数理解析研究所講究録
1139 巻 2000 年 19-29

A NOTE ON GENERAL SETTING OF WHITE NOISE TRIPLE AND POSITIVE GENERALIZED FUNCTIONS

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1. Introduction

Let $\mathcal{E}^*$ be the space of tempered distributions and μ be the standard Gaussian measure on $\mathcal{E}^*$. Being motivated by the distribution theory on infinite dimensional space by Cochran, Kuo and Sengupta (CKS) [6], Asai, Kubo and Kuo (AKK) have recently determined the best possible class $C^{(2)}_{+,\frac{1}{2},1}$ of functions u to constract white noise triple,

$$[\mathcal{E}]_u \subset L^2(\mathcal{E}^*, \mu) \subset [\mathcal{E}]^*_u,$$

and to characterize white noise test function space $[\mathcal{E}]_u$ and generalized function space $[\mathcal{E}]^*_u$ in the series of papers [1],[2],[3],[4],[5]. The notion of Legendre transformation plays important roles to examine relationships between the growth order of holomorphic functions (S-transform) and the CKS-space of white noise test and generalized functions. It is well-known that a positive generalized function is induced by a Hida measure ν (generalized measure). A Hida measure can be characterized by integrability conditions on a function inducing the above triple ([5]). See also [20],[21],[25] for an overview of other recent developments in white noise analysis.

This short note is organized as follows. In Section 2, we give a short summary of white noise analysis including AKK's results. A certain class of convex functions will be introduced to make use of Legendre transformation and dual functions for our purposes. In Section 3, we restate the characterization theorems of the spaces of white noise test and generalized functions given in [3],[5]. In Section 4, we give a quick review of the basic facts on the theory of positive generalized functions [12],[19],[29]. Finally, we discuss the characterization of a Hida measure (Theorem 4.5). In this connection, we present the grey noise and the Poisson noise measures as typical examples inducing positive generalized functions (Examples 4.7 and 4.8, respectively). Moreover, we mention breifly the relationship between $[\mathcal{E}]_u$ and L^s-space on $(\mathcal{E}^*, \nu)$ (Proposition 4.6).

2. Preliminaries

Let us start with taking a special choice of a Gel'fand triple:

$$\mathcal{E} = \mathcal{S}(\mathbb{R}) \subset \mathcal{E}_0 = L^2(\mathbb{R}, dt) \subset \mathcal{E}^* = \mathcal{S}^*(\mathbb{R})$$

just for convenience where $\mathcal{S}$ is the Schwartz space of rapidly decreasing functions and $\mathcal{S}^*$ is the space of tempered distributions. Consult excellent books [19],[24] for more general constraction. Let $A = 1 + t^2 - d^2/dt^2$. It is well-known that A is a densely defined positive self-adjoint operator on $\mathcal{E}_0$. So there exists an orthonormal basis $\{e_j\}_{j=0}^{\infty} \subset \mathcal{E}$ for $\mathcal{E}_0$ satisfying $Ae_j = (2j+2)e_j$. For each $p \geq 0$ we define $|f|_p = |A^p f|_0$ and let $\mathcal{E}_p = \{f \in \mathcal{E}_0 \; ; \; |f|_p < \infty, p \geq 0\}$. Note that $\mathcal{E}_p$ is the completion of $\mathcal{E}$ with respect to the norm $|\cdot|_p$. Moreover,

$$\rho = \|A^{-1}\|_{op} = \frac{1}{2}, \quad \|i_{q,p}\|_{HS}^2 = \sum_{j=0}^{\infty}(2j+2)^{-(q-p)} < \infty$$

for any $q > p \geq 0$. Then the projective limit space $\mathcal{E}$ of $\mathcal{E}_p$ is a nuclear space and the dual space of $\mathcal{E}$ is nothing but $\mathcal{E}^*$. Hence we have the following continuous inclusions:

$$\mathcal{E} \subset \mathcal{E}_p \subset \mathcal{E}_0 \subset \mathcal{E}_p^* \subset \mathcal{E}^*, \quad p \geq 0.$$

where the norm on $\mathcal{E}_p^*$ is given by

$$|f|_{-p} = |A^{-p} f|_0.$$

Troughout this paper, we denote the complexification of a real space X by X_c. Let μ be the standard Gaussian measure on $\mathcal{E}^*$ given by

$$e^{-\frac{1}{2}|\xi|_0^2} = \int_{\mathcal{E}^*} e^{i\langle x,\xi\rangle}\mu(dx), \qquad \xi \in \mathcal{E}^*.$$

The probability space $(\mathcal{E}^*, \mu)$ is called the *white noise space* or *Gaussian space*. Let $(L^2) = L^2(\mathcal{E}^*, \mu)$ denote the Hilbert space of μ-square integrable functions on $\mathcal{E}^*$. By the Wiener-Itô theorem each φ in (L^2) can be uniquely expressed as

$$\varphi = \sum_{n=0}^{\infty} I_n(f_n) = \sum_{n=0}^{\infty} \langle :\cdot^{\otimes n}:, f_n\rangle, \qquad f_n \in \mathcal{E}_{0,c}^{\widehat{\otimes} n}, \tag{2.1}$$

and the (L^2)-norm $\|\varphi\|_0$ of φ is given by

$$\|\varphi\|_0 = \left(\sum_{n=0}^{\infty} n!|f_n|_0^2\right)^{1/2}.$$

We now briefly recall notions and results discussed by Asai et al. [2],[3].

Definition 2.1. Let u be a positive continuous function on $[0, \infty)$. The function u is called *(log, x^2)-convex* if $\log u(x^2)$ is convex on $[0, \infty)$.

Definition 2.2.

(1) Let $C_{+,\log}$ be the class of all positive continuous functions on $[0,\infty)$ and

$$\lim_{r\to\infty} \frac{\log u(r)}{\log r} = \infty. \tag{2.2}$$

(2) Let $C_{+,\frac{1}{2}}$ be the class of all positive continuous functions on $[0,\infty)$ and

$$\lim_{r\to\infty} \frac{\log u(r)}{\sqrt{r}} = \infty. \tag{2.3}$$

(3) Let $C_{+,\frac{1}{2},1}$ be the class of functions $u \in C_{+,\frac{1}{2}}$ such that

$$\lim_{r\to\infty} \frac{\log u(r)}{r} < \infty. \tag{2.4}$$

(4) Let $C^{(2)}_{+,\frac{1}{2}}$ (or $C^{(2)}_{+,\frac{1}{2},1}$) be the class of all $(\log, x^2)$-convex functions in $C_{+,\frac{1}{2}}$ (or $C_{+,\frac{1}{2},1}$), respectively.

Definition 2.3. The *Legendre transform* ℓ_u of a function $u \in C_{+,\log}$ is defined by

$$\ell_u(t) := \inf_{r>0} \frac{u(r)}{r^t}, \quad t \geq 0 \tag{2.5}$$

Definition 2.4. The *dual Legendre function* u^* of a function $u \in C_{+,\frac{1}{2}}$ is given by

$$u^*(r) := \sup_{s>0} \frac{e^{2\sqrt{rs}}}{u(s)} \in C^{(2)}_{+,\frac{1}{2}}. \tag{2.6}$$

Definition 2.5. Two positive sequences $\{a(n)\}_{n=0}^{\infty}$ and $\{b(n)\}_{n=0}^{\infty}$ are called *equivalent* (denoted by $a(n) \sim b(n)$) if there exist positive constants K_1, K_2, c_1, c_2 such that

$$K_1 c_1^n a(n) \leq b(n) \leq K_2 c_2^n a(n) \quad \text{for any } n \in \mathbb{N}. \tag{2.7}$$

Definition 2.6. A positive sequence $\{a(n)\}_{n=0}^{\infty}$ is a dual sequence of $\{b(n)\}_{n=0}^{\infty}$ if $a(n)b(n) \sim (n!)^{-2}$ holds.

According to Theorem 4.6 in [3], for a fucntion $u \in C^{(2)}_{+,\frac{1}{2}}$ we have

$$\ell_{u^*}(n) \sim \frac{1}{\ell_u(n)(n!)^2}. \tag{2.8}$$

Thus, $\{\ell_{u^*}(n)\}_{n=0}^{\infty}$ is a dual sequence of $\{\ell_u(n)\}_{n=0}^{\infty}$.

Definition 2.7. Two positive functions $u(r)$ and $v(r)$ on $[0, \infty)$ are called *equivalent* (denoted by $u(r) \approx v(r)$) if there exist positive constants a_1, a_2, c_1, c_2 such that

$$c_1 u(a_1 r) \le v(r) \le c_2 u(a_2 r) \quad \text{for any } r \ge 0. \tag{2.9}$$

The condition (2.3) is required for both u and u^* to be equivalent to entire functions, respectively. This requirement is essential for Theorem 3.1 and Theorem 3.2 which will be discussed later.

Next, we describe the spaces of test and generalized functions on the space $\mathcal{E}^*$ introduced by Cochran et al. in a recent paper [6]. Let $\{\alpha(n)\}_{n=0}^{\infty}$ be a weight sequence satisfying the following *two conditions* [2],[4],[6]:

(A1) $\alpha(0) = 1$ and $\inf_{n \ge 0} \alpha(n)\sigma^n > 0$ for some $\sigma \ge 1$.

(A2) $\lim_{n\to\infty} \left(\frac{\alpha(n)}{n!}\right)^{1/n} = 0$.

Let $\varphi \in (L^2)$ be represented as in Equation (2.1). For $p \ge 0$ and a given function $u \in C^{(2)}_{+,\frac{1}{2},1}$, define

$$\|\varphi\|_{u,p} = \left(\sum_{n=0}^{\infty} \frac{1}{\ell_u(n)} |f_n|_p^2\right)^{1/2}, \qquad f_n \in \mathcal{E}_{p,c}^{\widehat{\otimes} n}. \tag{2.10}$$

where $|f_n|_p = |(A^{\otimes n})^p f_n|_0^2$. Note that since $\ell_u(n)$ and $\ell_{u^*}(n)$ are dual sequences of one another, we choose $\alpha(n) = (n!\ell_u(n))^{-1} \sim n!\ell_{u^*}(n)$ as a weight sequence. It is easy to check by (2.3), (2.4) and (2.5) that the above weight sequence with assumptions, $u \in C^{(2)}_{+,\frac{1}{2},1}$ and $\inf\{u(r); r \ge 0\} = 1$, satisfies the conditions (A1) and (A2). Let $[\mathcal{E}_p]_u = \{\varphi \in (L^2)\,;\, \|\varphi\|_{u,p} < \infty\}$. Define the space $[\mathcal{E}]_u$ of test functions to be the projective limit of the family $\{[\mathcal{E}_p]_u; p \ge 0\}$. Its dual space $[\mathcal{E}]_u^*$ is the space of generalized functions. By identifying (L^2) with its dual we get the following continuous inclusion maps:

$$[\mathcal{E}]_u \subset [\mathcal{E}_p]_u \subset (L^2) \subset [\mathcal{E}_p]_u^* \subset [\mathcal{E}]_u^*, \quad p \ge 0.$$

Note that the condition (2.4) is needed in order to have the continuous inclusion $[\mathcal{E}_p]_u \subset (L^2)$ for $p \ge 0$. The canonical bilinear form on $[\mathcal{E}]_u^* \times [\mathcal{E}]_u$ is denoted by $\langle\langle \cdot, \cdot \rangle\rangle$. For each $\Phi \in [\mathcal{E}_p]_u^*$ there exists a unique $F_n \in (\mathcal{E}_{p,c}^{\otimes n})^*_{symm}$ such that

$$\langle\langle \Phi, \varphi \rangle\rangle = \sum_{n=0}^{\infty} n! \langle F_n, f_n \rangle$$

and

$$\|\varphi\|_{u^*,-p} = \left(\sum_{n=0}^{\infty} \frac{1}{\ell_{u^*}(n)} |F_n|_{-p}^2\right)^{1/2}. \tag{2.11}$$

The Gel'fand triple $[\mathcal{E}]_u \subset (L^2) \subset [\mathcal{E}]_u^*$ is called the *Cochran-Kuo-Sengupta space* (CKS-space for short) associated with a given weight function $u \in C^{(2)}_{+,\frac{1}{2},1}$ (see [4],[5]). We remark that some similar results have been obtained independently by Gannoun et al. [7].

Example 2.8. Let $0 \le \beta < 1$. If $v(r) = \exp((1+\beta)r^{\frac{1}{1+\beta}})$, then $v \in C^{(2)}_{+,\frac{1}{2},1}$. In addition, the dual function is $v^*(r) = \exp((1-\beta)r^{\frac{1}{1-\beta}})$. By Stirling formula, we have for any $n \ge 0$

$$\frac{1}{(n!)^{1+\beta}} \le \ell_v(n) = \left(\frac{e}{n}\right)^{(1+\beta)n} \le \left(\frac{e2^{n/2}}{n!}\right)^{1+\beta} \tag{2.12}$$

That is, $\ell_v(n) \sim (n!)^{-(1+\beta)}$. On the other hand, similarly we get $\ell_{v^*}(n) \sim (n!)^{-(1-\beta)}$. Hence $\ell_{u_1}(n)$ and $\ell_{v^*}(n)$ are dual sequences of one another. This example induces the *Hida-Kubo-Takenaka space* with $\beta = 0$ [9],[17],[18],[24],

$$(\mathcal{E})_0 \subset (L^2) \subset (\mathcal{E})_0^*,$$

and the *Kondratiev-Streit space* [13],[14],[19],

$$(\mathcal{E})_\beta \subset (L^2) \subset (\mathcal{E})_\beta^*, \quad 0 \le \beta < 1.$$

See [10], [11] if $\beta = 1$.

Example 2.9. Let

$$u_k(r) = \frac{\exp_k(r)}{\exp_k(0)} = \sum_{n=0}^{\infty} \frac{b_k(n)}{n!} r^n.$$

where

$$\exp_1(r) := \exp(r), \quad \exp_k(r) := \exp(\exp_{k-1}(r)) \ \text{ for } k \ge 2$$

and $b_k(n)$ is *the k-th order Bell number* [1],[2],[3],[4],[6],[15]. Then its dual Legendre function u_2^* is equivalent to the function

$$u_k^*(r) \approx \exp\left[2\sqrt{r \log_{k-1}\sqrt{r}}\right],$$

where $\log_k(r)$ is given by

$$\log_1(r) := \log(\max\{e, r\}), \quad \log_k(r) := \log_1(\log_{k-1}(r)) \ \text{ for } k \ge 2.$$

Then, $u_k^* \in C^{(2)}_{+,\frac{1}{2},1}$. Details of $\ell_{u_k}(n)$ can be found in [15],[16]. The Gel'fand triple

$$[\mathcal{E}]_{u_k} \subset (L^2) \subset [\mathcal{E}]_{u_k}^*$$

is called the *Bell number space of order k* [6] and

$$[\mathcal{E}]_{u_k} \subset (\mathcal{E})_\beta \subset (L^2) \subset (\mathcal{E})_\beta^* \subset [\mathcal{E}]_{u_k}^*.$$

Remark. We point out that

$$(\mathcal{E})_1 \subset [\mathcal{E}]_u \subset (\mathcal{E})_0 \subset (L^2) \subset (\mathcal{E})_0^* \subset [\mathcal{E}]_u^* \subset (\mathcal{E})_1^*$$

holds for any $u \in C^{(2)}_{+,\frac{1}{2},1}$.

3. General Characterization Theorems

The *exponential function* (or *coherent state*) $\phi_\xi(\cdot)$ is given by

$$\phi_\xi = e^{\langle\cdot,\xi\rangle - \frac{1}{2}|\xi|_0^2} = \sum_{n=0}^{\infty} \frac{1}{n!}\langle :\cdot^{\otimes n}:, \xi^{\otimes n}\rangle, \qquad \xi \in \mathcal{E}_c.$$

Since it is well-known that the exponential functions $\{\phi_\xi\ ;\ \xi \in \mathcal{E}_c\}$ span a dense subspace of $[\mathcal{E}]_u$, $\Phi \in [\mathcal{E}]_u^*$ is uniquely determined by its *S-transform*:

$$S\Phi(\xi) := \langle\langle \Phi, \phi_\xi \rangle\rangle, \quad \xi \in \mathcal{E}_c.$$

Now we are in a position to state the characterization theorems of test and generalized functions under very general assumptions. We remark that these theorems were examined in [22],[27] for the Hida-Kubo-Takenaka space case and [13],[14] for the Kondratiev-Streit space case.

Theorem 3.1 ([3],[5]). *Let $u \in C^{(2)}_{+,\frac{1}{2},1}$ be increasing with $u(0) = 1$. A function $F : \mathcal{E}_c \to \mathbb{C}$ is the S-transform of some $\varphi \in [\mathcal{E}]_u$ if and only if*
$(F1) z \mapsto F(z\xi + \eta)$ is entire holomorphic in $z \in \mathbb{C}$ for any $\xi, \eta \in \mathcal{E}_c$,
$(F2)'$ For any $a, p \geq 0$, there exists a constant $K > 0$ such that

$$|F(\xi)| \leq K u(a|\xi|_{-p}^2)^{\frac{1}{2}} \quad \textit{for any } \xi \in \mathcal{E}_c. \tag{3.1}$$

In addition, in that case,

$$\|\varphi\|_{u,q}^2 \leq K^2 \left(1 - ae^2 \|i_{p,q}\|_{HS}^2\right)^{-1} \tag{3.2}$$

for any $q < p$ satisfying $ae^2\|i_{p,q}\|_{HS}^2 < 1$.

Theorem 3.2 ([5]). *Suppose that $u \in C^{(2)}_{+,\frac{1}{2},1}$ and $\inf_{r>0} u(r) = 1$. A function $F : \mathcal{E}_c \to \mathbb{C}$ is the S-transform of some $\varphi \in [\mathcal{E}]_u^*$ if and only if*
$(F1) z \mapsto F(z\xi + \eta)$ is entire holomorphic in $z \in \mathbb{C}$ for any $\xi, \eta \in \mathcal{E}_c$,
$(F2)$ there exist nonnegative constants K, a and p such that

$$|F(\xi)| \leq K u^*(a|\xi|_p^2)^{\frac{1}{2}} \quad \textit{for any } \xi \in \mathcal{E}_c. \tag{3.3}$$

In addition, in that case,

$$\|\varphi\|_{u^*,-q}^2 \leq K^2 \left(1 - ae^2 \|i_{q,p}\|_{HS}^2\right)^{-1} \tag{3.4}$$

for any $q > 0$ satisfying $ae^2\|i_{q,p}\|_{HS}^2 < 1$.

Remark. In [8], Hida introduced the infinite dimensional analogue of Fourier transform, so-called $\mathcal{T}$-*transform*, given by

$$\mathcal{T}\Phi(\xi) = \langle\langle \Phi, e^{i\langle \cdot, \xi\rangle}\rangle\rangle, \quad \xi \in \mathcal{E}_c. \tag{3.5}$$

It is well-known that the $\mathcal{T}\Phi(z\xi + \eta)$ is entire holomorphic in $z \in \mathbb{C}$ for any $\xi, \eta \in \mathcal{E}_c$. In addition, there is a nice relationship between $\mathcal{T}$-transform and S-transform:

$$S\Phi(\xi) = \mathcal{T}(-i\xi)\exp[-\frac{1}{2}\langle \xi, \xi\rangle], \quad \xi \in \mathcal{E}_c. \tag{3.6}$$

and

$$\mathcal{T}\Phi(\xi) = S(i\xi)\exp[-\frac{1}{2}\langle \xi, \xi\rangle], \quad \xi \in \mathcal{E}_c. \tag{3.7}$$

Therefore, Theorems 3.1 and 3.2 remain valid even if the S-transform is replaced by the $\mathcal{T}$-transform.

4. Characterization of Positive Generalized Functions

Definition 4.1. A measure ν on $\mathcal{E}^*$ is called a *Hida measure* if $[\mathcal{E}]_u \subset L^1(\nu)$ and the mapping $\varphi \mapsto \int_{\mathcal{E}^*} \varphi(x)\nu(dx)$ is coninuous on $[\mathcal{E}]_u$.

Suppose $\Phi \in [\mathcal{E}]_u^*$ is induced by a Hida measure ν on $\mathcal{E}^*$. Then

$$\langle\langle \Phi, \varphi\rangle\rangle = \int_{\mathcal{E}^*} \varphi(x)\nu(dx), \ \varphi \in [\mathcal{E}]_u.$$

In paticular, take $\varphi = e^{i<x,\xi>}$, $\xi \in \mathcal{E}$. Then, we have the $\mathcal{T}$-transform restricted to $\mathcal{E}$ by

$$\mathcal{T}\Phi(\xi)|_{\mathcal{E}} := \langle\langle \Phi, e^{i<x,\xi>}\rangle\rangle = \int_{\mathcal{E}^*} e^{i<x,\xi>}\nu(dx), \quad \xi \in \mathcal{E}.$$

It is clearly seen that $\mathcal{T}\Phi(\xi)|_{\mathcal{E}}$ is equal to the characteristic function $C(\xi)$ of ν. Thus in order to see the existance of a measure ν, we only need to check that the function $\mathcal{T}\Phi(\xi)|_{\mathcal{E}}$ satisfies the following conditions:
(1) $\mathcal{T}\Phi(\xi)|_{\mathcal{E}}$ is continuous on $\mathcal{E}$.
(2) $\mathcal{T}\Phi(\xi)|_{\mathcal{E}}$ is positive definite on $\mathcal{E}$.

Definition 4.2. A generalized function $\Phi \in [\mathcal{E}]_u^*$ is called *positive* if $\langle\langle \Phi, \varphi\rangle\rangle \geq 0$ for all nonnegative test functions $\varphi \in [\mathcal{E}]_u$.

Hence for any nonnegative test function $\varphi \in [\mathcal{E}]_u$,

$$\langle\langle \Phi, \varphi\rangle\rangle = \int_{\mathcal{E}^*} \varphi(x)\nu(dx) \geq 0.$$

Thus Φ is a positive generalized function. In the following, Theorem 4.3 says that all of positive generalized functions is generalized functions induced by Hida measures on $\mathcal{E}^*$.

Theorem 4.3. *Let $u \in C^{(2)}_{+,\frac{1}{2},1}$ and $\Phi \in [\mathcal{E}]^*_u$. Then the following statements are equivalent:*

(a) *Φ is positive.*

(b) *$\mathcal{T}\Phi$ is positive definite on $\mathcal{E}$.*

(c) *Φ is induced by a Hida measure. That is, there exists a finite measure ν on $\mathcal{E}^*$ such that $[\mathcal{E}]_u \subset L^1(\nu)$ and*

$$\langle\langle \Phi, \varphi \rangle\rangle = \int_{\mathcal{E}^*} \varphi(x)\nu(dx), \quad \varphi \in [\mathcal{E}]_u.$$

Remark. For the Kondratiev-Streit space, the equivalence of $(a) \sim (c)$ has been discussed in [19]. The equivalence of (a) and (c) was examined originally in [12] and [29] only for the case of the Hida-Kubo-Takenaka space.

Next, our problem is how to characterize Hida measures on $\mathcal{E}^*$. For this purpose, based on Lee's idea [23] (see also [3],[5],[19],[26]), we shall define another norm as follows. Let $\mathcal{A}_{u,p}$ be the space of all functions φ on $\mathcal{E}^*_c$ satisfying the following conditions:

(A1) φ is an analytic function on $\mathcal{E}^*_{p,c}$.

(A2) There exists a nonnegative constant C such that

$$|\varphi(x)|^2 \leq Cu(|x|^2_{-p}) \ \text{ for any } x \in \mathcal{E}^*_{p,c}.$$

For $\varphi \in \mathcal{A}_{u,p}$, its norm is defined by

$$|||\varphi|||_{\mathcal{A}_{u,p}} := \sup_{x \in \mathcal{E}^*_{p,c}} |\varphi(x)| u(|x|^2_{-p})^{-\frac{1}{2}}. \tag{4.1}$$

for a function $u \in C_{+,\log}$. Define the space $\mathcal{A}_u$ of test functions on $\mathcal{E}^*$ by

$$\mathcal{A}_u = \operatorname{proj}\lim_{p\to\infty} \mathcal{A}_{u,p}.$$

Proposition 4.4. *Suppose that $u \in C^{(2)}_{+,\frac{1}{2},1}$. Then $\{|||\cdot|||_{\mathcal{A}_{u,p}}; p \geq 1\}$ is equivalent to $\{\|\cdot\|_{u,p}; p \geq 1\}$. As a result, $\mathcal{A}_u = [\mathcal{E}]_u$ as vector spaces*

Remark. This proposition implies that

$$\mathcal{A}^*_u = [\mathcal{E}]^*_u$$

and they have the same inductive limit topology. Moreover, for the constraction of spaces $\mathcal{A}_u$ and $\mathcal{A}^*_u$ the Wiener-Itô decomposition theorem is not used and a measure on $\mathcal{E}^*$ is not refered at all.

We are in a position to state our characterization theorem of Hida measure in terms of positivity and integrability condition. The proof of Theorem 4.5 is based on simple applications of S-transform ($\mathcal{T}$-transform, equivalently the Bochner-Minlos

Theorem), the dual functions given in (2.6) and some technical estimations. See [5] for details.

Theorem 4.5 ([5]). *Suppose that $u \in C^{(2)}_{+,\frac{1}{2},1}$. A measure ν on $\mathcal{E}^*$ is a Hida measure inducing a positive generalized function $\Phi_\nu \in [\mathcal{E}]^*_u$ if and only if ν is supported in $\mathcal{E}^*_p$ for some $p \geq 1$ and*

$$\int_{\mathcal{E}^*_p} u(|x|^2_{-p})^{\frac{1}{2}} \nu(dx) < \infty.$$

Remark. See [23] for the Hida-Kubo-Takenaka space and [19] for the Kondratiev-Streit space.

Proposition 4.6. *Let $u \in C^{(2)}_{+,\frac{1}{2},1}$ and ν be a Hida measure on $\mathcal{E}^*$ inducing a generalized function Φ_ν in $[\mathcal{E}]^*_u$. Then $[\mathcal{E}]_u \subset \bigcap_{1\leq s<\infty} L^s(\mathcal{E}^*, \nu)$. In addition, for each $1 \leq s < \infty$, the inclusion mapping $[\mathcal{E}]_u \hookrightarrow L^s(\nu)$ is continuous.*

Example 4.7. (Grey noise measure)
Let $0 < \lambda \leq 1$. The grey noise measure on $\mathcal{E}^*$ is the measure ν_λ having the characteristic function

$$L_\lambda(|\xi|^2_0) = \int_{\mathcal{E}^*} e^{i\langle x,\xi\rangle} \nu_\lambda(dx), \quad \xi \in \mathcal{E},$$

where $L_\lambda(t)$ is the Mittag-Leffler function with parameter λ;

$$L_\lambda(t) = \sum_{n=0}^{\infty} \frac{(-t)^n}{\Gamma(1+\lambda n)}.$$

Here Γ is the Gamma function. This measure was introduced by Schneider [28]. It is shown in [19] that ν_λ is a Hida measure which induces a generalized function Φ_{ν_λ} in $(\mathcal{E})^*_{1-\lambda}$. Therefore by Theorem 4.5 and Example 2.8 the grey noise measure ν_λ satisfies

$$\int_{\mathcal{E}^*_p} \exp\Big(\frac{1}{2}(2-\lambda)|x|^{\frac{2}{2-\lambda}}_{-p}\Big) \nu_\lambda(dx) < \infty$$

for some p.

Example 4.8. (Poisson measure)
Let $\mathcal{P}$ be the Poisson measure on $\mathcal{E}^*$ given by

$$\exp\Big(\int_{\mathbb{R}} (e^{i\xi(t)} - 1)dt\Big) = \int_{\mathcal{E}^*} e^{i\langle x,\xi\rangle} \mathcal{P}(dx), \quad \xi \in \mathcal{E}^*.$$

It has been shown [6] that the Poisson noise measure induces a generalized function in the Bell number space of order 2. Thus by Theorem 4.5 and Example 2.9 we have the integrability condition

$$\int_{\mathcal{E}_p^*} \exp\Big(|x|_{-p}\sqrt{\log |x|_{-p}}\Big)\mathcal{P}(dx) < \infty$$

for some p.

Acknowledgements The author is grateful for financial supports from the Daiko Foundation and Kamiyama Foundation.

References

1. Asai, N., Kubo, I., and Kuo, H.-H.: Bell numbers, log-concavity, and log-convexity; in: *Classical and Quantum White Noise*, L. Accardi et al. (eds.) Kluwer Academic Publishers (1999)
2. Asai, N., Kubo, I., and Kuo, H.-H.: Characterization of test functions in CKS-space; in: *Proc. International Conference on Mathematical Physics and Stochastic Processes*, A. Albeverio et al. (eds.) World Scientific (1999)
3. Asai, N., Kubo, I., and Kuo, H.-H.: Log-concavity, log-convexity, and growth order in white noise analysis; *Preprint* (1999)
4. Asai, N., Kubo, I., and Kuo, H.-H.: CKS-space in terms of growth functions; *Preprint* (1999)
5. Asai, N., Kubo, I., and Kuo, H.-H.: General characterization theorems and intrinsic topologies in white noise analysis; *Preprint* (1999)
6. Cochran, W. G., Kuo, H.-H., and Sengupta, A.: A new class of white noise generalized functions; *Infinite Dimensional Analysis, Quantum Probability and Related Topics* **1** (1998) 43–67
7. Gannoun, R., Hachaichi, R., Ouerdiane, H., and Rezgui, A.: Un Thórème de Dualite Entré Espaces de Fonctions Holomorphes à Croissance Exponentiele; to appear in *J. Funct. Anal.*
8. Hida, T., *Analysis of Brownian Functionals.* Carleton Mathematical Lecture Notes **13**, 1975
9. Hida, T., Kuo, H.-H., Potthoff, J., and Streit, L.: *White Noise: An Infinite Dimensional Calculus.* Kluwer Academic Publishers, 1993
10. Holden, H., Øksendal, B., Ubøe, J., and Zhang, T.: *Stochastic Partial Differential Equations.* Birkhäuser, 1996
11. Kondratiev, Yu. G., Leukert, P., and Streit, L.: Wick calculus in white noise analysis; *Acta Appl. Math* **44** (1996) 269–294
12. Kondratiev, Yu. G. and Samoilenko, Yu. S.: Integral representation of generalized positive definite kernels of an infinite number of variables; *Soviet Math. Dolk.* **17** (1976) 517-521
13. Kondratiev, Yu. G. and Streit, L.: A remark about a norm estimate for white noise distributions; *Ukrainian Math. J.* **44** (1992) 832–835
14. Kondratiev, Yu. G. and Streit, L.: Spaces of white noise distributions: Constructions, Descriptions, Applications. I; *Reports on Math. Phys.* **33** (1993) 341–366

15. Kubo, I.: Entire functionals and generalized functionals in white noise analysis; in:"*Analysis on Infinite Dimensional Lie Group and Algebras,*" World Scientific, 1998
16. Kubo, I., Kuo, H.-H., and Sengupta, A.: White noise analysis on a new space of Hida distributions; to appear in *Infinite Dimensional Analysis, Quantum Probability and Related Topics*
17. Kubo, I. and Takenaka, S.: Calculus on Gaussian white noise I; *Proc. Japan Acad.* **56A** (1980) 376–380
18. Kubo, I. and Takenaka, S.: Calculus on Gaussian white noise II; *Proc. Japan Acad.* **56A** (1980) 411–416
19. Kuo, H.-H.: *White Noise Distribution Theory.* CRC Press, 1996
20. Kuo, H.-H.: Some aspects of white noise analysis; to appear in: *Proc. Volterra International School on White Noise Approach to Classical and Quantum Stochastic Calculi*, L. Accardi (eds.)
21. Kuo, H.-H.: White noise theory; to appear in: *Handbook of Stochastic Analysis and Applications*, D. Kannan and V. Lakshmikantham (eds.) Marcel Dekker, Inc.
22. Kuo, H.-H., Potthoff, J., and Streit, L.: A characterization of white noise test functionals; *Nagoya Math. J.* **121** (1991) 185–194
23. Lee, Y.-J.: Analytic version of test functionals, Fourier transform and a characterization of measures in white noise calculus; *J. Funct. Anal.* **100** (1991) 359–380
24. Obata, N.: *White Noise Calculus and Fock Space.* Lecture Notes in Math. **1577**, Springer-Verlag, 1994
25. Obata, N.: White noise operator theory and applications; to appear in: *Proc. Volterra International School on White Noise Approach to Classical and Quantum Stochastic Calculi*, L. Accardi (eds.)
26. Ouerdiane, H.: Noyaux et Symboles d'opérateurs sur des Functionnelles Analytiques Gaussiennes; *Japan. J. Math.* **21**, (1995) 223–233
27. Potthoff, J. and Streit, L.: A characterization of Hida distributions; *J. Funct. Anal.* **101** (1991) 212–229
28. Schneider, W. R.: Grey noise; in: *Stochastic Processes, Physics and Geometry*, S. Albeverio et al. (eds.) World Scientific (1990)
29. Yokoi, Y.: Positive generalized white noise functionals; *Hiroshima Math. J.* **20**, (1990) 137–157