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Facial structure of convex sets and some applications

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§1 Introduction

Let Ω be a measure space and let $S(\Omega)$ be the space of all measurable functions f on Ω such that $f(t)<\infty \quad (a.e.t\in\Omega)$. An operator $F: X\supset D(F)\longrightarrow S(\Omega)$ is called a convex operator if $D(F)$ is a convex set in a real vector space X, and for each x, $y\in D(F)$ and $0<\alpha<1$,

$$F((1-\alpha)x+\alpha y)(t)\leq(1-\alpha)F(x)(t)+\alpha F(y)(t) \qquad (a.e.t\in\Omega).$$

On the other hand, a function $f: X\times\Omega\longrightarrow\mathbb{R}\cup\{\infty\}$ is called a convex integrand if for each $t\in\Omega$ the function $f(\cdot,t)$ is convex on $\mathbb{R}^{d}$. The convex integrand theory is well known and there are many applications. (See [7] for example.) We say that a convex integrand f represents a convex operator F if

$$f(x,t)=\begin{cases} F(x)(t) & \text{for a.e.} t\in\Omega \qquad x\in D(F)\\ \infty & \qquad\qquad\qquad\quad x\notin D(F)\end{cases} \tag{1}$$

In two of the author's previous paper [3, 4], many applications of integrand representations of convex operators were demonstrated. However, the existense of integrand representation is nontrivial, and it is known only in some special cases. When X is the d-dimensional Euclidian space $\mathbb{R}^{d}$, the represenstion theorem has been proved in [3]. In this note, we apply the theory of the faces of convex sets, and give a new method of the proof which is expected to have an advantage in extending the reperesentation theorem to infinite dimensional cases.

§2 Faces of convex sets

Let $\mathbb{R}^{d}$ be the d-dimensional Euclidean space. When x, $y\in\mathbb{R}^{d}$ are distinct points, then the set $[x,y]=\{(1-t)x+ty\mid 0\leq t\leq 1\}$ is called the closed segment between x and y. Half open segments $(x,y]$, $[x,y)$ and open segments (x,y) are defined analogously. Through this section, we fix a nonempty closed convex set D in $\mathbb{R}^{d}$. A convex subset C of D is called a face of D if

$$(2) \qquad \left\{ \begin{array}{r} x, y \in D \\ (x,y) \cap C \neq \emptyset \end{array} \right\} \quad \text{implies} \quad [x,y] \subset C.$$

By $\mathfrak{F}(D)$, we denote the set of all faces of D. For $C \in \mathfrak{F}(D)$, $\dim C$ is defined to be the dimension of $\operatorname{aff} C$ (the affine hull of C). It is clear that $x \in D$ is an extreme point of D if and only if $\{x\}$ is a 0-dimensional face of D. For preparation, we will state some fundamental properties of faces in the following propositions whose proofs are given in [1].

Proposition 1. *If $C_\lambda \in \mathfrak{F}(D)$, $(\lambda \in \Lambda)$, then $\cap_{\lambda \in \Lambda} C_\lambda \in \mathfrak{F}(D)$, and also there exists a smallest face of D containing $\cup_{\lambda \in \Lambda} C_\lambda$. Hence $(\mathfrak{F}(D), \subset)$ forms a complete lattice.*

Propositon 2. *Let C_1 be a face of D and suppose that $C_2 \subset C_1$. Then $C_2 \in \mathfrak{F}(D)$ if and only if $C_2 \in \mathfrak{F}(C_1)$.*

For a convex set C in $\mathbb{R}^d$, $\mathring{C}$ denotes the relative interior of C, which means the interior of C with respect to the relative topology of $\operatorname{aff} C$. It is easy to see that every face of D is a closed set. Indeed, if x is a point of the closure of a face C and $x_0 \in \mathring{C}$, the convexity of C yields $[x_0, x) \subset \mathring{C} \subset C$. Since C is a face of D, x must be in C.

Proposition 3. *If $C_1, C_2 \in \mathfrak{F}(D)$, and $C_1 \subsetneqq C_2$, then $C_1 \cap \mathring{C}_2 = \emptyset$.*

Proposition 4. *Let x be a point of D and let C be a face of D. Then C is the smallest face of D containing x if and only if $x \in \mathring{C}$.*

Proposition 5. *Let C_1 be a face of D and let x be a relative boundary point of C_1. If C_2 is the smallest face of D containing x, then C_2 is contained by the relative boundary of C_1.*

From these propositons we obtain the following decomposition of a convex set by its faces.

Proposition 6. *For a closed convex set D in $\mathbb{R}^d$,*

$$D = \cup \{\mathring{C}_\lambda \mid C_\lambda \in \mathfrak{F}(D)\}$$

and the union is disjoint.

We say that a collection $\{C_\lambda\}_{\lambda\in\Lambda} \subset \mathfrak{F}(D)$ is normal if $\lambda \in \Lambda$ and $C_\lambda \subset C_\mu \in \mathfrak{F}(D)$ imply $\mu \in \Lambda$. Now we define

$$\mathfrak{A} = \{A = \bigcup_{\lambda\in\Lambda} \overset{\circ}{C}_\lambda \mid \{C_\lambda\}_{\lambda\in\Lambda} \text{ is normal}\}.$$

Since $\{\overset{\circ}{D}\}$ is normal and $\overset{\circ}{D} \in \mathfrak{A}$, $\mathfrak{A}$ is at least nonempty. It is easy to see that if each A_λ $(\lambda \in \Lambda)$ is a member of $\mathfrak{A}$, then so are $\bigcup_{\lambda\in\Lambda} A_\lambda$ and $\bigcap_{\lambda\in\Lambda} A_\lambda$, and therefore $(\mathfrak{A}, \subset)$ is a complete lattice.

Lemma 1. *If $A \in \mathfrak{A}$, then A is a convex set.*

proof. We write $A = \bigcup_{\lambda\in\Lambda} \overset{\circ}{C}_\lambda$ and let x, y be arbitrary points of A. Then there exist λ and μ such that $x \in \overset{\circ}{C}_\lambda$ and $y \in \overset{\circ}{C}_\mu$. Let z be an arbitrary point of the open segment (x, y), and let C_ν be the smallest face containing z. Since C_ν is a face, we have $[x, y] \subset C_\nu$. By Proposition 4, C_λ is the smallest face containing x, and it follows that $C_\lambda \subset C_\nu$. Since the collection $\{C_\lambda\}_{\lambda\in\Lambda}$ is normal, we obtain $\overset{\circ}{C}_\nu \subset A$. This means that $z \in A$, and thus A is convex.

§3 Representation of Convex Operators

In this section, we prove a representation theorem of convex operators. Let $D(F)$ be a convex set in $\mathbb{R}^d$ and let $F : D(F) \longrightarrow S(\Omega)$ be a convex operator. We can assume without loosing generality that the interor of $D(F)$ is nonempty. Through this section, D denotes the closure of $D(F)$. First we state the main theorem.

Theorem 1. *Every convex operator $F : \mathbb{R}^d \supset D(F) \longrightarrow S(\Omega)$ has at least a representation. That is, there exists a convex integrand $f : \mathbb{R}^d \times \Omega \longrightarrow \mathbb{R} \cup \{\infty\}$ such that* (1) *holds.*

For $D = \overline{D(F)}$, we define $\mathfrak{A}$ as in §2. For $A \in \mathfrak{A}$, a convex integrand $f : A \times \Omega \longrightarrow \mathbb{R} \cup \{\infty\}$ is said to represent F on A, if

$$f(x,t) = \begin{cases} F(x)(t) & \text{for a.e.} t \in \Omega \qquad x \in A \cap D(F) \\ \infty & \qquad\qquad\qquad\quad x \in A \setminus D(F). \end{cases}$$

Definition. *For a convex operator F, we define*

$$\tilde{\mathfrak{A}} = \{(A, f) \mid A \in \mathfrak{A}, \text{ and } f \text{ represents } F \text{ on } A\}.$$

Moreover, for (A_1, f_1), $(A_2, f_2) \in \tilde{\mathfrak{A}}$, we write $(A_1, f_1) \leq (A_2, f_2)$ when $A_1 \subset A_2$ and f_2 is an extension of f_1 to A_2.

Lemma 2. $(\tilde{\mathfrak{A}}, \leq)$ *is inductively ordered.*

proof. Let $\{(A_\lambda, f_\lambda)\}_{\lambda \in \Lambda}$ be a totally ordered subset of $\tilde{\mathfrak{A}}$. Then $A = \bigcup_{\lambda \in \Lambda} A_\lambda$ is an element of $\mathfrak{A}$. Moreover we can define a convex integrand f on $A \times \Omega$ satisfying $f = f_\lambda$ on $A_\lambda \times \Omega$ for every $\lambda \in \Lambda$. Clearly, $(A, f) \in \tilde{\mathfrak{A}}$ and it is an upper bound of $\{(A_\lambda, f_\lambda)\}_{\lambda \in \Lambda}$.

Lemma 3. *For $A \in \mathfrak{A}$ such that $A \neq D$, we define $\mathfrak{S}_A = \{C \in \mathfrak{F}(D) \mid C \cap A = \emptyset\}$. Then $(\mathfrak{S}_A, \subset)$ is inductively ordered.*

proof. Let $\{C_\lambda\}_{\lambda \in \Lambda}$ be a totally ordered subset of $\mathfrak{S}_A$. If we put $C = \bigcup_{\lambda \in \Lambda} C_\lambda$, then C is a convex set and $C \cap A \neq \emptyset$. Moreover $C \in \mathfrak{F}(D)$. Indeed, if we assume $(x, y) \cap C \neq \emptyset$, then there exists $\lambda \in \Lambda$ such that $(x, y) \cap C_\lambda \neq \emptyset$. Hence it follows that $[x, y] \subset C_\lambda \subset C$. Thus $C \in \mathfrak{S}_A$ and it is an upper bound of $\{C_\lambda\}_{\lambda \in \Lambda}$.

Lemma 4. *Let A be an element of $\mathfrak{A}$, and assume that $A \neq D$. Then there exists $C \in \mathfrak{S}_A$ such that $A \cup \overset{\circ}{C} \in \mathfrak{A}$.*

proof. By Lemma 3 and Zorn's lemma, $\mathfrak{S}_A$ has at least a maximal element C. It is sufficient to show that $A \cup \overset{\circ}{C} \in \mathfrak{A}$. Put $A = \bigcup_{\lambda \in \Lambda} \overset{\circ}{C}_\lambda$, and take $C_1 \in \mathfrak{F}(D)$, such that $C_1 \supset C$. Since C is a maximal element of $\mathfrak{S}_A$, we have $C_1 \notin \mathfrak{S}_A$ and hence $C_1 \cap A \neq \emptyset$. Therefore we can choose $\lambda \in \Lambda$ such that $\overset{\circ}{C}_\lambda \cap C_1 \neq \emptyset$. It follows from Proposition 3 that, $C_\lambda \subset C_1$ holds. Since the collection $\{C_\lambda\}_{\lambda \in \Lambda}$ is normal, $\overset{\circ}{C}_1 \subset A \subset A \cup \overset{\circ}{C}$. This shows that the collection $\{C_\lambda\}_{\lambda \in \Lambda} \cup \{C\}$ is also normal, and $A \cup \overset{\circ}{C} \in \mathfrak{A}$.

Lemma 5. $\tilde{\mathfrak{A}}$ *is not empty. In other words, there exists $A \in \mathfrak{A}$ such that F has a representation f on A.*

The proof can be done by constructing a convex integrand f which represents F on $\overset{\circ}{D}$. The method of construction is an analogy of that in [4].

Lemma 6. *Suppose that* $(A, f) \in \tilde{\mathfrak{A}}$ *and* $A \neq D$. *Let* $C \in \mathfrak{S}_A$ *is a face such that* $A \cup \overset{\circ}{C} \in \mathfrak{A}$ *as in Lemma 4. Then* f *has an extension* f_1 *defined on* $(A \cup \overset{\circ}{C}) \times \Omega$ *such that* $(A \cup \overset{\circ}{C}, f_1) \in \tilde{\mathfrak{A}}$.

The proof of this lemma is an analogy of one provide in a previous paper by the author [3].

proof of Theorem 1. By Lemma 3, Lemma 5 and Zorn's lemma, $\tilde{\mathfrak{A}}$ has at least a maximal element (A_0, f_0). Moreover, Lemma 6 shows that $A_0 = D$, and this means that f_0 represents F on D. Defining $f_0 = \infty$ on $D^c \times \Omega$, we complete the construction of a representation of F.

§4 Normal Representations

A convex integrand $f : \mathbb{R}^d \times \Omega \longrightarrow \mathbb{R} \cup \{\infty\}$ is said to be normal if $f(\cdot, t)$ is lower semicontinuous for every $t \in \Omega$ and there exists a coutable family of measurable functions $\xi_n : \Omega \longrightarrow \mathbb{R}^d$ $(n = 1, 2, \cdots)$ such that

(1) for each n, $f(\xi_n(t), t)$ is measurable in $t \in \Omega$,

(2) for each $t \in \Omega$, $\{\xi_n(t)\}_{n=1}^{\infty}$ is dense in $D(f(\cdot, t))$,

where $D(f(\cdot, t)) = \{x \in \mathbb{R}^d \mid f(x, t) < \infty\}$. If a convex integrand f is normal, then $f(\xi(t), t)$ is measurable in $t \in \Omega$ whenever $\xi : \Omega \longrightarrow \mathbb{R}^d$ is measurable. A convex operator F is said to have a normal representation if there exists a normal convex integrand which represents F. We will find some conditions under which a convex operator has a normal representation. By the conjugate of a convex integrand f, we mean the convex integrand $f^* : \mathbb{R}^d \times \Omega \longrightarrow \mathbb{R} \cup \{\infty\}$ defined by

$$f^*(\xi, t) = \sup_{x \in \mathbb{R}^d} \{< x, \xi > - f(x, t)\}.$$

Also the biconjugate $f^{**} : \mathbb{R}^d \times \Omega \longrightarrow \mathbb{R} \cup \infty$ is given by

$$f^{**}(x, t) = \sup_{\xi \in \mathbb{R}^d} \{< x, \xi > - f^*(\xi, t)\}.$$

If a convex integrand f is normal, then so are f^* and f^{**}. We note that if a convex integrand f represents a convex operator F then $D(f(\cdot, t))$ does not depend on $t \in \Omega$.

Lemma 7. *Let $f : \mathbb{R}^d \times \Omega \longrightarrow \mathbb{R} \cup \{\infty\}$ be a representation of some onvex operator. Then f is normal if and only if $f(\cdot, t)$ is lower semicontinuous, in other words, $f^{**} = f$ on $\mathbb{R}^d \times \Omega$.*

proof. Let $D = D(f(\cdot, t))$ and take a conuntable subset $\{a_n\}$ of D. If we put $\xi_n(t) = a_n$ for all $t \in \Omega$ and $n = 1, 2, \cdots$, then the family $\{\xi_n\}$ satisfies the definition of nomality.

Remark. If a convex integrand f satisfiies
(1) for each $x \in \mathbb{R}^d$, $f(x, \cdot)$ is measurable, and
(2) $\overline{D(\cdot, t))}$ does not depend on $t \in \Omega$,
the conclusion of Lemma 7 is also valid.

Let $L(\mathbb{R}^d, S(\Omega))$ denotes the space of all linear mapping from $\mathbb{R}^d$ to $S(\Omega)$. We identify $L(\mathbb{R}^d, S(\Omega))$ with the set $S(\Omega)^d = \{\xi = (\xi_1, \cdots, \xi_d) \mid \xi_i \in \S(\Omega), i = 1, \cdots d\}$ by corresponding $S(\Omega)^d \ni (\xi_1, \cdots, \xi_d)$ to the mapping $\varphi : \mathbb{R}^d \ni (x_1, \cdots, x_d) \longrightarrow < x, \xi >= x_1\xi_1 +, \cdots, +x_d\xi_d \in S(\Omega)$. The conjugate operator $F^* : L(\mathbb{R}^d, S(\Omega)) \supset D(F^*) \longrightarrow S(\Omega)$ of F is defined by

$$F^*(\xi) = \bigvee_{x \in D(F^*)} (< x, \xi > -F(x))$$

where $\bigvee$ means the lattice supremumin the space $S(\Omega)$, and $D(F^*)$ is the set of all $\xi \in S(\Omega)^d$ such that the supremum F^* exists. The bi-conjugate operator F^{**} is defined on the space $L(L(\mathbb{R}^d, S(\Omega)), S(\Omega)) = L(S(\Omega)^d, S(\Omega))$, and we regard $S(\Omega)^d$ and $\mathbb{R}^d$ as the subspaces of this by corresponding $\eta \in S(\Omega)^d$ and $x \in \mathbb{R}^d$ to $< \eta, \cdot >$ and $< x, \cdot > \in L(S(\Omega)^d, S(\Omega))$ respectively. For $x \in \mathbb{R}^d$ and $\eta \in S(\Omega)$, F^{**} is defined by

$$F^{**}(x) = \bigvee_{\xi \in D(F^*)} (< x, \xi > -F^*(\xi)),$$

$$F^{**}(\eta) = \bigvee_{\xi \in D(F^*)} (< \eta, \xi > -F^*(\xi)).$$

They are only defined on the domain $D(F^{**})$ where these suprema exist.

Theoem 2. *Let $F : \mathbb{R}^d \supset D(F) \longrightarrow S(\Omega)$ be a convex operator and let $f : \mathbb{R}^d \times \Omega \longrightarrow \mathbb{R} \cup \{\infty\}$ be a representation of F. Then the convex integrand f^* and f^{**} are normal representations of F^* and F^{**} respectively. Moreover for $\xi \in D(F^*)$ and $\eta \in D(F^{**})$,*

$$(F^*(\xi))(t) = f^*(\xi(t), t)$$

$$(F^{**}(\eta))(t) = f^{**}(\eta(t), t)$$

holds for almost every $t \in \Omega$.

This theorem is due to the following lemma.

Lemma 8. *Let* $F : \mathbb{R}^d \supset D(F) \longrightarrow S(\Omega)$ *be a convex operator, and let* $f : \mathbb{R}^d \times \Omega \longrightarrow \mathbb{R}^d \cup \{\infty\}$ *be a representation of* F. *Let* U *be a convex subset of* $D(F)$ *and suppose that* $\inf_{x\in U} f(x,t) > -\infty$ *for almost every* $t \in \Omega$. *Then* $\bigwedge_{x\in U} F(x) \in S(\Omega)$ *exists and*

$$(\bigwedge_{x\in U} F(x))(t) = \inf_{x\in U} f(x,t).$$

proof. Let E be a countable dense set in U. Then we have

$$\inf_{x\in U} f(x,t) = \inf_{x\in E} f(x,t)$$

for $a.e.t \in \Omega$. Hence $\inf_{x\in U} f(x,t)$ is measurable in t and

$$\begin{aligned}(\bigwedge_{x\in U} F(x))(t) &\le (\bigwedge_{x\in E} F(x))(t) \\ &= \inf_{x\in E} f(x,t) \\ &= \inf_{x\in U} f(x,t) \\ &= (\bigwedge_{x\in U} F(x))(t)\end{aligned}$$

for $a.e.t \in \Omega$, and the lemma is proved.

proof of Theorem 2. By Lemma 8 we have

$$\begin{aligned}(F^*(\xi))(t) &= \bigvee_{x\in D(F)} (<\xi, x> - F(x))(t) \\ &= \sup_{x\in D(F)} (<\xi(t), x> - f(x,t)) \\ &= f^*(\xi(t), t) \quad (a.e.t \in \Omega),\end{aligned}$$

for every $\xi \in D(F^*) \subset S(\Omega)^d$. The latter statement can be obtained by analogy.

Combining Lemma 7 and Theorem 2, we obtain the following result.

Theorem 3. *A convex operator* $F : \mathbb{R}^d \supset D(F) \longrightarrow S(\Omega)$ *satisfies*

$$F^{**}(x) = F(x)$$

for every $x \in D(F)$, *if and only if* F *has a normal representation.*

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