z) \ne 0\} \subset Z = \{(u,z) \in \mathbb{R}\times\mathbb{R}; \varphi(u,z) \ne 0\}$. Indeed, $\varphi(u,z) = 0$ implies $\varphi_z(u,z) = 0$, since $\varphi(u,z) \ge 0$. Then we get

$$\begin{aligned} J_3 &= -5\int_{Z_z} \varphi_z(u,u_x)D_x^2uD_x^3uD_x^4u\ dx \\ &\le \frac{1}{4}\int_Z \varphi(u,u_x)(D_x^4u)^2\ dx \\ &\qquad + 25\int_Z \frac{\varphi_z(u,u_x)^2}{\varphi(u,u_x)}|D_x^2u|^2|D_x^3u|^2\ dx. \end{aligned}$$

Here, by assumption (A.2), $\frac{(\varphi_z(u,z))^2}{\varphi(u,z)} \le g_1(u,u_x)$, there exists a positive constant K_1 depending on $\|u\|_{L^\infty(0,T_1;L^\infty(\mathrm{R}))}$ and $\|u_x\|_{L^\infty(0,T_1;L^\infty(\mathrm{R}))}$ such that

$$\left|\frac{(\varphi_z(u,z))^2}{\varphi(u,z)}\right|_{L^\infty(\mathbf{R}\times\mathbf{R})} \leq |g_1(u,u_x)|_{L^\infty(\mathbf{R}\times\mathbf{R})} \leq K_1. \tag{16}$$

Consequently we obtain

$$J_3 \;\leq\; \frac{1}{4}J_0 + 16K_1(C_{11})^2\|D_x^3u\|^2_{L^2(\mathbf{R})}. \tag{17}$$

To estimate J_4, we now note that $Z'_{zz} = \{(u,z) \in \mathbb{R}\times\mathbb{R};\ \varphi_{zz}(u,z)z \neq 0\} \subset Z'_z = \{(u,z) \in \mathbb{R}\times\mathbb{R}; \varphi_{zz}(u,z) \neq 0\}$. In fact, since $\varphi_z(u,z)z \geq 0$, $\varphi_z(u,z)z = 0$ implies that $\Big(\varphi_z(u,z)z\Big)_z = \varphi_{zz}(u,z)z + \varphi_z(u,z) = 0$. Then $\varphi_z(u,z) = 0$ or $z = 0$ gives $\varphi_{zz}(u,z)z = 0$. Hence we find that $\varphi_z(u,z)z = 0$ implies $\varphi_{zz}(u,z)z = 0$. Then we get

$$\begin{aligned} J_4 &= -3\int_{Z'_{zz}} \varphi_{zz}(u,u_x)u_xD_x^2uD_x^3uD_x^4u\ dx \\ &\leq \frac{1}{4}\int_{Z'_z}\varphi_z(u,u_x)u_x(D_x^4u)^2\ dx \\ &\qquad + 9\int_{Z'_z}\frac{\Big(\varphi_{zz}(u,u_x)u_x\Big)^2}{\varphi_z(u,u_x)u_x}|D_x^2u|^2|D_x^3u|^2\ dx. \end{aligned}$$

Here, by assumption (A.2), $\frac{(\varphi_{zz}(u,z)z)^2}{\varphi_z(u,z)z} \leq g_2(u,u_x)$, there exists a positive constant K_2 depending on $\|u\|_{L^\infty(0,T_1;L^\infty(\mathbf{R}))}$ and $\|u_x\|_{L^\infty(0,T_1;L^\infty(\mathbf{R}))}$ such that

$$\left|\frac{(\varphi_{zz}(u,z)z)^2}{\varphi_z(u,z)z}\right|_{L^\infty(\mathbf{R}\times\mathbf{R})} \leq |g_2(u,u_x)|_{L^\infty(\mathbf{R}\times\mathbf{R})} \leq K_2. \tag{18}$$

Consequently we obtain

$$J_4 \;\leq\; \frac{1}{4}J_0 \;+\; 9K_2(C_{11})^2\|D_x^3u\|^2_{L^2(\mathbf{R})}. \tag{19}$$

Thus, in view of (12),(14) (15), (17) and (19), we derive

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|D_x^3u(t)\|^2_{L^2(\mathbf{R})} \\ &\quad\leq\; 2C_8C_9\|D_x^3u(t)\|_{L^2(\mathbf{R})} \\ &\qquad\qquad + \Big(C_8 + C_{10} + (16K_1 + 9K_2)(C_{11})^2\Big)\|D_x^3u(t)\|^2_{L^2(\mathbf{R})}, \\ &\qquad\qquad\qquad\qquad \text{for all } t \in [0.T_1]. \end{aligned} \tag{20}$$

We divide both sides of (20) by $\|D_x^3u(t)\|_{L^2(\mathrm{R})}$ to get,

$$\begin{aligned}
&\frac{d}{dt}\|D_x^3u(t)\|_{L^2(\mathrm{R})} \\
&\quad\leq 2C_8C_9 + \Big(C_8 + C_{10} + (16K_1 + 9K_2)(C_{11})^2\Big)\|D_x^3u(t)\|_{L^2(\mathrm{R})}
\end{aligned}$$

Then Gronwall's inequality gives

$$\sup_{0\leq t\leq T}\|D_x^3u(t)\|_{L^2(\mathrm{R})} \leq (C_{12} + \|D_x^3u_0\|_{L^2(\mathrm{R})})e^{C_{13}T} \tag{21}$$

Here we note that the constants appeared in this step depend on $\|u_0\|_{L^\infty(\mathrm{R})}, \|u_{0x}\|_{L^\infty(\mathrm{R})}, \|D_x^2u_0\|_{L^\infty(\mathrm{R})}, K_1, K_2$ but not on ε.

(vi) Estimate for $\sup_{0\leq t\leq T_1}\|D_x^4u(x,t)\|_{L^2(\mathrm{R})}$:

In order to obtain the a priori bound for $\|D_x^4u(t)\|^2_{L^2(\mathrm{R})}$, we multiply $D_x^3(\mathrm{E})^\varepsilon$ by D_x^5u and repeat much the same arguments as above to get

$$\frac{1}{2}\frac{d}{dt}\|D_x^4u(t)\|^2_{L^2(\mathrm{R})} = -\int D_x^4(\varphi(u,u_x)u_x)D_x^5u\ dx,$$

$$\begin{aligned}
&D_x^4(\varphi(u,u_x)u_x) \\
&= (\varphi + \varphi_z u_x)D_x^5u + d_0(u,u_x,D_x^2u) + d_1(u,u_x,D_x^2u)D_x^3u \\
&\quad + \Big(5\varphi_z(u,u_z) + 3\varphi_{zz}u_x\Big)(D_x^3u)^2 \\
&\qquad + \Big(6\varphi_z(u,u_x) + 4\varphi_{zz}(u,u_x)u_x\Big)D_x^2uD_x^4u \\
&\qquad\quad + d_2(u,u_x)D_x^4u.
\end{aligned}$$

So we have

$$\frac{1}{2}\frac{d}{dt}\|D_x^4u(t)\|^2_{L^2(\mathrm{R})} + J_0 = J_1 + J_2 + J_3 + J_4 + J_5 \tag{22}$$

$$\begin{aligned}
J_0 &= \int \Big(\varphi(u,u_x) + \varphi_z(u,u_x)u_x\Big)(D_x^5u)^2 \geq 0, \\
J_1 &= -\int d_0(u,u_x,D_x^2u)D_x^5u\ dx, \\
J_2 &= -\int d_1(u,u_x,D_x^2u)D_x^3uD_x^5u\ dx, \\
J_3 &= -6\int \varphi_z(u,u_z)\Big((D_x^3u)^2 + \frac{5}{6}D_x^2uD_x^4u\Big)D_x^5u\ dx,
\end{aligned}$$

$$J_4 = -4\int \varphi_{zz} u_x \Big((D_x^3 u)^2 + \frac{3}{4} D_x^2 u D_x^4 u\Big) D_x^5 u \; dx,$$
$$J_5 = -\int d_2(u, u_x) D_x^4 u D_x^5 u \; dx.$$

The integration by parts and estimates (2), (5) and (10) yield

$$\begin{aligned} J_1 &= -\int d_0(u, u_x, D_x^2 u) D_x^5 u \; dx \\ &= \int \Big(d_{0u}(u, u_x, D_x^2 u) u_x + d_{0z}(u, u_x, D_x^2 u) D_x^2 u \\ &\qquad + d_{0v}(u, u_x, D_x^2 u) D_x^3 u\Big) D_x^4 u \; dx \\ &\le C_{14} \int (|u_x| + |D_x^2 u| + |D_x^3 u|)|D_x^4 u| \; dx \end{aligned}$$

Therefore, by virtue of (13) and (21), we get

$$J_1 \le 3 C_{14} C_{15} \|D_x^4 u(t)\|_{L^2(\mathrm{R})}. \tag{23}$$

Similarly we get

$$\begin{aligned} J_2 &= -\int d_1(u, u_x, D_x^2 u) D_x^3 u D_x^5 u \; dx \\ &= \int \Big(d_{1u}(u, u_x, D_x^2 u) u_x + d_{1z}(u, u_x, D_x^2 u) D_x^2 u \\ &\qquad\qquad + d_{1v}(u, u_x, D_x^2 u) D_x^3 u\Big) D_x^3 u D_x^4 u \; dx \\ &\quad + \int d_1(u, u_x, D_x^2 u)(D_x^4 u)^2 \; dx \\ &\le C_{16} \int \Big(1 + |D_x^3 u|\Big)|D_x^3 u||D_x^4 u| \; dx + C_{17} \int (D_x^4 u)^2 \; dx \\ &\le C_{16} \|D_x^3 u\|_{L^2(\mathrm{R})} \|D_x^4 u\|_{L^2(\mathrm{R})} \\ &\quad + C_{16} \|D_x^3 u\|_{L^4(\mathrm{R})}^2 \|D_x^4 u\|_{L^2(\mathrm{R})} + C_{17} \|D_x^4 u\|_{L^2(\mathrm{R})}^2 \end{aligned}$$

Here we note

$$\begin{aligned} \int (D_x^3 u)^3 D_x^3 u \; dx &= \int 3 (D_x^3 u)^2 D_x^4 u D_x^2 u \; dx \\ &\le 3 \|D_x^2 u\|_{L^\infty(\mathrm{R})} \|D_x^4 u\|_{L^2(\mathrm{R})} \|D_x^3 u\|_{L^4(\mathrm{R})}^2, \end{aligned}$$

whence follows

$$\|D_x^3 u\|_{L^4(\mathrm{R})}^2 \le 3 \|D_x^2 u\|_{L^\infty(\mathrm{R})} \|D_x^4 u\|_{L^2(\mathrm{R})}. \tag{24}$$

Then we get

$$\begin{aligned} J_2 &\le C_{16}\|D_x^3u\|_{L^2(\mathrm{R})}\|D_x^4u\|_{L^2(\mathrm{R})} \\ &\quad +\Big(3C_{16}\|D_x^2u\|_{L^\infty(\mathrm{R})}+C_{17}\Big)\|D_x^4u\|^2_{L^2(\mathrm{R})}. \end{aligned} \tag{25}$$

Using again the fact that $Z_z \subset Z$, we now have

$$\begin{aligned} J_3 &= -6\int_{Z_z}\varphi_z(u,u_z)\Big(\frac{5}{6}(D_x^3u)^2+D_x^2uD_x^4u\Big)D_x^5u\ dx \\ &\le \frac{1}{2}\int_Z \varphi(u,u_x)(D_x^5u)^2\ dx \\ &\quad + 36\int_Z \frac{\varphi_z(u,u_x)^2}{\varphi(u,u_x)}\Big(|D_x^3u|^4+|D_x^2u|^2|D_x^4u|^2\Big)\ dx. \end{aligned}$$

Hence by (16) and (24), we get

$$\begin{aligned} J_3 &\le \frac{1}{2}J_0+360K_1\|D_x^2u\|^2_{L^\infty(\mathrm{R})}\cdot\|D_x^4u\|^2_{L^2(\mathrm{R})} \\ &\le \frac{1}{2}J_0+C_{18}\|D_x^4u\|^2_{L^2(\mathrm{R})}. \end{aligned} \tag{26}$$

In order to estimate J_4, recall the fact that $Z'_{zz}\subset Z'_z$, then we have

$$\begin{aligned} J_4 &= -4\int \varphi_{zz}u_x\Big(\frac{3}{4}(D_x^3u)^2+D_x^2uD_x^4u\Big)D_x^5u\ dx \\ &\le \frac{1}{2}\int_Z \varphi_z(u,u_x)u_x(D_x^5u)^2\ dx \\ &\quad + 16\int_Z \frac{\Big(\varphi_{zz}(u,u_x)u_x\Big)^2}{\varphi_z(u,u_x)u_x}\Big(|D_x^3u|^4+|D_x^2u|^2|D_x^4u|^2\Big)\ dx. \end{aligned}$$

Hence by (18) and (24), we get

$$\begin{aligned} J_4 &\le \frac{1}{2}J_0\ +\ 160K_2\|D_x^2u\|^2_{L^\infty(\mathrm{R})}\|D_x^4u\|^2_{L^2(\mathrm{R})} \\ &\le \frac{1}{2}J_0+C_{19}\|D_x^4u\|^2_{L^2(\mathrm{R})} \end{aligned} \tag{27}$$

As for the last term J_5, the integration by parts gives

$$J_5 = -\int d_2(u,u_x)D_x^4uD_x^5u\ dx$$

$$
\begin{aligned}
&= -\int d_2(u,u_x)\cdot\frac{1}{2}\Big((D_x^4u)^2\Big)_x\,dx \\
&= \frac{1}{2}\int\Big(d_{2u}(u,u_x)u_x + d_{2z}(u,u_x)D_x^2u\Big)(D_x^4u)^2\,dx \\
&\le C_{20}\|D_x^4u\|^2_{L^2(\mathrm{R})} \qquad (28)
\end{aligned}
$$

Thus, in view of (22),(23),(25) (26), (27) and (28), we derive

$$
\frac{1}{2}\frac{d}{dt}\|D_x^4u(t)\|^2_{L^2(\mathrm{R})} \le C_{21}\|D_x^4u(t)\|_{L^2(\mathrm{R})} + C_{22}\|D_x^4u(t)\|^2_{L^2(\mathrm{R})}, \quad \text{for all } t\in[0.T_1]. \qquad (29)
$$

Consequently, by Gronwall's inequality, we obtain

$$
\sup_{0\le t\le T}\|D_x^4u(t)\|_{L^2(\mathrm{R})} \le (C_{21}+\|D_x^4u_0\|_{L^2(\mathrm{R})})e^{C_{22}T} \qquad (30)
$$

Here we note that the constants appeared in this step depend only on the initial data and so on but not ε.

(vii) Estimate for $\|D^m u(x,t)\|_{L^\infty}$ $(m\ge 4)$:

The basic method to establish a priori bounds for $\|D^m u(x,t)\|_{L^\infty}$ with $m\ge 4$ is essentially the same as that for the case $m=4$. So we here show how to derive the a priori bound only for the case $m=4$.
First we note that $D_x^3(\mathrm{E})^\varepsilon$ gives

$$
\begin{aligned}
(D_x^3u)_t &= \Big(\varphi+\varphi_z u_x\Big)D_x^5u + e_1(u,u_x,D_x^2u,D_x^3u) \\
&\quad + e_2(u,u_x,D_x^2u)D_x^4u
\end{aligned}
$$

Multiplying this by $-\Delta_r(D_x^3u) = -(|D_x^4u|^{r-2}D_x^4u)_x$, we obtain

$$
\begin{aligned}
&\|D_x^4u(t)\|^{r-1}_{L^r(\mathrm{R})}\frac{d}{dt}\|D_x^4u(t)\|_{L^r(\mathrm{R})} \\
&= J_0 + J_1 + J_2, \qquad (31) \\
&\quad J_0 = (r-1)\int\Big(\varphi(u,u_x)+\varphi_z(u,u_x)u_x\Big)|D_x^4u|^{r-2}(D_x^5u)^2\,dx \\
&\qquad\qquad \ge 0, \\
&\quad J_1 = -\int e_1(u,u_x,D_x^2u,D_x^3u)(|D_x^4u|^{r-2}D_x^4u)_x\,dx, \\
&\quad J_2 = -\int e_2(u,u_x,D_x^2u)D_x^4u(|D_x^4u|^{r-2}D_x^4u)_x\,dx.
\end{aligned}
$$

By the integration by parts, we get

$$\begin{aligned} J_1 &= \int \Big(e_{1u}(u,u_x,D_x^2u,D_x^3u)u_x + e_{1z}(u,u_x,D_x^2u,D_x^3u)D_x^2u \\ &\qquad + e_{1v}(u,u_x,D_x^2u,D_x^3u)D_x^3u \\ &\qquad\quad + e_{1w}(u,u_x,D_x^2u,D_x^3u)D_x^4u\Big)|D_x^4u|^{r-2}D_x^4u\, dx \\ &\le C_{23}\int \Big(|u_x| + |D_x^2u| + |D_x^3u| + |D_x^4u|\Big)|D_x^4u|^{r-1}\, dx. \\ &\le C_{23}\Big(\|u_x\|_{L^r(\mathrm{R})} + \|D_x^2u\|_{L^r(\mathrm{R})} + \|D_x^3u\|_{L^r(\mathrm{R})}\Big)\|D_x^4u\|_{L^r(\mathrm{R})}^{r-1} \\ &\qquad + C_{23}\|D_x^4u\|_{L^r(\mathrm{R})}^r, \end{aligned} \tag{32}$$

$$\begin{aligned} J_2 &= -\int e_2(u,u_x,D_x^2u)\cdot\frac{r-1}{r}(|D_x^4u|^r)_x\, dx. \\ &= -\frac{r-1}{r}\int e_2(u,u_x,D_x^2u)(|D_x^4u|^r)_x\, dx. \\ &= \int \Big(e_{2u}(u,u_x,D_x^2u)u_x + e_{2z}(u,u_x,D_x^2u)D_x^2u \\ &\qquad + e_{2v}(u,u_x,D_x^2u)D_x^3u\Big)|D_x^4u|^r\, dx \\ &\le C_{24}\int \Big(|u_x| + |D_x^2u| + |D_x^3u|\Big)|D_x^4u|^{r-1}\, dx. \\ &\le C_{24}\Big(\|u_x\|_{L^\infty(\mathrm{R})} + \|D_x^2u\|_{L^\infty(\mathrm{R})} + \|D_x^3u\|_{L^\infty(\mathrm{R})}\Big)\|D_x^4u\|_{L^r(\mathrm{R})}^r. \end{aligned} \tag{33}$$

Since $\|v\|_{L^r} \le \|v\|_{L^2}^{\frac{2}{r}}\|v\|_{L^\infty}^{\frac{r-2}{r}}$, (31) – (33) together with (2), (5), (10) and (13) give

$$\frac{d}{dt}\|D_x^4u(t)\|_{L^r(\mathrm{R})} \le C_{25}\Big(1 + \|D_x^4u\|_{L^r(\mathrm{R})}\Big).$$

Hence, by Gronwall's inequality, we get

$$\sup_{\substack{0\le t\le T_1 \\ 2\le r\le\infty}} \|D_x^4u(t)\|_{L^r(\mathrm{R})} \le (C_{25}T + \sup_{2\le r\le\infty}\|D_x^4u_0\|_{L^r(\mathrm{R})})e^{C_{25}T}. \tag{34}$$

Here we note that the constants depend only on the initial data but not on ε. Thus a priori estimates for solutions of approximate equations independent of ε are established. To complete the proof of Theorem, we have only to apply standard arguments for convergence.

We can derive the local estimate for $\|D_x^3 u(x,t)\|_{L^\infty(\mathrm{R}\times[0,T_1])}$ without assuming (A.2) as follows.

(viii) Local estimate for $\|D_x^3 u(x,t)\|_{L^\infty(\mathrm{R}\times[0,T_2])}$:

In order to derive the a priori bound for $\|D_x^3 u(x,t)\|_{L^\infty}$, we differentiate $(\mathrm{E})^\varepsilon$ twice with respect to x-variable to obtain $D_x^2(\mathrm{E})^\varepsilon$, and multiply $D_x^2(\mathrm{E})^\varepsilon$ by $-\Delta_r D_x^3 u = -(|D_x^3 u|^{r-2} D_x^3 u)_x$. Then we get

$$\frac{1}{r}\frac{d}{dt}\|D_x^3 u(t)\|_{L^r(\mathrm{R})}^r = -\int D_x^3(\varphi(u,u_x)u_x)(|D_x^3 u|^{r-2}D_x^3 u)_x\, dx,$$

$$\begin{aligned} D_x^3(\varphi(u,u_x)u_x) &= (\varphi+\varphi_z u_x)D_x^4 u + \Big(5\varphi_z + 3\varphi_{zz}u_x\Big)D_x^2 u D_x^3 u \\ &\quad + b_1(u,u_x,D_x^2 u) + b_2(u,u_x)D_x^3 u. \end{aligned}$$

So we have

$$\frac{1}{r}\frac{d}{dt}\|D_x^3 u(t)\|_{L^r(\mathrm{R})}^r + J_0 = J_1 + J_2 \tag{35}$$

$$J_0 = \int (r-1)\Big(\varphi(u,u_x) + \varphi_z(u,u_x)u_x\Big)|D_x^3 u|^{r-2}(D_x^4 u)^2 \geq 0,$$

$$J_1 = -\int b_1(u,u_x)(|D_x^3 u|^{r-2}D_x^3 u)_x\, dx,$$

$$J_2 = -\int b_2(u,u_x)D_x^3 u(|D_x^3 u|^{r-2}D_x^3 u)_x u\, dx,$$

$$J_3 = -\int \Big(5\varphi_z + 3\varphi_{zz}u_x\Big)u_x D_x^2 u D_x^3 u(|D_x^3 u|^{r-2}D_x^3 u)_x\, dx,$$

By virtue of (2), (5) and (10), it is easy to see that

$$\begin{aligned} J_1 &= \int \Big(b_{1u}(u,u_x)u_x + b_{1z}(u,u_x)D_x^2 u\Big)|D_x^3 u|^{r-2}D_x^3 u\, dx \\ &\leq C_{27}\int(|u_x| + |D_x^2 u|)|D_x^3 u|^{r-1}\, dx \\ &\leq C_{27}\Big(|u_x|_{L^r(\mathrm{R})} + |D_x^2 u|_{L^r(\mathrm{R})}\Big)|D_x^3 u|_{L^r(\mathrm{R})}^{r-1}, \end{aligned} \tag{36}$$

$$
\begin{aligned}
J_2 &= -\int b_2(u,u_x)\cdot\frac{r-1}{r}(|D_x^3u|^r)_x\,dx\\
&= \frac{r-1}{r}\int\Big(b_{2u}(u,u_x)u_x+b_{1z}(u,u_x)D_x^2u\Big)|D_x^3u|^r\\
&\le C_{28}|D_x^3u|^r_{L^r(\mathrm{R})},
\end{aligned}
\tag{37}
$$

and

$$
\begin{aligned}
J_3 &= -\int\Big(5\varphi_z+3\varphi_{zz}u_x\Big)D_x^2u\cdot\frac{r-1}{r}(|D_x^3u|^r)_x\,dx\\
&= \frac{5(r-1)}{r}\int\Big(\varphi_{uz}(u,u_x)u_x+\varphi_{zz}(u,u_x)D_x^2u\Big)D_x^2u|D_x^3u|^r\,dx\\
&\quad+\frac{3(r-1)}{r}\int\Big((\varphi_{uzz}(u,u_x)u_x+\varphi_{zzz}(u,u_x)D_x^2u)u_xD_x^2u\\
&\qquad\qquad+\varphi_{zz}(u,u_x)(D_x^2u)^2\Big)|D_x^3u|^r\,dx\\
&\quad+\frac{(r-1)}{r}\int\Big(5\varphi_z+3\varphi_{zz}(u,u_x)u_x\Big)D_x^3u|D_x^3u|^r\,dx\\
&\le C_{29}\int|D_x^3u|^r\,dx+C_{30}\int|D_x^3u|^{r+1}\,dx\\
&\le C_{29}|D_x^3u|^r_{L^r(\mathrm{R})}+C_{30}|D_x^3u|_{L^\infty(\mathrm{R})}|D_x^3u|^r_{L^r(\mathrm{R})}
\end{aligned}
\tag{38}
$$

Thus, in view of (35),(36) (37) and (38), we derive

$$
\begin{aligned}
&\|D_x^3u(t)\|^{r-1}_{L^r(\mathrm{R})}\frac{d}{dt}\|D_x^3u(t)\|_{L^r(\mathrm{R})}\\
&\le C_{27}\Big(|u_x|_{L^r(\mathrm{R})}+|D_x^2u|_{L^r(\mathrm{R})}\Big)\|D_x^3u(t)\|^{r-1}_{L^r(\mathrm{R})}\\
&\qquad+(C_{28}+C_{29}+C_{30}|D_x^3u|_{L^\infty(\mathrm{R}\times[0,T_1])})\|D_x^3u(t)\|^r_{L^r(\mathrm{R})},
\end{aligned}
$$

By dividing both-hands side by $\|D_x^3u(t)\|^{r-1}_{L^r(\mathrm{R})}$, we get

$$
\begin{aligned}
&\frac{d}{dt}\|D_x^3u(t)\|_{L^r(\mathrm{R})}\\
&\le C_{27}\Big(|u_x|_{L^r(\mathrm{R})}+|D_x^2u|_{L^r(\mathrm{R})}\Big)\\
&\qquad+(C_{28}+C_{29}+C_{30}|D_x^3u|_{L^\infty(\mathrm{R}\times[0,T_1])})\|D_x^3u(t)\|_{L^r(\mathrm{R})}
\end{aligned}
$$

Hence letting $r\to\infty$, we can obtain

$$
\begin{aligned}
&\|D_x^3u(t)\|_{L^\infty(\mathrm{R})}\\
&\le \|D_x^3u_0\|_{L^\infty(\mathrm{R})}+C_{31}\int_0^t\|D_x^3u(s)\|_{L^\infty(\mathrm{R})}+\|D_x^3u(s)\|^2_{L^\infty(\mathrm{R})}ds\\
&\qquad\qquad\text{for all } t\in[0.T_1].
\end{aligned}
$$

Here we note that the constants in this step depend only on the initial data but not on ε.

Then it is easy to show that there exists a positive number $T_2 = T_2(u_0\|_{L^\infty(\mathrm{R})}, \|u_{0x}\|_{L^\infty(\mathrm{R})}, \|D_x^2 u_0\|_{L^\infty(\mathrm{R})}, \|D_x^3 u_0\|_{L^\infty(\mathrm{R})}) \leq T_1$ such that

$$\|D_x^3 u(t)\|_{L^\infty(\mathrm{R})} \leq 2\|D_x^3 u_0\|_{L^\infty(\mathrm{R})} \quad \text{for all } t \in [0, T_2].$$

Concluding remarks

(1) Our arguments here can work also for initial boundary value problems for the same generalized porous medium equations. In fact, for Neumann problems with the homogeneous Neumann condition $u_x(x,t) = 0$ on the boundary ∂I, the same assertion of Theorem holds good, provided that the initial data satisfies the compatibility condition:

$$D_x^{2k-1} u_0(x,t) \mid_{\partial I} = 0, \quad \text{for } k = 0, 1, 2, \cdots.$$

and $\varphi(u, u_x)$ satisfies the additional assumption

$$D_u^i D_z^{2j+1} \varphi(\cdot, 0) = 0, \quad \text{for all } i, j = 0, 1, 2 \cdots.$$

As for the Dirichlet problems with the homogeneous Dirichlet boundary condition, in order to derive the same assertion of Theorem, we need to assume the additional assumption on $\varphi(u, u_x)$ such that

$$D_u^{2i+1} D_z^j \varphi(0, \cdot) = 0, \quad \text{for all } i, j = 0, 1, 2, \cdots,$$

together with the compatibility condition on u_0 :

$$D_x^{2k} u_0(x,t) \mid_{\partial I} = 0, \quad \text{for all } k = 0, 1, 2, \cdots.$$

(2) With slight modifications, our argument works also for the multi-dimension case. However we need much more heavy calculateions than those exploited here.

(3) The following perturbation problem can be also treated within our framework.

$$
\text{(GP)} \begin{cases} \dfrac{\partial u}{\partial t}(x,t) &= (\varphi(u,u_x)u_x)_x + f(u,u_x), \quad (x,t) \in \mathbb{R} \times [0,T], \\ u(x,0) &= u_0(x), \quad x \in \mathbb{R}, \end{cases}
$$

under the following condition: There exist functions $d_1(u,z), d_2(u,z) \in L^\infty_{\text{loc}}(\mathbb{R} \times \mathbb{R})$ such that

$$
\left| \frac{(f_z(u,z))^2}{\varphi(u,z)} \right| \leq d_1(u,z), \quad \left| \frac{(f_{zz}(u,z))^2}{\varphi(u,z)} \right| \leq d_2(u,z).
$$

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