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Some applications of computer algebra to vector bundles on projective spaces

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Abstract

During the past years the results and the techniques of computer algebra have become more and more useful in algebraic geometry, in particular in the study of algebraic vector bundles on complex projective spaces, which are strictly related (by means of presentations or resolutions by direct sums of line bundles) to matrices whose entries are homogeneous polynomials. The obvious strategy consists in translating the problems on vector bundles to problems on matrices (mostly related to computation of syzygies, which is the core of the current Computer Algebra systems intended for algebraic geometry).

Here we give some examples in the case of mathematical instanton bundles and their moduli spaces.

0 Notations

- $\mathbf{P}^d$: the complex d-dimensional projective space;
- $\mathcal{O}$: its structure sheaf;
- $\mathcal{O}(-1)$ (resp. $\mathcal{O}(1)$): the tautological line bundle (resp. its dual) on $\mathbf{P}^d$;
- S_j: the vector space of homogeneous polynomials of degree j in d variables (in particular $S_0 = \mathbf{C}$);
- $Mat(k, r; S_j)$: the vector space of $k \times r$-matrices with entries in S_j;
- E^*: the dual of a vector bundle (or a vector space) E;
- $E(-1)$ (resp. $E(1)$): the twisted bundle $E \otimes \mathcal{O}(1)$ (resp. $E \otimes \mathcal{O}(1)$).

1 Mathematical instanton bundles

Definition 1 *A (mathematical) instanton bundle E on $\mathbf{P}^{2n+1}$ with $c_2 = k$ is the cohomology bundle of a monad*

$$\mathcal{O}(-1)^k \xrightarrow{B^t} \mathcal{O}^{2n+2k} \xrightarrow{A} \mathcal{O}(1)^k \tag{1}$$

where A, B are matrices in the space $Mat(k, 2n+2k; S_1)$, i.e. their entries are homogeneous linear forms in the coordinates of $\mathbf{P}^{2n+1}$.

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The fact that (1) is a monad means the following two conditions on A, B:

i) A and B have rank k at every point x of $\mathbf{P}^{2n+1}$;

ii) $A \cdot B^t = 0$.

The condition i) shows that at every point x the linear map

$$B^t(x) : \mathbf{C}^k \to \mathbf{C}^{2n+2k}$$

is injective, and the linear map

$$A(x) : \mathbf{C}^{2n+2k} \to \mathbf{C}^k$$

is surjective; by ii) we get $ImB^t(x) \subset KerA(x)$; finally the fiber of E at x is given by

$$E(x) = KerA(x)/ImB^t(x).$$

There is an important relationship between the instanton bundles on $\mathbf{P}^3$ and the solutions of the Yang-Mills equation on the 4-dimensional sphere S^4; we refer to the fundamental paper [AW] for details.

Let S^* be the kernel of the map $\mathcal{O}^{2n+2k} \xrightarrow{A} \mathcal{O}(1)^k$ in (1); then the monad (1) gives rise to the exact sequences

$$0 \to S^* \to \mathcal{O}^{2n+2k} \xrightarrow{A} \mathcal{O}(1)^k \to 0 \tag{2}$$

$$0 \to \mathcal{O}(-1)^k \xrightarrow{B^t} S^* \to E \to 0 \tag{3}$$

The equations (2) , (3) are called *the display of the monad.*

An istanton bundle E is called *symplectic* if there is an isomorphism $\phi : E \to E^*$ with $\phi^* = -\phi$.

It is still an open problem whether a general instanton bundle E is stable. This is true on $\mathbf{P}^3$ (easy) and $\mathbf{P}^5$ ([AO1]); moreover the so called *special* symplectic instanton bundles are stable ([AO1]). The stable instanton bundles with $c_2 = k$ define a moduli space $MI_{\mathbf{P}^{2n+1}}(k)$ which is an open subset of the corresponding *Maruyama moduli scheme.* The closed points of $MI_{\mathbf{P}^{2n+1}}(k)$ correspond to isomorphism classes of bundles.

Example 2 *([OS]) Let $x_0, \ldots, x_n, y_0, \ldots, y_n$ be homogeneous coordinates on $\mathbf{P}^{2n+1}$; the following pair $A, B \in Mat(k, 2n+2k; S_1)$*

$$A = \begin{pmatrix} 0 & \ldots & 0 & y_n & \ldots & y_0 & 0 & \ldots & 0 & -x_n & \ldots & -x_0 \\ \ldots & 0 & y_n & \ldots & y_0 & 0 & \ldots & 0 & -x_n & \ldots & -x_0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & y_n & \ldots & y_0 & 0 & \ldots & 0 & -x_n & \ldots & -x_0 & 0 & \ldots \\ y_n & \ldots & y_0 & 0 & \ldots & 0 & -x_n & \ldots & -x_0 & 0 & \ldots & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} x_0 & \dots & x_n & 0 & \dots & 0 & y_0 & \dots & y_n & 0 & \dots & 0 \\ 0 & x_0 & \dots & x_n & 0 & \dots & 0 & y_0 & \dots & y_n & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & 0 & x_0 & \dots & x_n & 0 & \dots & 0 & y_0 & \dots & y_n & 0 \\ 0 & \dots & 0 & x_0 & \dots & x_n & 0 & \dots & 0 & y_0 & \dots & y_n \end{pmatrix}$$

represent an instanton bundle which is special symplectic, hence stable.

Theorem 3 *Two instanton bundles corresponding to pairs of matrices (A, B) and (C, D) are isomorphic if and only if there is a triple:*

$$(Q, P, R) \in GL(k) \times GL(2n + 2k) \times GL(k)$$

such that:

$$\begin{aligned} C &= QAP^t \\ D &= RBP^{-1}. \end{aligned}$$

2 Computation of $H^1(E \otimes E^*)$ and $H^2(E \otimes E^*)$

The Zariski tangent space to the moduli space $MI_{\mathbf{P}^{2n+1}}(k)$ at a point corresponding to the bundle E is the vector space $H^1(E \otimes E^*)$; it is possible to describe it in terms of matrices.

Let (A, B) be the pair which detects E, and $\epsilon \in \mathbf{C}$ be a parameter; let moreover $A', B' \in Mat(k, 2n + 2k; S_1)$ such that:

$$(A + \epsilon A') \cdot (B + \epsilon B')^t = 0 \ \ (mod\, \epsilon^2) \tag{4}$$

i.e.

$$A \cdot B'^t + A' \cdot B^t = 0$$

Definition 4 *A first order deformation of E is a pair $(A + \epsilon A', B + \epsilon B')$ verifying (4).*

We denote by $V_{(A,B)}$ the vector space of pairs (A', B') corresponding to first order deformations of of $E = (A, B)$:

$$V_{(A,B)} := \{(A', B') \in Mat(k, 2n + 2k; S_1) \mid A \cdot B'^t + A' \cdot B^t = 0\}.$$

From now on, let $r = 2n + 2k$.
The Lie group $GL(k) \times GL(r) \times GL(k)$ acts on the pairs $(A, B) \in Mat(k, r; S_1)^{\oplus 2}$ by

$$\begin{array}{ccc} GL(k) \times GL(r) \times GL(k) & \xrightarrow{\rho} & GL(Mat(k, r; S_1)^{\oplus 2}) \\ (Q, P, R) & \longrightarrow & \rho_{(Q,P,R)} \end{array}$$

where:

$$\begin{array}{ccc} Mat(k, r; S_1)^{\oplus 2} & \xrightarrow{\rho_{(Q,P,R)}} & Mat(k, r; S_1)^{\oplus 2} \\ (A, B) & \longrightarrow & (QAP^t, RBP^{-1}) \end{array}$$

By the theorem 3 two instantons E and F on $\mathbf{P}^{2n+1}$ are isomorphic if and only if the corresponding pairs are in the same orbit of ρ.
The action ρ induces an action ρ' of the Lie algebra $gl(k) \times gl(r) \times gl(k)$ on $V_{(A,B)}$ by

$$\rho'_{(Q,P,R)}(A', B') = (C', D')$$

with

$$\rho_{(I+\epsilon Q, I+\epsilon P, I+\epsilon R)}(A + \epsilon A', B + \epsilon B') = (A + \epsilon C', B + \epsilon D') \ \ (mod\ \epsilon^2)$$

that is

$$\begin{aligned} A + \epsilon C' &= (I + \epsilon Q) \cdot (A + \epsilon A') \cdot (I + \epsilon P)^t & (mod\ \epsilon^2) \\ B + \epsilon D' &= (I + \epsilon R) \cdot (B + \epsilon B') \cdot (I + \epsilon P)^{-1} & (mod\ \epsilon^2) \end{aligned}$$

Since

$$(I + \epsilon P)^{-1} = (I - \epsilon P) \ \ (mod\ \epsilon^2)$$

we get:

$$\begin{aligned} A + \epsilon C' &= A + \epsilon(A' + QA + AP^t) & (mod\ \epsilon^2) \\ B + \epsilon D' &= B + \epsilon(B' + RB - BP) & (mod\ \epsilon^2) \end{aligned}$$

It follows that (A', B') and $(C', D') \in V_{(A,B)}$ are equivalent under the action of ρ' if and only if there exists $(Q, P, R) \in gl(k) \times gl(r) \times gl(k)$ such that :

$$\begin{aligned} C' &= QA + A' + AP^t \\ D' &= RB + B' - BP. \end{aligned}$$

Let $\mathcal{U} := Mat(k, k; \mathbf{C}) \oplus Mat(r, r; \mathbf{C}) \oplus Mat(k, k; \mathbf{C})$; we define the following subspace of $V_{(A,B)}$:

$$W_{(A,B)} := \{(M, N) \in V_{(A,B)} \,|\, \exists (X, Z, Y) \in \mathcal{U} \,|\, M = XA + AZ;\ N = YB - BZ^t\}.$$

Then

Theorem 5 $H^1(E \otimes E^*) \simeq V_{(A,B)}/W_{(A,B)}$.

For the proof one uses the long cohomology exact sequences resulting from (2), (3). The details can be found in [A] or in [AO2].

An algorithm for constructing a basis of the vector space $H^1(E \otimes E^*) = V_{(A,B)}/W_{(A,B)}$ consists of the following three steps:

(1) costruction of a basis of $V_{(A,B)}$;
(2) costruction of a system of generators of $W_{(A,B)}$;
(3) construction of a basis of a complement $U_{(A,B)}$ of $W_{(A,B)}$ in $V_{(A,B)}$:

$$U_{(A,B)} \simeq H^1(E \otimes E^*)$$

The only nontrivial step is (1). For this let $T \in Mat(k^2, 2kr; S_1)$:

$$T = \begin{pmatrix} N_1 & N_2 \end{pmatrix}$$

where $N_1 \in Mat(k^2, kr; S_1)$ is defined by

$$N_1 = \begin{pmatrix} B & 0 & \dots & 0 \\ 0 & B & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & B \end{pmatrix}$$

and $N_2 \in Mat(k^2, kr; S_1)$:

$$N_2 = \begin{pmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & A_1 \\ \dots & \dots & \dots & \dots \\ A_k & 0 & \dots & 0 \\ 0 & A_k & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & A_k \end{pmatrix}$$

where A_j is the j-th row of A.

Theorem 6 *$V_{(A,B)}$ is isomorphic to the vector space $Syz_1(T)$ of the linear syzygies of the matrix T.*

It is easy to implement an algorithm on Macaulay [BS] that computes a basis of $V_{(A,B)}$ (the built-in command "tensor" constructs the matrix T from the matrices A and B).

Next we deal with $H^2(E \otimes E^*)$.
Let us consider the following vector space:

$$Z_{(A,B)} := \{D \in Mat(k,k;S_2) \mid \exists (E,F) \in Mat(k,r;S_1)^{\oplus 2} \mid D = AE^t + FB^t\}.$$

Then

Theorem 7 $H^2(E \otimes E^*) \simeq Mat(k,k;S_2)/Z_{(A,B)}$.

Again we refer to [A], [AO2] for the proof.

Hence the space $H^2(E \otimes E^*)$ can be easily computed by finding a complement of $Z_{(A,B)}$ in $Mat(k,k;S_2)$.

3 The Kuranishi map

Let E be an instanton corresponding to a pair (A, B); the *Kuranishi map*:

$$K\colon\ U \longrightarrow H^2(E \otimes E^*)$$

where $U \subset H^1(E \otimes E^*)$ is a neighborhood of the origin is a holomorphic map such that the germ at 0 of $K^{-1}(0)$ is the versal deformation of E [FK]. When E is stable the above germ is also isomorphic to the germ of the moduli space $MI_{\mathbf{P}^{2n+1}}(k)$ at the point corresponding to E. It follows that the point is smooth if and only if $K \equiv 0$. By 5 and 7 K can be seen as a map

$$K\colon\ U \subset \frac{V_{(A,B)}}{W_{(A,B)}} \longrightarrow \frac{Mat(k,k;S_2)}{Z_{(A,B)}}.$$

The vanishing of K at a point $(A_{(1)}, B_{(1)}) \in V_{(A,B)}$ means that the "first order" instanton $(A + \epsilon A_{(1)}, B + \epsilon B_{(1)})$ extends to a "formal" instanton (M, N) where

$$\begin{aligned} M &= \textstyle\sum_{j=0}^{\infty} \epsilon^j A_{(j)}, \quad A_{(0)} = A \\ N &= \textstyle\sum_{j=0}^{\infty} \epsilon^j B_{(j)}, \quad B_{(0)} = B \end{aligned}$$

are formal power series such that $M \cdot N^t \equiv 0$. The last identity means

$$\sum_{j=0}^{s} A_{(j)} \cdot B^t_{(s-j)} = 0 \ (s = 1, 2 \ldots)$$

In particular for $s = 2$ the second order obstruction for the formal extension is given by

$$A_{(1)} \cdot B^t_{(1)} + A_{(2)} \cdot B^t + A \cdot B^t_{(2)} = 0$$

Hence

Remark 8 *If the map*

$$K_2\colon\ \frac{V_{(A,B)}}{W_{(A,B)}} \longrightarrow \frac{Mat(k,k;S_2)}{Z_{(A,B)}}$$

defined as the product

$$K_2((A', B')\ mod\ W_{(A,B)}) = A' \cdot B'^t\ mod\ Z_{(A,B)}$$

is not identically zero, the point corresponding to E in the moduli space is singular.

The map K_2 can be explicitely computed.
Let $U_{(A,B)}$ and $U'_{(A,B)}$ be subspaces of $V_{(A,B)}$ and $Mat(k,k;S_2)$ respectively isomorphic to $H^1(E \otimes E^*)$ and $H^2(E \otimes E^*)$ ($U_{(A,B)}$ is a complement of $W_{(A,B)}$ in $V_{(A,B)}$, $U'_{(A,B)}$ a complement of $Z_{(A,B)}$ in $Mat(k,k;S_2)$). Let $\{(A_i, B_i)\}_{i=1,\ldots,s}$ be a basis of $U_{(A,B)}$, and $\{X_i\}_{i=1,\ldots,s}$ be the coordinates of an element (A', B') of $U_{(A,B)}$. Then

$$A' \cdot B'^t = (\sum_{i=1}^{s} X_i A_i) \cdot (\sum_{j=1}^{s} X_j \cdot B^t_j) = \sum_{i=1}^{s} \sum_{j=1}^{s} X_i X_j A_i \cdot B^t_j.$$

Let $\{C_i\}_{i=1,\dots,N}$ be a basis of $Mat(k,k;S_2)$ such that $\{C_1,\dots,C_t\}$ is a basis of $U'_{(A,B)}$. We write

$$A_i \cdot B_j^t = \sum_{l=1}^{N} Y_l^{ij} C_l$$

so that

$$A' \cdot B'^t = \sum_{i=1}^{s}\sum_{j=1}^{s}\sum_{l=1}^{N} X_i X_j Y_l^{ij} C_l$$

If we denote by $(A' \cdot B'^t)_{U'_{(A,B)}}$ the projection of $A' \cdot B'^t$ on $U'_{(A,B)}$, we get

$$(A' \cdot B'^t)_{U'_{(A,B)}} = \sum_{i=1}^{s}\sum_{j=1}^{s}\sum_{l=1}^{t} X_i X_j Y_l^{ij} C_l.$$

In the end, the computation of the map K_2 has been reduced to that of the coefficients Y_l^{ij} of the quadratic forms

$$\sum_{i=1}^{s}\sum_{j=1}^{s} Y_l^{ij} X_i X_j, \quad l = 1,\dots,t$$

(the corresponding algorithm on Macaulay [BS] has been implemented by G.Anzidei and A. Pizzotti: see [A]).

Remark 9 *One can also study the moduli spaces of symplectic bundles (which is a subspace of the full moduli space); then $H^j(E \otimes E^*)$ must be replaced by $h^j(S^2E)$ $(j = 1,2)$. It is easy to find results analogous to 5, 7, 8 and the corresponding algorithms.*

4 Examples of singular and reducible moduli spaces

In spite of the vast literature concerned with vector bundles on projective spaces, the first examples of of singular points of their moduli spaces were found by G. Ottaviani and the author in [AO2], performing some computation along the above lines. Of course one difficult point is to guess which bundles correspond to singular points. We give here two explicit examples.

Example 10 *Let E be the istanton of the example 2 with $k = 3$ on $\mathbf{P}^5$; then $h^1(E \otimes E^*) = 57$ and $h^2(E \otimes E^*) = 3$. We find for K_2 the following three equations*

$$\begin{aligned} z_1z_5 + z_2z_6 + z_3z_7 + z_4z_8 &= 0 \\ z_1z_9 + z_2z_{10} + z_3z_{11} + z_4z_{12} &= 0 \\ z_1z_{13} + z_2z_{14} + z_3z_{15} + z_4z_{16} &= 0 \end{aligned}$$

Example 11 *Let E be the instanton on $\mathbf{P}^5$ given by the following matrices*

$$A = \begin{pmatrix} 0 & -x_0 & x_5 & x_4 & x_3 & 0 & 0 & -x_2 & -x_1 & -x_0 \\ -x_0 & x_5 & x_4 & x_3 & 0 & 0 & -x_2 & -x_1 & -x_0 & 0 \\ x_5 & x_4 & x_3 & 0 & -x_3 & -x_2 & -x_1 & -x_0 & 0 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} x_0 & x_1 & x_2 & 0 & 0 & x_3 & x_4 & x_5 & -x_0 & 0 \\ 0 & x_0 & x_1 & x_2 & 0 & 0 & x_3 & x_4 & x_5 & -x_0 \\ 0 & 0 & x_0 & x_1 & x_2 & -x_3 & 0 & x_3 & x_4 & x_5 \end{pmatrix}$$

Then $h^1(E \otimes E^*) = 55$, $h^2(E \otimes E^*) = 1$ and K_2 is given by the equation

$$z_1 z_2 + z_3 z_4 = 0.$$

It follows by 8 that the above examples give singular points of the corresponding moduli spaces.

(After the paper [AO2] appeared, other singularities in moduli spaces of vector bundles on $\mathbf{P}^d$ were found: [M], [Ma], [AO3])

The algorithms explained above are also useful for the study of the irreducible components of the moduli spaces $MI_{\mathbf{P}^{2n+1}}(k)$.

Theorem 12 *([AO4]) $MI_{\mathbf{P}^5}(4)$ contains (at least) two irreducible components of dimension 65 and ≥ 68.*

Remark 13 *The same technique shows that for $k = 4, 5, 6, 7, 8$, $MI_{\mathbf{P}^5}(k)$ contains components of different dimensions, therefore it is reducible.*

We sketch the proof of the above theorem. First we explicitly exhibit an istanton F in $MI_{\mathbf{P}^5}(4)$ with $h^2(F \otimes F^*) = 0$ (see [AO2]). Hence $MI_{\mathbf{P}^5}(4)$ is smooth at F, of dimension $h^1(F \otimes F^*) = 65$. On the other hand we construct another pair (C, D) as follows; we define C as

$$C = \begin{pmatrix} x_0 & 2x_1 & x_2 & & & & y_0 & 3y_1 & y_2 & & & \\ & x_0 & x_1 & x_2 & & & & y_0 & y_1 & y_2 & & \\ & & x_0 & x_1 & x_2 & & & & y_0 & y_1 & y_2 & \\ & & & x_0 & x_1 & x_2 & & & & y_0 & y_1 & y_2 \end{pmatrix}$$

Then we put

$$J = \begin{pmatrix} & 1 & & & & \\ 1 & & & & & \\ & & & & & 1 \\ & & & & 1 & \\ & & & 1 & & \\ & & 1 & & & \end{pmatrix}$$

$$Q = \begin{pmatrix} 0 & J \\ -J^t & 0 \end{pmatrix}$$

and finally we define

$$D = A \cdot Q^t.$$

It is easy to check that the rank of C and D is 4 at every point of $\mathbf{P}^5$ and $C \cdot D^t = 0$. Hence C and D define an instanton bundle E, which is symplectic.
By using Macaulay [BS] one computes

$$h^2(S^2E) = 0$$

$$h^1(S^2E) = 68$$

From 9 it follows that E is a smooth point of the moduli space of symplectic bundles, whose dimension at the point E is 68. As a consequence, the full moduli space $MI_{\mathbf{P}^5}(4)$ has dimension ≥ 68 at E. In particular, E and F belong to different irreducible components.

5 The Brill-Noether locus

Let $\mathcal{Y}$ be a moduli space of stable bundles on $\mathbf{P}^d$ (or more generally on any algebraic variety), so that the points of $\mathcal{Y}$ are (isomorphism classes of) vector bundles. Then it is possible to define interesting subvarieties of $\mathcal{Y}$ just picking out the bundles satisfying a given property. In particular the set

$$\mathcal{Z} = BN(\mathcal{Y}, m) = \{F \in \mathcal{Y} : H^0(F) \geq m\}$$

is a closed subvariety of $\mathcal{Y}$ which we call the *Brill-Noether locus of level m.*

The Zariski tangent space to $\mathcal{Z} = BN(\mathcal{Y}, m)$ at a point F such that $H^0(F) = m$ is a subspace $T_{\mathcal{Z},F} \subset H^1(F \otimes F^*)$ which can be obtained in the following way. Using the Čech cohomology it is easy to find a natural bilinear map

$$\alpha : H^0(F) \times H^1(F \otimes F^*) \longrightarrow H^1(F) \tag{5}$$

which induces a linear map

$$\beta : H^1(F \otimes F^*) \longrightarrow (H^0(F))^* \otimes H^1(F).$$

Then

$$T_{\mathcal{Z},E} = Ker\ \beta$$

If E is a stable instanton bundle it is easy to check that $H^0(E)$ is always zero; hence in order to obtain a non trivial Brill-Noether locus we must replace E by $E(1)$, which does not affect the moduli space. Taking $\mathcal{Y} = MI_{\mathbf{P}^{2n+1}}(k)$ and $F = E(1)$ the Brill-Noether map (5) takes the form

$$\alpha : H^0(E(1)) \times H^1(E \otimes E^*) \longrightarrow H^1(E(1)) \tag{6}$$

We want to compute the above map starting with a pair of matrices (A, B) which detects E. Let us consider the following vector spaces (recall $r = 2n + 2k$):

$$S_A = \{M \in Mat(r, 1; S_1) \mid A \cdot M = 0\}$$

$$T_B = \{M \in Mat(r, 1; S_1) \mid \exists C \in Mat(k, 1; \mathbf{C}) \mid M = B^t \cdot C\}$$

$$R_A = \{R \in Mat(k, 1; S_2) \mid \exists N \in Mat(r, 1; S_1) \mid R = A \cdot N\}$$

then:

Theorem 14

i) $H^0(E(1)) = S_A/T_B$

ii) $H^1(E(1)) = Mat(k, 1; S_2)/R_A$.

Sketch of the proof. The long exact cohomology sequence associated to (2) shows that $H^0(S^*(1)) = S_A$, $H^1(S^*(1)) = Mat(k, 1; S_2)/R_A$, and $H^j(S^*(1)) = 0$ for $j \geq 2$; then the long exact sequence associated to (3) gives $H^0(E(1)) = H^0(S^*(1))/T_B$ and $H^1(E(1)) = H^1(S^*(1))$.

Recall that by the theorem 5 $H^1(E \otimes E^*) \simeq V_{(A,B)}/W_{(A,B)}$.
The partial multiplication map

$$\begin{array}{ccccccc} \mu: & S_A & \times & V_{(A,B)} & \longrightarrow & Mat(k, 1; S_2) \\ & M & & (A', B') & \longrightarrow & A' \cdot M \end{array}$$

satisfies

$$M \in T_B \Longrightarrow \mu(M, (A', B') \in R_A$$

and

$$(A', B') \in W_{(A,B)} \Longrightarrow \mu(M, (A', B') \in R_A$$

According to the theorem 14, μ induces a bilinear map

$$\sigma : H^0(E(1)) \times H^1(E \otimes E^*) \longrightarrow H^1(E(1)) \tag{7}$$

Theorem 15 *The bilinear map σ coincides with the Brill-Noether map (6).*

Proof. Let $m = dim_{\mathbf{C}} H^0(E(1)$. A pair $(A', B') \in H^1(E \otimes E^*)$ belongs to $T_{\mathcal{Z},E}$ if and only if any section M of $E(1)$ extends to a section of the first order deformation $(A + \epsilon A', B + \epsilon B')$. That is, for every $M \in Mat(r, 1; S_1)$ with $A \cdot M = 0$ there exists $M' \in Mat(r, 1; S_1)$ such that

$$(A + \epsilon A') \cdot (M + \epsilon M') \ \ (mod\ \epsilon^2)$$

or

$$A' \cdot M + A \cdot M' = 0,$$

which is exactly the thesis.

If $M_1, \ldots, M_m$ are representatives of a basis of $H^0(E(1)$, a pair $(A', B') \in Mat(k, 2n+2k; S_1)^{\oplus 2}$ belongs to the Brill-Noether locus if it satisfies the system

$$\begin{aligned} A \cdot B'^t + A' \cdot B^t &= 0 \\ A' \cdot M_j + A \cdot M'_j &= 0 \quad (j = 0 \ldots, m) \end{aligned}$$

(where the unknown are A', B' and $M'_1, \ldots, M'_m$). The above equations are equivalent to the computation of the linear syzygies of a suitable matrix; the corresponding algorithm can be easily implemented.

References

[A] V. Ancona, I fibrati istantoni sugli spazi proiettivi: teoria, metodi algoritmici e risultati sperimentali, Pubbl. dell'Istituto Naz. di Alta Matematica, Roma 1996

[AABOP] V. Ancona, G. Anzidei, P. Breglia, G. Ottaviani, A. Pizzotti, Mathematical instanton bundles on projective spaces: an algorithmic approach. Pubbl. del Dipartimento di Matematica dell'Universita' di Firenze n.24 (1996)

[AO1] V. Ancona, G. Ottaviani, Stability of special instanton bundles on $\mathbf{P}^{2n+1}$, Trans. Am. Math. Soc. 341, 677-693 (1994)

[AO2] V. Ancona, G. Ottaviani, On moduli of instanton bundles in $\mathbf{P}^{2n+1}$, Pacific Journal of math. 171, 343-351 (1995)

[AO3] V. Ancona, G. Ottaviani, On singularities of $M(c_1, c_2)$, International J. of Math. 9, 407-419 (1998)

[AO4] V. Ancona, G. Ottaviani, On the irreducible components of the moduli spaces of instanton bundles on P^5, to appear in Seminario di variabili complesse (S. Coen editor), Bologna 1999.

[AW] M.F.Atiyah, R.S.Ward, Instantons and algebraic geometry, Comm. Math. Phys 55. 117-124 (1977)

[BS] D. Bayer, M. Stillman, Macaulay, a computer algebra system for algebraic geometry.

[FK] O. Forster, K. Knorr, Über die Deformationen von Vectorraumbündeln auf Kompacten Komplexen Raümen, Math. Ann. 209, 291-346 (1974)

[M] R.M. Miro'-Roig, Singular moduli spaces of stable vector bundles on $\mathbf{P}^3$, Pacific Journal of math. 172, 477-482 (1996)

[Ma] M. Maggesi, $MI_{\mathbf{P}^3}(0, 2d^2)$ is singular, Forum mathematicum 8, 397-400 (1996)

[MO] R.M. Miro'-Roig, J.A. Orus-Lacort, On the smoothness of the moduli space of mathematical instanton bundles, Compositio math. 105, 109-119 (1997)

[OS] C. Okonek, H. Spindler, Mathematical instanton bundles on $\mathbf{P}^{2n+1}$, Journal reine angew. Math. 364, 35-50 (1986)

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