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Chaotic Behavior of Composition Operators

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1 Introduction

In Devaney's sense (see [3]), a continuous map on a metric space is called chaotic if it is topologically transitive, has sensitive dependence on initial conditions, and has dense periodic points.

On the other hand, we have the notion of hypercyclicity. Let T be a bounded linear operator on a Banach space X. T is called a hypercyclic on X if there exists a vector $x\in X$ such that its orbit $\mathrm{Orb}(T,x)=\{x,Tx,T^{2}x,\ldots\}$ is dense in X, and such a vector x is called hypercyclic for T. Clearly the hypercyclicity needs the separability of X. So in this paper we assume that X is separable. On separable Banach spaces, hypercyclicity is equivalent to topologically transitivity.

In general, it is known that if T is topologically transitive and has dense periodic points, then T has sensitive dependence on initial conditions, that is: there exists a constant $\delta>0$ such that, for any $x\in H$ and any neighborhood N of x, there exist $y\in N$ and $n\geq 0$ such that $|T^{n}(x)-T^{n}(y)|>\delta$.

Now we have the following.

Proposition. *Let T be a bounded linear operator on a separable Banach space X. Then T is chaotic on X in Devaney's sense if and only if*

(i) T is hypercyclic on X,

(ii) the set $\mathrm{Per}(T)$ of periodic points is dense in X.

Example (Non-chaotic Operator, but is Hypercyclic, [4]). Let β be the positive valued function on the non-negative integers such that

$$\sup_{n\geq 0}\frac{\beta(n+1)}{\beta(n)}<\infty$$

and $l^2(\beta)$ be the weighted l^2-space normed by

$$\|a\|^2 = \sum_{n=0}^{\infty} |a_n|^2 \beta(n)$$

where $a = (a_n) \in l^2(\beta)$.

If $\lim_{n\to\infty} \beta(n) = 0$, then the backward shift operator is hypercyclic on $l^2(\beta)$. But if β fails the following condition;

$$\sum_{n=0}^{\infty} \beta(n) < \infty$$

(for example, $\beta(n) = 1/(n+1)$), then the backward shift operator has no non-zero periodic point, and so is not chaotic.

2 Chaotic Composition Operators on $H^2(D)$

We will study the chaotic behavior of composition operators induced by the holomorphic self-map φ of the open unit disk D, which act on the Hardy space $H^2(D)$.

The Hardy space $H^2(D)$ is the separable Hilbert space of functions holomorphic on D whose Taylor coefficients in the expansion about 0 form a square summable sequence. More precisely,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \in H^2(D) \iff \|f\|^2 = \sum_{n=0}^{\infty} |a_n|^2 < \infty.$$

If φ maps D into itself, φ is called self-map of D. Let φ be a holomorphic self-map of D. Then for any holomorphic function f on D, the composition $f \circ \varphi$ is also holomorphic on D. By Littlewood's Theorem (see [5]), we can see that the composition operator C_φ, defined by $C_\varphi f = f \circ \varphi$, is a bounded linear operator on $H^2(D)$.

If φ is not univalent on D, then the orthogonal complement of the range of C_φ is infinite dimensional. Thus C_φ is not hypercyclic. Next suppose that φ has a fixed point p in D. Then $C_\varphi^n f(p) = f(p)$ for every $f \in H^2(D)$ and every $n \geq 0$. Thus for every $g \in \overline{\mathrm{Orb}(C_\varphi, f)}$, $g(p) = f(p)$ and this implies that C_φ is not hypercyclic. So in the following we always assume that φ is a univalent holomorphic self-map of D with no fixed point in D.

Let ψ be an automorphism of D with no fixed point in D. Then ψ is either hyperbolic or parabolic and in both cases C_ψ is hypercyclic (see [1], [5]). Moreover we can see that C_ψ has dense periodic points. So we obtain the following theorem.

Theorem 1. *If the automorphism ψ of D has no fixed point in D, then C_ψ is chaotic on $H^2(D)$.*

Example (Chaos induced by automorphism). The following automorphisms of D induce the chaotic composition operators on $H^2(D)$.

(i) hyperbolic automorphism: $\psi(z) = \dfrac{2z+1}{2+z}$.

(ii) parabolic automorphism: $\psi(z) = \dfrac{(i-1)z+1}{(i+1)-z}$.

Next we prepare some notation to see the chaotic behavior of the composition operators induced by non-automorphisms.

Definition 1. The point $p \in \overline{D}$ is called Denjoy-Wolff point of φ if the iteration φ^n converges to p uniformly on compact subsets of D.

Remark. Denjoy-Wolff theorem (see [2]) ensures the existence and the uniqueness of the Denjoy-Wolff point of φ which is not an elliptic automorphism.

Definition 2. If φ is a holomorphic self-map of D which is continuous and univalent on $\overline{D}$, has Denjoy-Wolff point $p \in \partial D$, and $\varphi(\overline{D}) \subset D \cup \{p\}$, then we say that φ is shrinking to p.

Definition 3. Let φ be a holomorphic self-map of D with Denjoy-Wolff point p on ∂D and $\varphi^{(k)}(p)$ be the k-th angular derivative of φ at p. If for $\varepsilon \in [0,1)$, φ has the expansion

$$\varphi(z) = \sum_{k=0}^{n} \frac{\varphi^{(k)}(p)}{k!}(z-p)^k + \gamma(z), \tag{1}$$

where $\gamma(z) = o(|z-p|^{n+\varepsilon})$ as $z \to p$ in D, then we denote that $\varphi \in C^{n+\varepsilon}(p)$.

Definition 4. Suppose that φ is shrinking to p. We say that

(i) φ is hyperbolically shrinking to p if $\varphi \in C^{1+\varepsilon}(p)$ and $\varphi'(p) < 1$,

(ii) φ is parabolically shrinking to p if $\varphi \in C^{3+\varepsilon}(p)$ and $\varphi'(p) = 1$.

Now we can state the following theorem.

Theorem 2. *Let φ be a holomorphic self-map of D.*

(i) If φ is hyperbolically shrinking to p, then the composition operator C_φ induced by φ is chaotic on $H^2(D)$.

(ii) If φ is parabolically shrinking to p and $\varphi''(p)$ is non-zero purely imaginary, then C_φ is chaotic on $H^2(D)$.

Example. (i) $\varphi(z) = \dfrac{3z+2}{z+4}$ is hyperbolically shrinking to 1. Thus C_φ is chaotic on $H^2(D)$.

(ii) Let Π be the right-half plane of $\mathbb{C}$ and $T(z) = \dfrac{1+z}{1-z}$ be the map of D onto Π which takes 1 to ∞. For $0 < \alpha < 1$, let $\Phi_\alpha : \overline{\Pi} \to \Pi$ be defined by

$$\Phi_\alpha(w) = w + i + \frac{1}{\alpha(w+1)^\alpha}, \tag{2}$$

where $w \in \overline{\Pi}$. Then $\varphi_\alpha = T^{-1} \circ \Phi_\alpha \circ T$ is shrinking to 1 with $\varphi''_\alpha(1) = i$ and $\varphi_\alpha \in C^{2+\alpha}(1)$. But C_{φ_α} is not chaotic.

Suppose that φ is a linear-fractional self-map of D. In [5] and [1], it is shown that if C_φ is hypercyclic on $H^2(D)$, then φ is either a hyperbolic map or a parabolic automorphism of D. On the other hand, if φ is a linear-fractional hyperbolic non-automorphism of D, then φ is hyperbolically shrinking. So C_φ is chaotic. Hence we obtain the following corollary.

Corollary 1. *Let φ be a linear-fractional self-map of D. If C_φ is hypercyclic on $H^2(D)$, then C_φ is chaotic on $H^2(D)$.*

References

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