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On the coupled system of nonlinear wave equations with different propagation speeds in two space dimensions

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1 Introduction and Main results

In the present paper, we treat the coupled system of wave equations with different propagation speeds:

$$(\partial_t^2-\Delta)f=F(f,\partial f,g,\partial g),\qquad x\in\mathbb{R}^n,t\in\mathbb{R},\tag{1.1}$$

$$(\partial_t^2-s^2\Delta)g=G(f,\partial f,g,\partial g),\qquad x\in\mathbb{R}^n,t\in\mathbb{R},\tag{1.2}$$

$$f(x,0)=f_0(x)\in H^a,\qquad \partial_t f(x,0)=f_1(x)\in H^{a-1},\qquad x\in\mathbb{R}^n,\tag{1.3}$$

$$g(x,0)=g_0(x)\in H^a,\qquad \partial_t g(x,0)=g_1(x)\in H^{a-1},\qquad x\in\mathbb{R}^n,\tag{1.4}$$

where $\partial=\partial_{x_j}(1\leq j\leq n)$ or ∂_t and s is a propagation speed of (1.2) with $s>1$. The nonlinear terms are as follows:

$$F=\Sigma_{j=1}^3\alpha_jF_j,\qquad \alpha_j\in\mathbb{C},$$
$$G=\Sigma_{j=1}^3\beta_jG_j,\qquad \beta_j\in\mathbb{C},$$
$$F_1=g\partial g,\quad F_2=f\partial g,\quad F_3=g\partial f,$$
$$G_1=f\partial f,\quad G_2=f\partial g,\quad G_3=g\partial f.$$

Our aim is to prove the time local well-posedness with the low regularity initial datas. Physically, this system describe the Klein-Gordon-Zakharov equations (K-G-Z) and the coupled system of complex scalar field and Maxwell equations (C-M). we can derive the time local well-posedness of (K-G-Z) and (C-M) from the time local well-posednwss of this system.

In the case of $n \geq 4$, we can prove the time local well-posedness with $a \geq (n-1)/2$ by the Strichartz estimate. This proof is indepent of the difference of the speeds. In the case of $n = 3$, we can prove the time local well-posedness with $a > 1$ by the Strichartz estimate. To prove the time local well-posedness with $a = 1$ in this argument, we need the limiting case of the Strichartz estimate, which fails. But, Ozawa, Tsutaya and Tsutsumi[5] proved the time local well-posedness in the case of $F = F_1, G = G_3$ with $a = 1$ by using the difference of speeds and Fourier restriction norm method. By this result and the energy conservation, they proved the time global well-posedness of (K-G-Z). By the same argument, T[7] proved the time local well-posedness in the case of $F = F_2, F = F_3, G = G_1$ with $a = 1$. By this result and the energy conservation, we had the time global well-posedness of (C-M).

Fourier restriction norm method was developed by Bourgain [1] and [2] to study the nonlinear Schrödinger equation and the KdV equation, and it was improved for the one dimensional case by Kenig, Ponce and Vega[4]. The related method was developed by Klainerman and Machedon [3] for the nonlinear wave equations.

In the case of $n = 2$, it seem to be difficult to prove the time local well-posedness with $a < 3/4$ by the Strichartz estimate. But, in the present paper, we have the time local well-posedness with $a > 1/2$ by using the difference of speeds and Fourier restriction norm method.

Before we state the theorem, we give several notations. For a function $u(t,x)$, we denote by $\widetilde{u}(\tau,\xi)$ the Fourier transform in both x and t variables of u. For $a,b \in \mathbb{R}, s > 0$ and $l = +$ or $-$, we define the spaces $X_{s,l}^{a,b}$ as follows:

$$X_{s,l}^{a,b} = \{u \in \mathcal{S}'(\mathbb{R}^3) \mid \|u\|_{X_{s,l}^{a,b}} < \infty\}$$

$$\|u\|_{X_{s,l}^{a,b}} = \| < \xi >^a P_{s,l}^b(\tau,\xi)\widetilde{u}\|$$

where $P_{s,l}(\tau,\xi) = (1 + |\tau + sl|\xi||), < \xi >= \sqrt{1+|\xi|^2}$ and $\|\cdot\| = \|\cdot\|_{L^2_{\tau,\xi}}$. For $T > 0$, we denote the cut function $\chi(t),\chi_T(t) \in C_0^\infty$ as follows:

$$\chi(t) = \begin{cases} 1 & \text{for } |t| \leq 1/2, \\ 0 & \text{for } |t| > 1, \end{cases}$$

$$\chi_T(t) = \chi(t/T).$$

For $s > 0$, we define $W_{s,\pm}(t) = e^{\mp ist\omega}$, where $\omega = \sqrt{1-\Delta}$. We put

$$\langle f, g\rangle = \int_{\mathbb{R}^3} f(t,x)\overline{g(t,x)}dtdx.$$

Theorem 1.1. *Let $s > 1$ or $1 > s > 0, a > 1/2$ and $2a - 1/2 > b > 1/2$, then there exist $T > 0$ and problem (1.1)-(1.4) has time local unique solution satisfying*

$$f, g \in C([-T,T] : H^a(\mathbb{R}^2)) \cap C^1([-T,T] : H^{a-1}(\mathbb{R}^2)), \tag{1.5}$$

$$\chi_T(t)\big(f \pm i\omega^{-1}\partial_t f\big) \in X^{a,b}_{1,\pm},$$
$$\chi_T(t)\big(g \pm i(s\omega)^{-1}\partial_t g\big) \in X^{a,b}_{s,\pm}.$$

Furthermore, this solution depends continuously on initial datas in the topology of (1.5).

2 The proof of the Theorem

We first put

$$f_\pm = f \pm i\omega^{-1}\partial_t f,$$
$$g_\pm = g \pm i(s\omega)^{-1}\partial_t g.$$

Then, (1.1)-(1.4) are rewritten as follows:

$$(i\partial_t \mp D)f_\pm = \mp\omega^{-1}F \mp (D-\omega)f_\pm, \tag{2.1}$$
$$(i\partial_t \mp sD)g_\pm = \mp(s\omega)^{-1}G \mp s(D-\omega)g_\pm, \tag{2.2}$$
$$f_\pm(0) = f_{\pm 0}, \quad g_\pm(0) = g_{\pm 0}, \tag{2.3}$$

where

$$f_{\pm 0} = f_0 \pm i\omega^{-1}f_1 \in H^a,$$
$$g_{\pm 0} = g_0 \pm i(s\omega)^{-1}g_1 \in H^a.$$

We try to solve (2.1)-(2.3) locally in time. For that purpase, we consider the following integral equations associated with (2.1)-(2.3):

$$f_\pm(t) = \chi(t)W_{1,\pm}(t)f_{\pm 0} \mp i\chi_T(t)\int_0^t W_{1,\pm}(t-s)\{\omega^{-1}F + (D-\omega)f_\pm\}ds, \tag{2.4}$$
$$g_\pm(t) = \chi(t)W_{s,\pm}(t)g_{\pm 0} \mp i\chi_T(t)\int_0^t W_{s,\pm}(t-s)\{\omega^{-1}G + (D-\omega)g_\pm\}ds. \tag{2.5}$$

If we try to apply the Fourier restriction norm method to (2.4)-(2.5), we have only to prove the following estimates:

$$\|F_1\|_{X^{a-1,b-1+\epsilon}_{1,l}} \le C\Sigma_{j,k}\|g_j\|_{X^{a,b}_{s,j}}\|g_k\|_{X^{a,b}_{s,k}}, \tag{2.6}$$
$$\|F_2\|_{X^{a-1,b-1+\epsilon}_{1,l}} \le C\Sigma_{j,k}\|f_j\|_{X^{a,b}_{1,j}}\|g_k\|_{X^{a,b}_{s,k}}, \tag{2.7}$$
$$\|F_3\|_{X^{a-1,b-1+\epsilon}_{1,l}} \le C\Sigma_{j,k}\|f_j\|_{X^{a,b}_{1,j}}\|g_k\|_{X^{a,b}_{s,k}}, \tag{2.8}$$
$$\|G_1\|_{X^{a-1,b-1+\epsilon}_{s,l}} \le C\Sigma_{j,k}\|f_j\|_{X^{a,b}_{1,j}}\|f_k\|_{X^{a,b}_{1,k}}, \tag{2.9}$$
$$\|G_2\|_{X^{a-1,b-1+\epsilon}_{s,l}} \le C\Sigma_{j,k}\|f_j\|_{X^{a,b}_{1,j}}\|g_k\|_{X^{a,b}_{s,k}}, \tag{2.10}$$
$$\|G_3\|_{X^{a-1,b-1+\epsilon}_{s,l}} \le C\Sigma_{j,k}\|f_j\|_{X^{a,b}_{1,j}}\|g_k\|_{X^{a,b}_{s,k}}, \tag{2.11}$$

where $a > 1/2, 2a - 1/2 > b > 1/2$ and $\epsilon > 0$ which is sufficiently small and j, k and l denote either of $+$ or $-$ sign. Without loss of generality, we can assume $\widetilde{f_j}$ and $\widetilde{g_k} > 0$. Here,we note that

$$\begin{aligned} f &= 1/2(f_+ + f_-), \\ \partial_t f &= \frac{\omega}{2i}(f_+ - f_-), \\ g &= 1/2(g_+ + g_-), \\ \partial_t g &= \frac{s\omega}{2i}(g_+ - g_-). \end{aligned}$$

Therefore, the left hand side of (2.6) is bounded by

$$\Sigma_{j,k} \| g_j \omega g_k \|_{X_{1,l}^{a-1,b-1+\epsilon}}. \tag{2.12}$$

To prove (2.6), we have only to prove

$$\| g_j \omega g_k \|_{X_{1,l}^{a-1,b-1+\epsilon}} \leq C \| g_j \|_{X_{s,j}^{a,b}} \| g_k \|_{X_{s,k}^{a,b}},$$

which is equivalent to

$$\langle g_j g_k, h \rangle \leq C \| g_j \|_{X_{s,j}^{a,b}} \| g_k \|_{X_{s,k}^{a-1,b}} \| h \|_{X_{1,l}^{1-a,1-b-\epsilon}}$$

by duality argument.We obtain this inequality by interpolating between (2.13) and (2.14). In the same manner, we obtain (2.7)-(2.11) from Proposition 2.1.

Proposition 2.1. *Assume that $a > 1/2, b > 1/4, 4a+2b > 3$ and $s > 1$ or $0 < s < 1$. Then the following inequalities hold.*

$$|\langle f, gh \rangle| \leq C \| f \|_{X_{1,j}^{-a,b}} \| g \|_{X_{s,k}^{a,b}} \| h \|_{X_{1,l}^{a,b}} \tag{2.13}$$

$$|\langle f, gh \rangle| \leq C \| f \|_{X_{s,j}^{-a,b}} \| g \|_{X_{1,k}^{a,b}} \| h \|_{X_{1,l}^{a,b}} \tag{2.14}$$

where j,k and l denote either of $+$ or $-$ sign.

Remark 2.1. This inequalities hold with $a = 1/2, b > 1/2$.But, because of $b > 1/2$, we can't apply Proposition 2.1 to (1.1)-(1.4).

Before we prove Proposition 2.1, we mention an essential lemma.

Lemma 2.1. *Assume that $a > 1/2, b > 1/4, 4a + 2b > 3$ and $s > 1$ or $0 < s < 1$. Then, there is a positive constant C and the following inequalities hold.*

$$\sup_{\tau,\xi} < \xi >^{2a} P_{1,j}^{-2b}(\tau,\xi) \big(< \xi >^{-2a} P_{s,k}^{-2b}(\tau,\xi) *_{\tau,\xi} < \xi >^{-2a} P_{1,l}^{-2b}(\tau,\xi) \big) < C$$

where j,k and l denote either of $+$ or $-$ sign.

Proof of Proposition 2.1. Without loss of generality, we can assume $\widetilde{f}$,$\widetilde{g}$ and $\widetilde{h} > 0$. We first prove (2.13). By the duality argument, (2.13) is equivalent to

$$\| < \xi >^a P_{1,j}^{-b}(\tau,\xi)\widetilde{gh}\|_{L_{\tau,\xi}} \leq C\| < \xi >^a P_{s,k}^{b}(\tau,\xi)\widetilde{g}\|\| < \xi >^a P_{s,l}^{b}(\tau,\xi)\widetilde{h}\|,$$

which is equivalent to

$$\| < \xi >^a P_{1,j}^{b}(\tau,\xi)\big(< \xi >^{-a} P_{s,k}^{-b}(\tau,\xi)\widetilde{g} *_{\tau,\xi} < \xi >^{-a} P_{s,l}^{-b}(\tau,\xi)\widetilde{h}\big)\| \leq C\|\widetilde{g}\|^2\|\widetilde{h}\|^2. \tag{2.15}$$

By the Schwartz's inequality and Lemma 2.1, the left hand side of (2.15) is bounded by

$$\begin{aligned}
\int_{\mathbb{R}^3} < \xi >^{2a} P_{1,j}^{2b}(\tau,\xi)\big(< \xi >^{-2a} P_{s,k}^{-2b}(\tau,\xi) *_{\tau,\xi} < \xi >^{-2a} P_{s,l}^{-2b}(\tau,\xi)\big)\big(\widetilde{g}^2 *_{\tau,\xi} \widetilde{h}^2\big) d\tau d\xi \\
\leq C \int_{\mathbb{R}^3} \widetilde{g}^2 *_{\tau,\xi} \widetilde{h}^2 d\tau d\xi \\
\leq C\|\widetilde{g}^2\|_{L^1_{\tau,\xi}}\|\widetilde{h}^2\|_{L^1_{\tau,\xi}} \\
\leq C\|\widetilde{g}\|\|\widetilde{h}\|.
\end{aligned}$$

We next prove (2.14).From (2.13), we have

$$\begin{aligned}
|\langle f, gh\rangle| &= |\langle \omega^{-2a} f, \omega^{2a}(gh)\rangle| \\
&\leq |\langle \omega^{2a} g, (\omega^{-2a} f)\overline{h}\rangle| + |\langle \omega^{2a} h, (\omega^{-2a} f)\overline{g}\rangle| \\
&\leq C\|g\|_{X_{1,j}^{a,b}}\|f\|_{X_{s,k}^{-a,b}}\|h\|_{X_{1,l}^{a,b}}.
\end{aligned}$$

□

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