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A weighted extension of Carleman's inequality

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1 Introduction

In 1923, T. Carleman [2] presented the following inequality: for a sequence of positive real numbers $\{x_n\}$ with $\sum_{n=1}^{\infty} x_n < +\infty$,

$$\sum_{n=1}^{\infty}(x_1 x_2 \cdots x_n)^{1/n} < e\sum_{n=1}^{\infty} x_n.$$

Recently, K.S. Kedlaya [4] proved the following weighted extension:

Theorem. (Kedlaya) *Let $\{\alpha_n\}$ be a sequence of positive real numbers satisfying*

$$\frac{\alpha_1}{\alpha_1} \geq \frac{\alpha_2}{\alpha_1+\alpha_2} \geq \frac{\alpha_3}{\alpha_1+\alpha_2+\alpha_3} \geq \cdots . \qquad (*)$$

If a sequence of positive real numbers $\{x_n\}$ satisfies $\sum_{n=1}^{\infty} \alpha_n x_n < +\infty$, then

$$\sum_{n=1}^{\infty} \alpha_n x_1^{\alpha_1/(\alpha_1+\cdots+\alpha_n)} \cdots x_n^{\alpha_n/(\alpha_1+\cdots+\alpha_n)} < e\sum_{n=1}^{\infty} \alpha_n x_n.$$

In [4] he obtained the above extension as a consequence of some weighted mixed-mean inequality. The condition $(*)$ are necessary for proving the weighted mixed-mean inequality.

Our purpose of this paper is to give the other weighted extension and the Kedlya type extension without the condition $(*)$.

2 A weighted extension

In this section we consider the other weighted extension of Carlemans's inequality. The first result is a weighted extension of finite type.

Theorem 2.1. *Let $\{\alpha_n\}$ and $\{x_n\}$ be sequences of positive real numbers. For any $N \in \mathbb{N}$,*

$$\lambda_N \mu_N \sum_{n=1}^{N} x_1^{\alpha_1/(\alpha_1+\cdots+\alpha_n)} \cdots x_n^{\alpha_n/(\alpha_1+\cdots+\alpha_n)} < \sum_{n=1}^{N} x_n$$

is valid, where

$$\begin{aligned}
\lambda_N &= \min\{\alpha_2, \cdots, \alpha_{N+1}\}, \\
\mu_N &= \min_{1\le n\le N} \beta_1^{\alpha_1/(\alpha_1+\cdots+\alpha_n)} \cdots \beta_n^{\alpha_n/(\alpha_1+\cdots+\alpha_n)}, \\
\beta_k &= \frac{1}{\alpha_k}\Big(\frac{\alpha_1+\cdots+\alpha_k}{\alpha_1+\cdots+\alpha_{k+1}}\Big)^{(\alpha_1+\cdots+\alpha_k)/\alpha_k}.
\end{aligned}$$

Proof. We shall apply H. Alzer's technique observed in [1]. Consider the weighted arithmetic mean and geometric mean inequality:

$$(y_1^{\alpha_1} \cdots y_n^{\alpha_n})^{1/(\alpha_1+\cdots+\alpha_n)} \le \frac{\alpha_1 y_1 + \cdots + \alpha_1 y_n}{\alpha_1 + \cdots + \alpha_n},$$

and replace $y_k = (\alpha_1 + \cdots + \alpha_k)x_k/\alpha_k$ for each $1 \le k \le n$. Moreover divide both sides of the obtained inequality by $\alpha_1 + \cdots + \alpha_{n+1}$ and sum form $n = 1$ to $n = N$. Let L_N and R_N denote the left side and right side of resulting inequality, respectively. Then

$$\begin{aligned}
R_N &= \sum_{n=1}^{N} \frac{\alpha_1 x_1 + (\alpha_1+\alpha_2)x_2 + \cdots + (\alpha_1+\cdots+\alpha_n)x_n}{(\alpha_1+\cdots+\alpha_n)(\alpha_1+\cdots+\alpha_{n+1})} \\
&= x_1 \sum_{n=1}^{N} \frac{\alpha_1}{(\alpha_1+\cdots+\alpha_n)(\alpha_1+\cdots+\alpha_{n+1})} \\
&\quad + x_2 \sum_{n=2}^{N} \frac{\alpha_1+\alpha_2}{(\alpha_1+\cdots+\alpha_n)(\alpha_1+\cdots+\alpha_{n+1})} \\
&\quad + \cdots + x_N \frac{\alpha_1+\cdots+\alpha_N}{(\alpha_1+\cdots+\alpha_N)(\alpha_1+\cdots+\alpha_{N+1})} \\
&\le \frac{\alpha_1}{\lambda_N} x_1 \Big\{\frac{1}{\alpha_1} - \frac{1}{\alpha_1+\cdots+\alpha_{N+1}}\Big\} \\
&\quad + \frac{\alpha_1+\alpha_2}{\lambda_N} x_2 \Big\{\frac{1}{\alpha_1+\alpha_2} - \frac{1}{\alpha_1+\cdots+\alpha_{N+1}}\Big\} \\
&\quad + \cdots + \frac{\alpha_1+\cdots+\alpha_N}{\lambda_N} x_N \Big\{\frac{1}{\alpha_1+\cdots+\alpha_N} - \frac{1}{\alpha_1+\cdots+\alpha_{N+1}}\Big\} \\
&= \frac{1}{\lambda_N}\Big\{\sum_{n=1}^{N} x_n - \frac{1}{\alpha_1+\cdots+\alpha_{N+1}} \sum_{n=1}^{N} (\alpha_1+\cdots+\alpha_n)x_n\Big\},
\end{aligned}$$

removing the term $(\alpha_1+\cdots+\alpha_{N+1})^{-1}\sum_{n=1}^{N}(\alpha_1+\cdots+\alpha_n)x_n$, we can obtain $\lambda_N R_N < \sum_{n=1}^{N} x_n$. And put

$$\nu_N = \min_{1\le n\le N} \frac{\left(\frac{\alpha_1}{\alpha_1}\right)^{\alpha_1/(\alpha_1+\cdots+\alpha_n)}\left(\frac{\alpha_1+\alpha_2}{\alpha_2}\right)^{\alpha_2/(\alpha_1+\cdots+\alpha_n)}\cdots\left(\frac{\alpha_1+\cdots+\alpha_n}{\alpha_n}\right)^{\alpha_n/(\alpha_1+\cdots+\alpha_n)}}{\alpha_1+\cdots+\alpha_{n+1}},$$

then

$$\begin{aligned} L_N &= \sum_{n=1}^{N} \frac{\left(\frac{\alpha_1}{\alpha_1}x_1\right)^{\alpha_1/(\alpha_1+\cdots+\alpha_n)}\cdots\left(\frac{\alpha_1+\cdots+\alpha_n}{\alpha_n}x_n\right)^{\alpha_n/(\alpha_1+\cdots+\alpha_n)}}{\alpha_1+\cdots+\alpha_{n+1}} \\ &\ge \nu_N \sum_{n=1}^{N} x_1^{\alpha_1/(\alpha_1+\cdots+\alpha_n)}\cdots x_n^{\alpha_n/(\alpha_1+\cdots+\alpha_n)}. \end{aligned}$$

Hence we can obtain

$$\lambda_N\nu_N \sum_{n=1}^{N} x_1^{\alpha_1/(\alpha_1+\cdots+\alpha_n)}\cdots x_n^{\alpha_n/(\alpha_1+\cdots+\alpha_n)} < \sum_{n=1}^{N} x_n,$$

so it only remains to show $\mu_N = \nu_N$. But it will be sufficient to show that

$$\begin{aligned} &\beta_1{}^{\alpha_1/(\alpha_1+\cdots+\alpha_n)}\cdots\beta_n^{\alpha_n/(\alpha_1+\cdots+\alpha_n)} \\ &= \frac{\left(\frac{\alpha_1}{\alpha_1}\right)^{\alpha_1/(\alpha_1+\cdots+\alpha_n)}\left(\frac{\alpha_1+\alpha_2}{\alpha_2}\right)^{\alpha_2/(\alpha_1+\cdots+\alpha_n)}\cdots\left(\frac{\alpha_1+\cdots+\alpha_n}{\alpha_n}\right)^{\alpha_n/(\alpha_1+\cdots+\alpha_n)}}{\alpha_1+\cdots+\alpha_{n+1}} \end{aligned} \quad (1)$$

for each n. In fact,

$$\begin{aligned} \beta_1^{\alpha_1}\cdots\beta_n^{\alpha_n} &= \left(\frac{1}{\alpha_1}\right)^{\alpha_1}\left(\frac{1}{\alpha_2}\right)^{\alpha_2}\cdots\left(\frac{1}{\alpha_n}\right)^{\alpha_n} \\ &\quad\cdot\left(\frac{\alpha_1}{\alpha_1+\alpha_2}\right)^{\alpha_1}\left(\frac{\alpha_1+\alpha_2}{\alpha_1+\alpha_2+\alpha_3}\right)^{\alpha_1+\alpha_2}\cdots\left(\frac{\alpha_1+\cdots+\alpha_n}{\alpha_1+\cdots+\alpha_{n+1}}\right)^{\alpha_1+\cdots+\alpha_n} \\ &= \left(\frac{1}{\alpha_1}\right)^{\alpha_1}\left(\frac{1}{\alpha_2}\right)^{\alpha_2}\cdots\left(\frac{1}{\alpha_n}\right)^{\alpha_n} \\ &\quad\cdot\alpha_1^{\alpha_1}(\alpha_1+\alpha_2)^{\alpha_2}\cdots(\alpha_1+\cdots+\alpha_n)^{\alpha_n}\frac{1}{(\alpha_1+\cdots+\alpha_{n+1})^{\alpha_1+\cdots+\alpha_n}} \\ &= \left(\frac{\alpha_1}{\alpha_1}\right)^{\alpha_1}\left(\frac{\alpha_1+\alpha_2}{\alpha_2}\right)^{\alpha_2}\cdots\left(\frac{\alpha_1+\cdots+\alpha_n}{\alpha_n}\right)^{\alpha_n}\frac{1}{(\alpha_1+\cdots+\alpha_{n+1})^{\alpha_1+\cdots+\alpha_n}}, \end{aligned}$$

raising to the power $1/(\alpha_1+\cdots+\alpha_n)$, (1) is hold. □

In the proof of Theorem 2.1 we remove the term $(\alpha_1+\cdots+\alpha_{N+1})^{-1}\sum_{n=1}^{N}(\alpha_1+\cdots+\alpha_n)x_n$. If $\sum_{n=1}^{\infty}\alpha_n=+\infty$ and $\sum_{n=1}^{\infty}x_n<+\infty$, then this term are sufficiently small, that is,

$$\lim_{N\to\infty}\frac{1}{\alpha_1+\cdots+\alpha_{N+1}}\sum_{n=1}^{N}(\alpha_1+\cdots+\alpha_n)x_n = 0.$$

To see this, let ε be an arbitrary positive number. Since $\sum_{n=1}^{\infty} x_n < +\infty$, there exists a number N_0 such that $\sum_{n=N_0+1}^{\infty} x_n < \varepsilon$. For any $N \geq N_0$,

$$\begin{aligned}
&\frac{1}{\alpha_1 + \cdots + \alpha_{N+1}} \sum_{n=1}^{N} (\alpha_1 + \cdots + \alpha_n) x_n \\
&\qquad \leq \frac{(\max_n x_n) \sum_{n=1}^{N_0} (\alpha_1 + \cdots + \alpha_n)}{\alpha_1 + \cdots + \alpha_{N+1}} + \sum_{n=N_0+1}^{N} \frac{\alpha_1 + \cdots + \alpha_n}{\alpha_1 + \cdots + \alpha_{N+1}} x_n \\
&\qquad \leq \frac{(\max_n x_n) \sum_{n=1}^{N_0} (\alpha_1 + \cdots + \alpha_n)}{\alpha_1 + \cdots + \alpha_{N+1}} + \varepsilon.
\end{aligned}$$

By taking the upper limit with respect to N and letting $\varepsilon \downarrow 0$, we obtain the desired equality.

Next we shall consider the infinite case. By Theorem 2.1, put $\gamma = \inf_N \lambda_N \mu_N$, then we have

$$\gamma \sum_{n=1}^{\infty} x_1^{\alpha_1/(\alpha_1 + \cdots + \alpha_n)} \cdots x_n^{\alpha_n/(\alpha_1 + \cdots + \alpha_n)} \leq \sum_{n=1}^{\infty} x_n. \tag{2}$$

whenever $\sum_{n=1}^{\infty} x_n < +\infty$. When $\inf_n \alpha_n = 0$, $\gamma = 0$ and the inequality (2) is trivial. If $\lim_{n \to \infty} \alpha_n = +\infty$, then

$$\begin{aligned}
\beta_1^{\alpha_1/(\alpha_1 + \cdots + \alpha_n)} \cdots \beta_n^{\alpha_n/(\alpha_1 + \cdots + \alpha_n)} &\leq \left(\frac{1}{\alpha_1}\right)^{\alpha_1/(\alpha_1 + \cdots + \alpha_n)} \cdots \left(\frac{1}{\alpha_n}\right)^{\alpha_n/(\alpha_1 + \cdots + \alpha_n)} \\
&\leq \frac{\alpha_1}{\alpha_1 + \cdots + \alpha_n} \cdot \frac{1}{\alpha_1} + \cdots + \frac{\alpha_n}{\alpha_1 + \cdots + \alpha_n} \cdot \frac{1}{\alpha_n} \\
&= \frac{n}{\alpha_1 + \cdots + \alpha_n} \to 0 \quad (n \to \infty),
\end{aligned}$$

which implies $\inf_n \beta_1^{\alpha_1/(\alpha_1 + \cdots + \alpha_n)} \cdots \beta_n^{\alpha_n/(\alpha_1 + \cdots + \alpha_n)} = 0$, and so $\gamma = 0$. Consequently, when $\lim_{n \to \infty} \alpha_n = +\infty$, the inequality (2) is also trivial. We shall consider the case when $\{\alpha_n\}$ is bounded.

Theorem 2.2. *Let $\{\alpha_n\}$ be a sequence of positive real numbers satisfying $0 < m \leq \alpha_n \leq M < +\infty$ $(n = 1, 2, \cdots)$. If a sequence of positive real numbers $\{x_n\}$ satisfies $\sum_{n=1}^{\infty} x_n < +\infty$ then*

$$\frac{m}{M} \left(\frac{1}{e}\right)^{M/m} \sum_{n=1}^{\infty} x_1^{\alpha_1/(\alpha_1 + \cdots + \alpha_n)} \cdots x_n^{\alpha_n/(\alpha_1 + \cdots + \alpha_n)} \leq \sum_{n=1}^{\infty} x_n. \tag{3}$$

Proof. Put $t_k = (\alpha_1 + \cdots + \alpha_k)/\alpha_{k+1}$, then for any $k \in \mathbb{N}$

$$\begin{aligned}\beta_k &= \frac{1}{\alpha_k}\Big(\frac{\alpha_1 + \cdots + \alpha_k}{\alpha_1 + \cdots + \alpha_{k+1}}\Big)^{(\alpha_1+\cdots+\alpha_k)/\alpha_k} \\ &\geq \frac{1}{M}\Big(\frac{1}{1+1/t_k}\Big)^{\alpha_{k+1}t_k/\alpha_k} \\ &> \frac{1}{M}\Big(\frac{1}{e}\Big)^{M/m}.\end{aligned}$$

Since $\lambda_N = \inf_{2\leq n\leq N+1} \alpha_n \geq m$ and

$$\mu_N = \inf_{1\leq n\leq N} \beta_1^{\alpha_1/(\alpha_1+\cdots+\alpha_n)} \cdots \beta_n^{\alpha_n/(\alpha_1+\cdots+\alpha_n)} > \frac{1}{M}\Big(\frac{1}{e}\Big)^{M/m},$$

this completes the proof. □

The inequality (3) reduces to Carleman's inequality in case of $m = M$.

3 A weighted extension of Hardy's inequality

In the book [3], Carleman's inequality can be obtained from Hardy's inequality. We shall consider the weighted extension of Hardy's inequality.

Lemma 3.1. *If $\{\alpha_n\}$ and $\{x_n\}$ are sequences of positive real numbers, then for any $p > 1$ and $N \in \mathbb{N}$*

$$\sum_{n=1}^{N} \alpha_n \Big(\frac{\alpha_1 x_1 + \cdots + \alpha_n x_n}{\alpha_1 + \cdots + \alpha_n}\Big)^p < \Big(\frac{p}{p-1}\Big)^p \sum_{n=1}^{N} \alpha_n x_n^p. \tag{4}$$

Proof. Let q be the positive real number with $1/p + 1/q = 1$. Consider

$$s_n = \alpha_1 + \cdots + \alpha_n, \qquad A_n = \frac{\alpha_1 x_1 + \cdots + \alpha_n x_n}{s_n},$$

then for any $n \geq 2$

$$\begin{aligned}A_n^p - \frac{p}{p-1}A_n^{p-1}x_n &= A_n^p - \frac{p}{\alpha_n(p-1)}A_n^{p-1}\alpha_n x_n \\ &= A_n^p - \frac{p}{\alpha_n(p-1)}A_n^{p-1}\{s_n A_n - s_{n-1}A_{n-1}\} \\ &= A_n^p\Big\{1 - \frac{p s_n}{\alpha_n(p-1)}\Big\} + \frac{s_{n-1}}{\alpha_n(p-1)}pA_n^{p-1}A_{n-1}.\end{aligned}$$

By Young's inequality we have

$$A_n^{p-1}A_{n-1} \leq \frac{1}{q}A_n^{q(p-1)} + \frac{1}{p}A_{n-1}^p = \frac{1}{q}A_n^p + \frac{1}{p}A_{n-1}^p,$$

and so

$$A_n^p - \frac{p}{p-1}A_n^{p-1}x_n \le A_n^p\Big\{1 - \frac{ps_n}{\alpha_n(p-1)}\Big\} + \frac{s_{n-1}}{\alpha_n(p-1)}\{(p-1)A_n^p + A_{n-1}^p\}$$
$$= \frac{1}{\alpha_n(p-1)}A_n^p\{\alpha_n(p-1) - ps_n + s_{n-1}(p-1)\} + \frac{s_{n-1}}{\alpha_n(p-1)}A_{n-1}^p$$
$$= \frac{1}{\alpha_n(p-1)}(s_{n-1}A_{n-1}^p - s_nA_n^p).$$

Hence

$$\alpha_nA_n^p - \frac{p}{p-1}A_n^{p-1}\alpha_nx_n \le \frac{1}{p-1}(s_{n-1}A_{n-1}^p - s_nA_n^p),$$

which implies

$$\sum_{n=1}^N \alpha_nA_n^p - \frac{p}{p-1}\sum_{n=1}^N A_n^{p-1}\alpha_nx_n$$
$$\le \alpha_1A_1^p - \frac{p}{p-1}A_1^{p-1}\alpha_1x_1 + \frac{1}{p-1}\sum_{n=2}^N(s_{n-1}A_{n-1}^p - s_nA_n^p)$$
$$= -\frac{1}{p-1}s_NA_N^p < 0.$$

According to Hölder's inequality

$$\sum_{n=1}^N \alpha_nA_n^p < \frac{p}{p-1}\sum_{n=1}^N A_n^{p-1}\alpha_nx_n$$
$$= \frac{p}{p-1}\sum_{n=1}^N \alpha_n^{1/q}A_n^{p-1}\cdot\alpha_n^{1/p}x_n$$
$$\le \frac{p}{p-1}\Big(\sum_{n=1}^N \alpha_nA_n^p\Big)^{1/q}\Big(\sum_{n=1}^N \alpha_nx_n^p\Big)^{1/p}. \tag{5}$$

Dividing both sides by $\left(\sum_{n=1}^N \alpha_nA_n^p\right)^{1/q}$ and raising the result to the p-th power, we obtain the inequality (4). □

Theorem 3.2. *Let $\{\alpha_n\}$ be a sequence of positive real numbers and $p > 1$. If a sequence of positive real numbers $\{x_n\}$ satisfies $\sum_{n=1}^\infty \alpha_nx_n^p < +\infty$, then*

$$\sum_{n=1}^\infty \alpha_n\Big(\frac{\alpha_1x_1 + \cdots + \alpha_nx_n}{\alpha_1 + \cdots + \alpha_n}\Big)^p < \Big(\frac{p}{p-1}\Big)^p\sum_{n=1}^\infty \alpha_nx_n^p.$$

Proof. Make $N \to \infty$ in Lemma 3.1, we can obtain

$$\sum_{n=1}^{\infty} \alpha_n \Big(\frac{\alpha_1 x_1 + \cdots + \alpha_n x_n}{\alpha_1 + \cdots + \alpha_n}\Big)^p \le \Big(\frac{p}{p-1}\Big)^p \sum_{n=1}^{\infty} \alpha_n x_n^p. \tag{6}$$

So it will be sufficient to show that the equality does not occur. Make $N \to \infty$ in (5), we have

$$\sum_{n=1}^{\infty} \alpha_n A_n^p \le \frac{p}{p-1} \sum_{n=1}^{\infty} \alpha_n^{1/q} A_n^{p-1} \cdot \alpha_n^{1/p} x_n \le \frac{p}{p-1} \Big(\sum_{n=1}^{\infty} \alpha_n A_n^p\Big)^{1/q} \Big(\sum_{n=1}^{\infty} \alpha_n x_n^p\Big)^{1/p}.$$

Assume that the equality occurs in (6), then the second term is equal to the third one. According to the equality condition for Hölder's inequality, $\{\alpha_n A_n^p\}$ and $\{\alpha_n x_n^p\}$ are proportional, i.e. there exists a constant c such that $\alpha_n A_n^p = c \cdot \alpha_n x_n^p$ $(n = 1, 2, \cdots)$. Hence $x_1 = x_2 = x_3 = \cdots$, and so $1 = \{p/(p-1)\}^p$. This is a contradiction. □

Using Lemma 3.1 we can obtain the Kedlya type extension of Carleman's inequality without the condition $(*)$.

Theorem 3.3. *Let $\{\alpha_n\}$ be a sequence of positive real numbers. If a sequence of positive real numbers $\{x_n\}$ satisfies $\sum_{n=1}^{\infty} \alpha_n x_n < +\infty$, then*

$$\sum_{n=1}^{\infty} \alpha_n x_1^{\alpha_1/(\alpha_1+\cdots+\alpha_n)} \cdots x_n^{\alpha_n/(\alpha_1+\cdots+\alpha_n)} \le e \sum_{n=1}^{\infty} \alpha_n x_n.$$

Proof. Let $p > 1$ and replace x_n by $x_n^{1/p}$ in the inequality (4), we can obtain

$$\sum_{n=1}^{N} \alpha_n \Big(\frac{\alpha_1 x_1^{1/p} + \cdots + \alpha_n x_n^{1/p}}{\alpha_1 + \cdots + \alpha_n}\Big)^p < \Big(\frac{p}{p-1}\Big)^p \sum_{n=1}^{N} \alpha_n x_n. \tag{7}$$

Since

$$\begin{aligned}\lim_{p\to\infty} \Big(\frac{\alpha_1 x_1^{1/p} + \cdots + \alpha_n x_n^{1/p}}{\alpha_1 + \cdots + \alpha_n}\Big)^p &= \lim_{t\to+0} \Big(\frac{\alpha_1 x_1^t + \cdots + \alpha_n x_n^t}{\alpha_1 + \cdots + \alpha_n}\Big)^{1/t} \\ &= x_1^{\alpha_1/(\alpha_1+\cdots+\alpha_n)} \cdots x_n^{\alpha_n/(\alpha_1+\cdots+\alpha_n)}\end{aligned}$$

and $\lim_{p\to\infty}\{p/(p-1)\}^p = e$, making $p \to \infty$ in (7),

$$\sum_{n=1}^{N} \alpha_n x_1^{\alpha_1/(\alpha_1+\cdots+\alpha_n)} \cdots x_n^{\alpha_n/(\alpha_1+\cdots+\alpha_n)} \le e \sum_{n=1}^{N} \alpha_n x_n.$$

Hence, making $N \to \infty$, this completes the proof. □

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