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Correlation Dimensions of Quasi-Periodic Trajectories for Evolution Equations

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1. Introduction

In our previous papers ([4], [5], [6]) we have estimated dimensions for quasi periodic orbits by using Diophantine approximations. In the present paper, for a Banach space valued 1-periodic function $g:\mathbf{R}\rightarrow X$, and for an irrational number τ, we consider a discrete quasi-periodic orbit

$$\Sigma=\{\varphi(n):\varphi(n)=g(n\tau),\quad n\in \mathbf{N}\}\subset X.$$

Our purpose is to estimate its correlation dimension in the following cases, which are classified by the algebraic properties of the frequency τ.
(i) Constant type; there exists a constant $c_{0}>0$ such that

$$|\tau-\frac{r}{q}|\geq\frac{c_{0}}{q^{2}} \tag{1.1}$$

for every positive integers r, q.
(ii) quasi Roth number type; there exists a constant $\alpha_{0}>0$ such that for every $\alpha\geq\alpha_{0}$ there exists a constant $c_{\alpha}>0$ which satisfies

$$|\tau-\frac{r}{q}|\geq\frac{c_{\alpha}}{q^{2+\alpha}} \tag{1.2}$$

for every positive integers r, q.
(iii) Roth number type; for every $\varepsilon>0$, there exists a constant $c_{\varepsilon}>0$ which satisfies

$$|\tau-\frac{r}{q}|\geq\frac{c_{\varepsilon}}{q^{2+\varepsilon}} \tag{1.3}$$

for every positive integers r, q.

Definition of correlation dimension. Let $S=\{x_{1}, x_{2}, \ldots, x_{n}, \ldots\}$ be an infinte sequence of elements in X and, for a small number $\varepsilon>0$, define

$$\begin{aligned}\underline{N}(\varepsilon) &= \liminf_{n\rightarrow\infty}\frac{1}{n^{2}}\sum_{i,j=1}^{n}H(\varepsilon-\|x_{i}-x_{j}\|),\\ \overline{N}(\varepsilon) &= \limsup_{n\rightarrow\infty}\frac{1}{n^{2}}\sum_{i,j=1}^{n}H(\varepsilon-\|x_{i}-x_{j}\|),\end{aligned}$$

where $H(\cdot)$ is a Heaviside function:

$$H(a) = \begin{cases} 1 & a \geq 0 \\ 0 & a < 0. \end{cases}$$

and if the limit exits, $N_\varepsilon := \underline{N}(\varepsilon) = \overline{N}(\varepsilon)$. The upper and lower correlation dimension of S, $\overline{C}(S), \underline{C}(S)$, are defined as follows:

$$\begin{aligned} \overline{C}(S) &= \limsup_{\varepsilon \downarrow 0} \frac{\log \underline{N}(\varepsilon)}{\log \varepsilon}, \\ \underline{C}(S) &= \liminf_{\varepsilon \downarrow 0} \frac{\log \overline{N}(\varepsilon)}{\log \varepsilon}. \end{aligned}$$

If N_ε exists and $\overline{C}(S) = \underline{C}(S)$, we say that S has the correlation dimension $C(S) = \overline{C}(S) = \underline{C}(S)$.

Assuming Hölder's continuity on the function $g(\cdot)$, we estimate the dimensions by using Hölder's exponents.

(G1) There exist constants $\delta_1, c_1 : 0 < \delta_1 \leq 1, c_1 > 0$:

$$|g(t) - g(t')| \leq c_1 |t - t'|^{\delta_1}, \quad t, t' \in \mathbf{R}$$

Since we try to estimate the correlation dimension from below, we also need the following Hölder conditions.

(G2) There exist constants $\delta_2, c_2 : 0 < \delta_2 \leq 1, c_2 > 0$:

$$|g(t) - g(t')| \geq c_2 |t - t'|^{\delta_2}, \quad t, t' \in \mathbf{R} : |t - t'| < 1/2.$$

The plan of this paper is as follows; In section 2 we estimate the correlation dimensions of the quasi Roth numbers. In section 3, we give some examples of Roth numbers and quasi Roth numbers. In section 4, as an application, we study q.p. attractors given by an abstract evolution equation with a quasi periodic perturbations, which is given by a Weierstrass type function.

2. Roth numbers case

Consider the following continued fraction of the number τ:

$$\tau = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}} \qquad (a_i \in \mathbf{N}) \tag{2.1}$$

and take the rational approximation as follows. Let $m_0 = 1, n_0 = 0, m_{-1} = 0, n_{-1} = 1$ and define the pair of sequences of natural numbers

$$m_i = a_i m_{i-1} + m_{i-2}, \tag{2.2}$$

$$n_i = a_i n_{i-1} + n_{i-2}, \quad i \geq 1, \tag{2.3}$$

then the elementary number theory gives the Diophantine approximation

$$\frac{1}{m_i(m_{i+1} + m_i)} < |\tau - \frac{n_i}{m_i}| < \frac{1}{m_i m_{i+1}} < \frac{1}{m_i^2}. \tag{2.4}$$

First we consider the case of quasi-Roth number type. Then we can obtain the following estimate:

$$\|\varphi(m) - \varphi(n)\| \geq c_2 (\frac{c_\alpha}{|m-n|^{1+\alpha}})^{\delta_2}, \quad \forall \alpha \geq \alpha_0 \tag{2.5}$$

for every $m, n \in \mathbf{N} : m \neq n$. In fact, since we can find an integer n':

$$|m\tau - n\tau - n'| < \frac{1}{2}$$

(in case $m > n$), Hypothesis **(G2)** and the periodicity of g yield the following estimates.

$$\begin{aligned} \|\varphi(m) - \varphi(n)\| &= \|g(m\tau) - g(n\tau)\| \\ &= \|g(m\tau - n') - g(n\tau)\| \\ &\geq c_2 |(m-n)\tau - n'|^{\delta_2}. \end{aligned}$$

Thus (1.2) yields (2.5).

In order to estimate the correlation dimension from below, we need the following condition on the sequence $\{m_j\}$ in the rational approximations.

(B) There exist constants $\beta, K > 0$:

$$m_{j+1} \leq K m_j^{1+\beta}, \quad \forall j. \tag{2.6}$$

We can show the following lemmas.

Lemma 1. If the condition **(B)** is satisfied for an irrational number τ, then τ is a quasi Roth number for the constant

$$\alpha_0 = \beta(\beta + 3). \tag{2.7}$$

Proof. For every positive integer l, there exists a number j:

$$m_{j-1} \leq l < m_j < K m_{j-1}^{\beta+1} \leq K l^{\beta+1}. \tag{2.8}$$

Since n_j/m_j is a best approximation of τ, we have

$$\begin{aligned} |\tau - \frac{r}{l}| &\geq |\tau - \frac{n_j}{m_j}| \\ &\geq \frac{1}{(m_{j+1} + m_j)m_j} \\ &\geq \frac{1}{2m_{j+1}m_j} > \frac{c}{m_j^{\beta+2}} \\ &> \frac{c}{l^{(\beta+1)(\beta+2)}} \end{aligned}$$

where we denote by c a suitable constant in each term. Thus for every rational number r/l we have

$$|\tau - \frac{r}{l}| > \frac{c}{l^{2+\beta(\beta+3)}}. \qquad \square \tag{2.9}$$

Lemma 2. If τ is a quasi Roth number, then for every $\beta \geq \alpha_0$, there exists $K_\beta > 0$ which satisfies **(B)**:

$$m_{j+1} \leq K_\beta m_j^{1+\beta}, \quad \forall j. \tag{2.10}$$

Proof. It follows from the definition of quasi Roth numbers that, for every $\beta \geq \alpha_0$, there exists $K_\beta > 0$:

$$\frac{K_\beta^{-1}}{m_j^{2+\beta}} \leq |\tau - \frac{n_j}{m_j}| \leq \frac{1}{m_{j+1}m_j}. \tag{2.11}$$

Thus we obtain the conclusion. $\square$

For the quasi periodic sequence $\Sigma = \{\varphi(n) : n \in \mathbf{N}\}$, we can estimate its correlation dimension from below.

Theorem 1. Assume Hypotheses (ii) and **(G2)**. Then we have

$$\underline{C}(\Sigma) \geq \frac{1}{(1+\alpha_0)^2\delta_2}.$$

Proof. Let $k, i : k < i$ be sufficiently large numbers and consider a small constant ε_k, given by

$$\varepsilon_k = (\frac{1}{m_{k+1}})^{\delta_2}.$$

It follows from Lemma 2 that

$$\varepsilon_{k+1} = (\frac{1}{m_{k+2}})^{\delta_2} \quad > \quad K^{-\delta}\varepsilon_k^{1+\alpha_0}$$

where we can assume that $K < 1$. In fact, for every $\beta : \beta > \alpha_0$, from Lemma 2 we obtain

$$m_{j+1} < (Km_{j_0}^{\alpha_0-\beta})m_j^{1+\beta}, \quad \forall j > j_0 \tag{2.12}$$

for some j_0. Then $K \to Km_{j_0}^{\alpha_0-\beta}$. Following the argument below, we can obtain the conclusion for every $\beta : \beta > \alpha_0$.

Let $\alpha_1 > 0$; $\alpha_1 > \alpha_0$, be a constant, which satisfies

$$\alpha_1 + 1 > (1+\alpha_0)^2, \tag{2.13}$$

and, take a small constant $\varepsilon : \varepsilon_{k+1}^{1+\alpha_0} < \varepsilon < \varepsilon_k^{1+\alpha_0}$.
Then, since we have

$$\varepsilon_{k+1}^{1+\alpha_0} > (K^{-\delta_2})^{1+\alpha_0}\varepsilon_k^{(1+\alpha_0)^2} > (K^{-\delta_2})^{1+\alpha_0}\varepsilon_k^{1+\alpha_1}, \tag{2.14}$$

$\exists \alpha : \alpha_0 \leq \alpha \leq \alpha_1$, which satisfies

$$\varepsilon = (K^{-\delta_2})^{1+\alpha_0}\varepsilon_k^{1+\alpha} \tag{2.15}$$

Now, consider an ε-neighborhood $B_\varepsilon := B_\varepsilon(\varphi(1))$.
Then, for a large integer $n \in \mathbf{N}$ and
$l \in I_n = \{1, ..., n\}$, define

$$M_n(\varepsilon) := \#\{\varphi(l) \in B_\varepsilon : \ l \in I_n\}.$$

Assume that $\varphi(n_1) \in B_\varepsilon$ for some $n_1 \in I_n$.
Then, for any $m \in I_n$, $m \neq n_1$, we can estimate

$$\begin{aligned}\|\varphi(m)-\varphi(1)\| &\geq \|\varphi(m)-\varphi(n_1)\| - \|\varphi(n_1)-\varphi(1)\| \\ &\geq c_2 c_\alpha^{\delta_2}(\frac{1}{|m-n_1|})^{(1+\alpha)\delta_2} - \varepsilon, \quad \forall \alpha \geq \alpha_0.\end{aligned}$$

It follows that, if

$$\begin{aligned}c_2 c_\alpha^{\delta_2}(\frac{1}{|m-n_1|})^{(1+\alpha)\delta_2} &\geq 2\varepsilon \\ &= 2(K^{-\delta_2})^{1+\alpha_0}\varepsilon_k^{1+\alpha} \\ &= 2(K^{-\delta_2})^{1+\alpha_0}(\frac{1}{m_{k+1}})^{\delta_2(1+\alpha)},\end{aligned}$$

that is, if

$$|m-n_1| \leq c_\alpha^{\frac{1}{1+\alpha}}(\frac{c_2}{2})^{\frac{1}{(1+\alpha)\delta_2}}(K^{-\delta_2})^{-\frac{1+\alpha_0}{(1+\alpha)\delta_2}} m_{k+1}$$

then $\varphi(m) \notin B_\varepsilon$. Thus we have

$$\begin{aligned}M_n(\varepsilon) &\leq c_\alpha^{-\frac{1}{1+\alpha}}(\frac{c_2}{2})^{-\frac{1}{(1+\alpha)\delta_2}}(K^{-\delta_2})^{\frac{1+\alpha_0}{(1+\alpha)\delta_2}} m_{k+1}^{-1} n \\ &< M_0 m_{k+1}^{-1} n, \\ & \quad M_0 = \sup_{\alpha_0<\alpha<\alpha_1} c_\alpha^{-\frac{1}{1+\alpha}}(\frac{c_2}{2})^{-\frac{1}{(1+\alpha)\delta_2}}(K^{-\delta_2})^{\frac{1+\alpha_0}{(1+\alpha)\delta_2}} m_{k+1}^{-1} n.\end{aligned}$$

Following the argument above for each $\varphi(l), l \in I_n$, we have

$$\frac{1}{n^2}\sum_{l,m=1}^{n} H(\varepsilon - \|\varphi(l)-\varphi(m)\|) \leq \frac{1}{n^2} n M_n(\varepsilon) = \frac{M_n(\varepsilon)}{n}.$$

Thus we have

$$\begin{aligned}\frac{1}{n^2}\sum_{l,m=1}^{n} H(\varepsilon - \|\varphi(l)-\varphi(m)\|) &\leq M_0(\frac{1}{m_{k+1}}) \\ &= M_0 \varepsilon_k^{\frac{1}{\delta_2}} \\ &= M_0((K^{-\delta_2})^{-(1+\alpha_0)}\varepsilon)^{\frac{1}{(1+\alpha)\delta_2}} \\ &\leq M_0 K \varepsilon^{\frac{1}{(1+\alpha_1)\delta_2}}.\end{aligned}$$

It follows that

$$\overline{N}(\varepsilon) = \limsup_{n\to\infty} \frac{1}{n^2}\sum_{l,m=1}^{n} H(\varepsilon - \|\varphi(l)-\varphi(m)\|) \leq c\varepsilon^{\frac{1}{\delta_2(1+\alpha_1)}}$$

for every $\varepsilon > 0$. From the definition we obtain

$$\begin{aligned}
\underline{C}(\Sigma) &= \liminf_{\varepsilon \downarrow 0} \frac{\log \overline{N}(\varepsilon)}{\log \varepsilon} \\
&\geq \liminf_{\varepsilon \downarrow 0} \frac{\log c\varepsilon^{\frac{1}{\delta_2(1+\alpha_1)}}}{\log \varepsilon} \\
&= \frac{1}{(1+\alpha_1)\delta_2}, \quad \forall \alpha_1 > (1+\alpha_0)^2 - 1. \qquad \square
\end{aligned}$$

3. Examples of quasi-Roth numbers

Lemma 3. Let $\{a_j\}$ be the partial quotients in the continued fraction expansion of τ. Assume that, for some $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$;

$$a_{j+1}a_j^2 \leq C_\varepsilon (a_{j-1}a_{j-2}\cdots a_1)^\varepsilon, \quad \forall j. \tag{3.1}$$

Then we have

$$|\tau - \frac{r}{l}| \geq \frac{c_\varepsilon}{l^{2+\varepsilon}}, \quad \forall l, r \in \mathbf{N} \tag{3.2}$$

where $c_\varepsilon = 1/(16C_\varepsilon)$.

Proof. Let $l \in \mathbf{N}$, then $\exists j : m_{j-1} \leq l \leq m_j$ and we have

$$m_{j-1} \leq l \leq m_j \leq (a_j+1)m_{j-1} \leq (a_j+1)l. \tag{3.3}$$

Since n_j/m_j is the best rational approximation, it follows that we have

$$\begin{aligned}
|\tau - \frac{r}{l}| &\geq |\tau - \frac{n_j}{m_j}| \geq \frac{1}{(m_{j+1}+m_j)m_j} \\
&\geq \frac{1}{2(a_{j+1}+1)m_j^2} \geq \frac{1}{2(a_{j+1}+1)(a_j+1)^2 l^2}
\end{aligned}$$

for every $r \in \mathbf{N}$. Since

$$(a_{j+1}+1)(a_j+1)^2 \leq 8a_{j+1}a_j^2,$$

it follows from Hypothesis that

$$(a_{j+1}+1)(a_j+1)^2 < 8C_\varepsilon(a_{j-1}a_{j-2}\cdots a_1)^\varepsilon.$$

On the other hand, we can estimate

$$\begin{aligned}
l \geq m_{j-1} &\geq a_{j-1}m_{j-2} \geq \cdots \\
&\geq a_{j-1}a_{j-2}\cdots a_1 m_0 \\
&= a_{j-1}a_{j-2}\cdots a_1.
\end{aligned}$$

Thus we obtain the conclusion. $\square$

For two sequences $\{a_j\}, \{b_j\}$, we write $a_j \sim b_j$ if there exist constants $c_1, c_2 > 0$:

$$c_1 a_j < b_j < c_2 a_j.$$

Example 1. If $a_j \sim j^\alpha, \quad \alpha > 0$, then τ is a Roth number.

In fact, for every $\varepsilon > 0$ there exists d_ε:

$$(j+1)^{\frac{3}{\varepsilon}} c_2^{\frac{3}{\alpha\varepsilon}} c_1^{-\frac{j-1}{\alpha\varepsilon}} \le d_\varepsilon (j-1)!, \quad \forall j. \tag{3.4}$$

It follows that

$$c_2^3 (j+1)^{3\alpha} \le d'_\varepsilon \{c_1^{j-1}(j-1)!\}^{\alpha\varepsilon}$$

and we have

$$a_{j+1}^3 < d'_\varepsilon (a_{j-1} a_{j-2} \cdots a_1)^\varepsilon.$$

Thus we can apply Lemma 3 for every $\varepsilon > 0$.

Example 2. If $a_j \sim K^j, \quad K > 1$, then τ is also a Roth number.

In fact, for every $\varepsilon > 0$ there exists j_ε:

$$c_2^3 K^{(3+\frac{\log c_1^{-\varepsilon}}{\log K})j_\varepsilon+1} < c_1^{-\varepsilon} K^{\frac{(j_\varepsilon-1)j_\varepsilon}{2}\varepsilon}.$$

Put

$$d_\varepsilon = c_2^3 K^{(3+\frac{\log c_1^{-\varepsilon}}{\log K})j_\varepsilon+1},$$

then we have

$$c_2^3 K^{3j+1} < d_\varepsilon (c_1^{j-1} K^{j-1} \cdots K^2 K^1)^\varepsilon, \quad \forall j,$$

which yields Hypothesis of Lemma 3.

Example 3. If $a_{j+1} \sim m_j^\beta, \quad \beta > 0$, then Hypothesis **(B)** is satisfied. Thus it follows from Lemma 1 that τ is a quasi Roth number: $\alpha_0 = \beta(\beta + 3)$.

Example 4. Here we consider the case that the growth rate of a_j has the order $M^{\kappa^j}, \quad M, \kappa > 1$.

Theorem 2. For constants $c_1, c_2, M, \kappa, \alpha :\ M, \kappa > 1,\ \ \alpha \geq 1$, assume that $\{a_j\}$ the partial quotients in the continued fraction expansion of τ satisfies

$$c_1 M^{\kappa^j} < a_j < c_2 (M^\alpha)^{\kappa^j}. \tag{3.5}$$

Then τ is a quasi-Roth number:

$$\alpha_0 = (\kappa - 1)(\kappa + 2)\alpha.$$

Proof. First we consider the case $c_1 > 1$.
Let $\varepsilon \geq (\kappa - 1)(\kappa + 2)$, then we have

$$\begin{aligned} &\frac{\kappa}{\kappa-1}(\kappa^{j-1} - 1)\varepsilon + \frac{\kappa}{\kappa-1}\varepsilon \\ &\geq \frac{\kappa}{\kappa-1}\kappa^{j-1}\varepsilon \\ &\geq \kappa\kappa^{j-1}(\kappa+2)\alpha. \end{aligned}$$

It follows that

$$(M^\alpha)^{\kappa^{j+1}}(M^\alpha)^{2\kappa^j} \leq M^{\frac{\kappa}{\kappa-1}\varepsilon} M^{(\kappa^1+\kappa^2+\cdots+\kappa^{j-1})\varepsilon}.$$

Thus we can apply Lemma 3, since we have

$$\begin{aligned} a_j^2 a_{j+1} &\leq c_2^3 (M^\alpha)^{\kappa^{j+1}}(M^\alpha)^{2\kappa^j} \\ &\leq c_2^3 M^{\frac{\kappa}{\kappa-1}\varepsilon} M^{(\kappa^1+\kappa^2+\cdots+\kappa^{j-1})\varepsilon} \\ &\leq C_\varepsilon (a_1 a_2 \cdots a_{j-1})^\varepsilon. \end{aligned}$$

Next we consider the case $0 < c_1 < 1$.
Take a constant $r : 0 < r < 1$ and put $M' = Mr$.
Then, for a large j_0, we have

$$c_1 (r^{-1})^{\kappa^{j_0}} > 1$$

and

$$c_1 (r^{-1})^{\kappa^{j_0}} M'^{\kappa^j} < a_j < c_2 M^{\alpha\kappa^j}$$

for every $j \geq j_0$. Since

$$M = M'^{\alpha \log M/(\log M + \log r)},$$

it follows from the above argument that $\exists C'_\varepsilon$:

$$a_{j+1} a_j^2 \leq C'_\varepsilon (a_1 a_2 \cdots a_{j-1})^\varepsilon$$

for every $j \geq j_0$. Put

$$C_\varepsilon = \max_{j=1,\ldots,j_0} \{C'_\varepsilon, a_{j+1} a_j^2 / (a_1 a_2 \cdots a_{j-1})^\varepsilon\}.$$

Then we can apply Lemma 3 for every ε, which satisfies

$$\varepsilon \geq (\kappa-2)(\kappa-1)\alpha \cdot \frac{\log M}{\log M + \log r}.$$

Since the above inequality holds for every $r : 0 < r < 1$, we can conclude that

$$\alpha_0 = (\kappa-2)(\kappa-1)\alpha. \qquad \square$$

4. Example of quasi periodic attractor

In this section we study an abstract evolution equation with a perturbation given by a Weierstrass type function. First we investigate the Hölder continuity of the Weierstrass type function.

Let H be a separable Hilbert space with its norm also denoted by $\|\cdot\|$ and $\{\varphi_i\}$ be a complete orthonormal system in H. We consider a H-valued W-type function $h : R \to H$ defined by

$$h(t) = \sum_{k=1}^{\infty} (\lambda^k)^{-\delta} e^{i2\pi\lambda^k t} \varphi_k \tag{4.1}$$

for some constants $\lambda > 1,\ 0 < \delta < 1$.

Lemma 4. *The function $h(t)$ satisfies*

$$\|h(t) - h(t')\| \leq d_1 |t-t'|^\delta, \tag{4.2}$$

$$\|h(t) - h(t')\| \geq d_2 |t-t'|^\delta \tag{4.3}$$

for $t, t' \in R : |t-t'| < (2\lambda)^{-1}$ and $d_1 = d_1(\lambda,\delta), d_2 = d_2(\lambda,\delta)$.

Proof. Since $|t-t'| < (2\lambda)^{-1}$, there exists an integer N such that

$$\frac{\lambda^{-(N+1)}}{2} \leq |t-t'| \leq \frac{\lambda^{-N}}{2}. \tag{4.4}$$

Using the above inequality and

$$2\pi\lambda^N |t-t'| \leq \pi, \quad |e^{i\vartheta} - 1| \leq |\vartheta|, \ \text{ for } \ |\vartheta| \leq \pi,$$

we obtain

$$\begin{aligned} \|h(t) - h(t')\|^2 &= \sum_{k=1}^{\infty} (\lambda^{2k})^{-\delta} |e^{i2\pi\lambda^k(t-t')} - 1|^2 \\ &\leq \sum_{k=1}^{N} (\lambda^{2k})^{-\delta} (2\pi\lambda^k)^2 |t-t'|^2 + \sum_{k=N+1}^{\infty} 4(\lambda^{2k})^{-\delta} \\ &\leq \frac{4\pi^2\lambda^{2N(1-\delta)}}{1-\lambda^{2(\delta-1)}} |t-t'|^2 + \frac{4\lambda^{-2(N+1)\delta}}{1-\lambda^{-2\delta}}. \end{aligned}$$

It follows from (4.4) that

$$\begin{aligned}
\|h(t)-h(t')\|^2 &\leq [\frac{\pi^2 2^{2\delta}}{1-\lambda^{2(\delta-1)}}+\frac{4\cdot 2^{2\delta}}{1-\lambda^{-2\delta}}]|t-t'|^{2\delta} \\
&\leq d_1^2|t-t'|^{2\delta}.
\end{aligned}$$

Next, assume that $t,t'\in R$ satisfy(4.4), then, applying an elementary inequality

$$|e^{i\vartheta}-1|\geq 2|\sin\frac{\vartheta}{2}|\geq\frac{2}{\pi}|\vartheta|,\quad -\pi\leq\vartheta\leq\pi,$$

we obtain

$$\begin{aligned}
\|h(t)-h(t')\|^2 &\geq \sum_{k=1}^{N}(\lambda^{2k})^{-\delta}|e^{i2\pi\lambda^k(t-t')}-1|^2 \\
&\geq \lambda^{-2N\delta}|e^{i2\pi\lambda^N(t-t')}-1|^2 \\
&\geq \lambda^{-2N\delta}(\frac{2}{\pi}2\pi\lambda^N(t-t'))^2 \\
&\geq 4\cdot 2^{2\delta}\lambda^{2(\delta-1)}|t-t'|^{2\delta}. \qquad \square
\end{aligned}$$

Now we consider a linear abstract equation on the Hilbert space H:

$$\begin{aligned}
&\frac{du}{dt}+Au=f^*(t),\quad t>0,\\
&u(0)=u_0.
\end{aligned}\tag{4.5}$$

We assume that A is a selfadjoint positive definite operator with dense domain $D(A)$ in H, and that A^{-1} exists and is compact. Then it is well known that there exist eigenvalues λ_j and corresponding eigenfunctions φ_j of the operator A satisfying the following conditions:

$$\begin{aligned}
&0<\lambda_1<\lambda_2<\cdots<\lambda_j<\cdots,\quad \lim_{j\to\infty}\lambda_j=\infty,\\
&A\varphi_j=\lambda_j\varphi_j,\quad j=1,2,\cdots,\\
&\{\varphi_j(\cdot)\}\quad \text{forms a complete orthonormal system in } H.
\end{aligned}$$

Here we assume that the perturbation $f^*(t)$ takes values in $D(A)^*$. Thus we consider (4.5) in the distribution sense. (In [3] we can find the various examples in the control theory where the perturbations or the control functions are given in the distribution sense.) Denote the inner product in H by $(\cdot,\cdot)$ and the dual pair between $D(A)$ and $D(A)^*$ by $<\cdot,\cdot>$. Define a W-type function $f:R\to H$ by

$$f(t)=\sum_{k=0}^{\infty}(\mu^{-\delta_1})^k e^{i2\pi\mu^k\tau t}\varphi_{j_k}+\sum_{k=0}^{\infty}(\nu^{-\delta_2})^k e^{i2\pi\nu^k\tau t}\varphi_{l_k}$$

where μ, ν are positive integers and the subsequences $\{j_k\}, \{l_k\} : \{j_k\} \cap \{l_k\} = \emptyset$ will be determined later. We consider a $D(A^*)$-valued functions f^* given by

$$f^*(t) \simeq \sum_{k=0}^{\infty} (\mu^{-\delta_1})^k \lambda_{j_k} e^{i2\pi\mu^k\tau t} \varphi_{j_k} + \sum_{k=0}^{\infty} (\nu^{-\delta_2})^k \lambda_{l_k} e^{i2\pi\nu^k\tau t} \varphi_{l_k},$$

which means that, for $u = \sum_{j=1}^{\infty} u_j \varphi_j \in D(A)$,

$$< f^*, u >= \sum_{k=0}^{\infty} (\mu^{-\delta_1})^k \lambda_{j_k} e^{i2\pi\mu^k\tau t} u_{j_k} + \sum_{k=0}^{\infty} (\nu^{-\delta_2})^k \lambda_{l_k} e^{i2\pi\mu^k\tau t} u_{l_k}. \tag{4.6}$$

Taking the dual pairs with $\varphi_{j_k}, \varphi_{l_k}$ in (4.5) and applying elementary calculations, we can show that the solution $u(t)$ converges to the following W-type function $u_\infty(t)$ in H as $t \to \infty$

$$\begin{aligned} u_\infty(t) &= \sum_{k=0}^{\infty} (\mu^{-\delta_1})^k \frac{\lambda_{j_k}}{\lambda_{j_k} + i2\pi\mu^k} e^{i2\pi\mu^k\tau t} \varphi_{j_k} + \sum_{k=0}^{\infty} (\nu^{-\delta_2})^k \frac{\lambda_{l_k}}{\lambda_{l_k} + i2\pi\nu^k} e^{i2\pi\nu^k\tau t} \varphi_{l_k} \\ &:= g_1(\tau t) + g_2(\tau t). \end{aligned}$$

In fact, for the ordinary differential equations

$$\begin{aligned} &\dot{u}_{j_k}(t) = -\lambda_{j_k} u_{j_k}(t) + \mu^{-\delta_1 k} \lambda_{j_k} e^{i2\pi\mu^k\tau t}, \\ &u_{j_k}(0) = u_{j_k,0}, \\ &\dot{u}_{l_k}(t) = -\lambda_{l_k} u_{l_k}(t) + \nu^{-\delta_2 k} \lambda_{l_k} e^{i2\pi\nu^k\tau t}, \\ &u_{l_k}(0) = u_{l_k,0}, \qquad k = 0, 1, 2, \ldots \end{aligned}$$

where $u(t) = \sum_k u_k(t)\varphi_k$, we have

$$\begin{aligned} u_{j_k}(t) &= e^{-\lambda_{j_k} t} u_{j_k,0} + \frac{\mu^{-\delta_1 k}\lambda_{j_k}}{\lambda_{j_k} + i2\pi\mu^k} \{e^{i2\pi\mu^k\tau t} - e^{-\lambda_{j_k} t}\}, \\ u_{l_k}(t) &= e^{-\lambda_{l_k} t} u_{l_k,0} + \frac{\nu^{-\delta_2 k}\lambda_{l_k}}{\lambda_{l_k} + i2\pi\nu^k} \{e^{i2\pi\nu^k\tau t} - e^{-\lambda_{l_k} t}\}. \end{aligned}$$

It follows that

$$\begin{aligned} \|u(t) - u_\infty(t)\|^2 &\le \sum_{k=0}^{\infty} [|u_{j_k,0} - \frac{\mu^{-\delta_1 k}\lambda_{j_k}}{\lambda_{j_k} + i2\pi\mu^k}|^2 e^{-2\lambda_{j_k} t} + |u_{l_k,0} - \frac{\nu^{-\delta_2 k}\lambda_{l_k}}{\lambda_{l_k} + i2\pi\nu^k}|^2 e^{-2\lambda_{l_k} t}] \\ &\quad + \sum_{j \notin \{j_k\}\cup\{l_k\}} |u_{j,0}|^2 e^{-2\lambda_j t} \to 0 \end{aligned}$$

as $t \to \infty$.

Next we show that $u_\infty(t) = g_1(\tau t) + g_2(\tau t)$ satisfies the Hölder conditions. Define a $1/\mu$-periodic function

$$h_1(t) = \sum_{k=0}^{\infty} (\mu^{-\delta_1})^k \frac{\lambda_{j_k}}{\lambda_{j_k} + i2\pi\mu^k} e^{i2\pi\mu^{k+1} t} \varphi_{j_k},$$

then it follows from Lemma 4 that $h_1(t)$ satisfies the Hölder conditions for $t, t' : |t - t'| < 1/2\mu$. In fact, choose a subsequence j_k, which satisfies

$$\mu^k \leq C\lambda_{j_k} \tag{4.7}$$

for some constant $C > 0$. Then, applying the proof of Lemma 4 with the following estimate

$$\frac{1}{\sqrt{1+(2\pi C)^2}} \leq |\frac{\lambda_{j_k}}{\lambda_{j_k} + i2\pi\mu^k}| \leq 1, \tag{4.8}$$

we can show Hölder continuity of $h_1(t)$. Since $g_1(t) = h_1(t/\mu)$, $g_1(t)$ is 1-periodic and $g_1(t)$ satisfies the Hölder conditions for $t, t' : |t - t'| < 1/2$. For the second function $g_2(t)$, by assuming

$$\nu^k \leq C\lambda_{l_k} \tag{4.9}$$

and considering the estimate

$$\frac{1}{\sqrt{1+(2\pi C)^2}} \leq |\frac{\lambda_{l_k}}{\lambda_{l_k} + i2\pi\nu^k}| \leq 1, \tag{4.10}$$

we can show the Hölder continuity of $g_2(t)$.

Thus, by applying Theorem 1 with

$$\delta_1 = \min\{\vartheta_1, \vartheta_2\}, \quad \delta_2 = \max\{\vartheta_1, \vartheta_2\},$$

according to the algebraic properties of the frequency τ, we can obtain the estimates of the correlation dimensions for the q.p. attractor.

$$\Sigma = \{\varphi(n) : \varphi(n) = u_\infty(\tau n), n \in \mathbf{N}\}.$$

as those in the previous sections.

Remark. Instead of (4.7) and (4.9) it is sufficient to assume that

$$\limsup_{k\to\infty} \frac{\mu^k}{\lambda_{j_k}} \leq C < \infty, \quad \limsup_{k\to\infty} \frac{\nu^k}{\lambda_{l_k}} \leq C < \infty,$$

since we can also obtain (4.8) and (4.10).

References

[1] J.W.S.Cassels, *An Introduction to Diophantine Approximation*, Cambridge Tracts in Math. and Math. Physics no.45, Cambridge Univ. Press, 1957.

[2] T.W.Cusick, *The Szekeres multidimensional continued fraction*, Math. Computation **31** (1977), 280-317.

[3] A.El Jai and A.J.Prichard, *Sensors and Controls in the Analysis of Distributed Systems*, John Wiley & Sons, Chichester, 1988.

[4] K.Naito, *Fractal dimensions of almost periodic attractors*, Ergodic Theory and Dynamical Systems **16** (1996), 791-803.

[5] K.Naito, *Dimensions of almost periodic trajectories for nonlinear evolution equations*, Yokohama Math. J., **44** (1997), 93-113.

[6] K.Naito, *Dimension estimate of almost periodic attractors by simultaneous Diophantine approximation*, J. Differential Equations, **141** (1997), 179-200.

[7] W.M.Schmidt, *Diophantine Approximation*, Springer Lecture Notes in Math. 785, 1980.

[8] R.Temam, *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*, Springer Appl. Math. Sci. 68, 1988.