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Cauchy Problems in White Noise Analysis and An Application to Finite Dimensional PDEs II

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1 Introduction

The white noise analysis initiated by Hida [8] has been considerably developed to an infinite dimensional analysis on Gaussian space with applications to many fields: stochastic analysis, Feynman path integral, quantum physics and infinite dimensional harmonic analysis and so on, see e.g. [10], [17], [18] and references cited therein. The mathematical framework of white noise analysis is the Gel'fand triple:

$$(E) \subset (L^2) \equiv L^2(E^*, \mu; \mathbb{C}) \subset (E)^*$$

associated with the Gel'fand triple $E \subset H \subset E^*$, where H is a real separable Hilbert space. In this paper, our work is based on a triplet:

$$\mathcal{G}(K) \subset (K) \subset \mathcal{G}(K)^* \tag{1.1}$$

where K is another complex Hilbert space such that the imbedding $K \hookrightarrow H_{\mathbb{C}}$ is contraction and (K) is the subspace of (L^2) corresponding to the Boson Fock space $\Gamma(K)$ over K under the Wiener-Itô-Segal isomorphism between (L^2) and the Boson Fock space $\Gamma(H_{\mathbb{C}})$ over $H_{\mathbb{C}}$. The particular case of triplet (1.1) with $K = H_{\mathbb{C}}$ has been studied by several authors with many applications (e.g. [7], [20]) and then $\mathcal{G}(H_{\mathbb{C}})$ and $\mathcal{G}(H_{\mathbb{C}})^*$ are called the spaces of regular test white noise functions and regular generalized white noise functions, respectively.

Gross [6] and Piech [19] initiated the study of the infinite dimensional Laplacians (the Gross Laplacian Δ_G and the number operator N, resp.), as infinite dimensional analogue of a finite dimensional Laplacian, in connection with the Cauchy problems in infinite

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dimensional abstract Wiener space. In the white noise analysis, Kuo [15] reformulated the Gross Laplacian Δ_G and N as continuous linear operators acting on the test white noise function space (E). In [9], Hida, Kuo and Obata proved that the adjoint group of the Fourier-Mehler group is a differentiable one-parameter transformation group with the infinitesimal generator $(i/2)\Delta_G + iN$. In the series papers [1]-[4], the existence and uniqueness of (explicit) solutions of Cauchy problems associated with the Gross Laplacian, the number operator and certain integral kernel operators have been investigated by using the theory of one-parameter transformation groups. In [3], a Lie group $\mathcal{H}$ associated with the five dimensional complex Lie algebra $\mathfrak{h} = \langle I, p_\zeta, q_\zeta, N, \Delta_G\rangle$ has been explicitly constructed, where p_ζ and q_ζ are defined by $p_\zeta = (1/2)(D_\zeta - D_\zeta^*)$ and $q_\zeta = i(D_\zeta + D_\zeta^*)$, $\zeta \in E_{\mathbb{C}}$. Here D_ζ and D_ζ^* are annihilation and creation operators, respectively. Recently, in [12], the integral kernel operators based on triplet (1.1) have been studied and then every integral kernel operator can be extended to $\mathcal{G}(K)^*$ as a continuous linear operator. Also, one-parameter transformation groups and Cauchy problems associated with certain integral kernel operator have been studied, and their application to finite dimensional partial differential equations has been discussed.

In this paper, we study one-parameter transformation groups and Cauchy problems associated with elements in the five dimensional complex Lie algebra $\mathfrak{h}$ based on triplet (1.1), and their application to finite dimensional partial differential equations will be discussed.

This paper is organized as follows. In Section 2, we briefly recall the spaces of white noise functions and integral kernel operators. In Section 3, we study one-parameter transformation groups with the infinitesimal generator $a_1 I + a_2 D_\zeta + a_3\Delta_G + a_4 N + a_5 D_\zeta^*$ for each $(a_1, \cdots, a_5) \in \mathbb{C}^5$. In Section 4, we investigate the unique solution of Cauchy problem:

$$\frac{du}{d\theta} = \Xi u, \qquad u(0) = \phi \in \mathcal{G}(K)^*, \qquad -\delta < \theta < \delta,$$

where $\delta > 0$ (depends on ϕ) and $\Xi = a_1 I + a_2 D_\zeta + a_3\Delta_G + a_4 N + a_5 D_\zeta^*$, $(a_1, \cdots, a_5) \in \mathbb{C}^5$. In Section 5, as an application, we discuss the finite dimensional partial differential equation:

$$\frac{\partial u(\theta, x)}{\partial\theta} = \mathbf{D}u(\theta, x), \qquad u(0, x) = \phi(x), \quad -\delta < \theta < \delta,$$

where ϕ is a certain distribution function on $\mathbb{R}^n$ and for each $(a_1, \cdots, a_5) \in \mathbb{C}^5$

$$\mathbf{D} = \sum_{j=1}^n \left(a_1 \frac{\partial^2}{\partial x_j^2} + \frac{a_1 + 4a_3}{2} x_j \frac{\partial}{\partial x_j} + a_2 \frac{\partial}{\partial x_j} + a_3 x_j^2 + a_4 x_j + a_5 \right).$$

2 White noise functions and integral kernel operators

Let T be a topological space with a Borel measure $\nu(dt) = dt$ and $H = L^2(T, \nu)$ be a real Hilbert space with norm $|\cdot|_0$. We assume that H is separable. From H and a positive selfadjoint operator A on H with $\|A^{-1}\|_{OP} < 1$ and $\|A^{-1}\|_{HS} < \infty$, a Gel'fand triple $E \subset H \subset E^*$ is constructed in the standard manner (see [10], [17], [18]). Then E is a nuclear space equipped with the Hilbertian norms $|\xi|_p = |A^p\xi|_0$, $p \in \mathbb{R}$. As usual we further assume that hypothesis (H1)–(H3) for having delta function δ_t in E^*, see [18]. The canonical bilinear form on $E^* \times E$ is denoted by $\langle \cdot, \cdot \rangle$. Let (E^*, μ) be the standard Gaussian space, where μ is the Gaussian measure whose characteristic function is given by

$$\int_{E^*} e^{i\langle x,\xi\rangle} \mu(dx) = \exp\left(-\frac{1}{2}|\xi|_0^2\right), \qquad \xi \in E.$$

Let $(L^2) \equiv L^2(E^*, \mu; \mathbb{C})$ be the Hilbert space of $\mathbb{C}$-valued square integrable functions on E^* with norm $\|\cdot\|_0$. Let $:x^{\otimes n}:$ be the Wick ordering of $x^{\otimes n}$, and $H_{\mathbb{C}}^{\widehat{\otimes} n}$ be the n-fold symmetric tensor product of the complexification of H and $|\cdot|_0$ denote the $H_{\mathbb{C}}^{\otimes n}$-norm for any n. Then by the Wiener-Itô decomposition theorem, each $\phi \in (L^2)$ admits the following expression:

$$\phi(x) = \sum_{n=0}^{\infty} \langle :x^{\otimes n}:, f_n \rangle, \quad x \in E^* \quad \text{with} \quad \|\phi\|_0 = \left(\sum_{n=0}^{\infty} n! |f_n|_0^2\right)^{1/2}, \quad f_n \in H_{\mathbb{C}}^{\widehat{\otimes} n}.$$

In this case, for the simple notation, we write $\phi \sim (f_n) \in (L^2)$. For each $\xi \in H_{\mathbb{C}}$ the element $\phi_\xi \sim (\xi^{\otimes n}/n!) \in (L^2)$ is called an *exponential vector.*

For each $p \geq 0$ we put

$$(E_p) = \left\{\phi \sim (f_n) \in (L^2); \| \phi \|_p^2 \equiv \sum_{n=0}^{\infty} n! \mid f_n \mid_p^2 < \infty\right\}$$

and let (E) be the projective limit of $\{(E_p); p \geq 0\}$. By a standard argument we see that the topological dual space $(E)^*$ with respect to (L^2) of (E) is isomorphic to the inductive limit of $\{(E_{-p}); p \geq 0\}$, where (E_{-p}) is the topological dual space with respect to (L^2) of (E_p). Then we obtain a Gel'fand triple:

$$(E) \subset (L^2) \subset (E)^*.$$

This Gel'fand triple is called the *Hida-Kubo-Takenaka space.* In this context elements of (E) and $(E)^*$ are called a *test white noise function* and a *generalized white noise function,* respectively. The canonical bilinear form on $(E)^* \times (E)$ is denoted by $\langle\langle \cdot, \cdot \rangle\rangle$.

Let K be another complex Hilbert space with norm $|\cdot|$ such that the following inclusions:

$$E_{\mathbb{C}} \subset K \subset H_{\mathbb{C}} \subset K^* \subset E_{\mathbb{C}}^*$$

are continuous, where K and K^* are dual each other with respect to $H_{\mathbb{C}}$. We assume that the imbedding $K \hookrightarrow H_{\mathbb{C}}$ is a contraction: $|\xi|_0 \leq |\xi|$, $\xi \in K$. We denote by (K) the subspace of (L^2) which is isomorphic to the Boson Fock space $\Gamma(K)$ over K.

For each $p \in \mathbb{R}$, we put

$$\mathcal{G}_p = \mathcal{G}_p(K) = \left\{\phi \sim (f_n) \in (K); \|\phi\|_p^2 = \sum_{n=0}^{\infty} n! e^{2pn} |f_n|^2 < \infty\right\}$$

and let $\mathcal{G}(K)$ be the projective limit of $\{\mathcal{G}_p; p \geq 0\}$. Let $\mathcal{G}(K)^*$ be the topological dual space with respect to (K) of $\mathcal{G}(K)$. Then $\mathcal{G}(K)^*$ is isomorphic to the inductive limit of

$\{\mathcal{G}_{-p} : p \geq 0\}$, where $\mathcal{G}_{-p}$ is the topological dual space with respect to (K) of $\mathcal{G}_p$. Then we have a triplet:

$$\mathcal{G}(K) \subset (K) \subset \mathcal{G}(K)^\star.$$

and natural inclusions:

$$(E) \subset \mathcal{G}(K) \subset (K) \subset \mathcal{G}(K)^\star \subset (E)^*. \tag{2.1}$$

In the following, we write $\mathcal{G} = \mathcal{G}(K)$ and $\mathcal{G}^\star = \mathcal{G}(K)^\star$ for the simple notation.

Let $\mathcal{L}(\mathfrak{X}, \mathfrak{Y})$ denote the space of all continuous linear operators from a locally convex space $\mathfrak{X}$ into another locally convex space $\mathfrak{Y}$. For the notational convenience, we write $\mathcal{L}(\mathfrak{X}) \equiv \mathcal{L}(\mathfrak{X}, \mathfrak{X})$. From (2.1), we have the following natural inclusions:

$$\mathcal{L}(\mathcal{G}) \subset \mathcal{L}(\mathcal{G}, \mathcal{G}^\star) \subset \mathcal{L}((E), (E)^*).$$

Let l, m be non-negative integers. Then for each $\kappa \in \mathcal{L}(E_{\mathbb{C}}^{\otimes m}, (E_{\mathbb{C}}^{\otimes l})^*)$, the *integral kernel operator* $\Xi_{l,m}(\kappa) \in \mathcal{L}((E), (E)^*)$ with kernel distribution κ (see [11], [18]) is defined by

$$\Xi_{l,m}(\kappa)\phi \sim \left(\frac{(n+m)!}{n!} \left(\kappa \otimes I^{\otimes n} f_{n+m} \right)_{\mathrm{sym}} \right), \qquad \phi \sim (f_n) \in (E),$$

where $(f)_{\mathrm{sym}}$ is the symmetrization of $f \in (E_{\mathbb{C}}^{\otimes l})^*$. In fact, $\Xi_{l,m}(\kappa)$ belongs to $\mathcal{L}((E))$ if and only if κ belongs to $\mathcal{L}(E_{\mathbb{C}}^{\otimes m}, E_{\mathbb{C}}^{\otimes l})$.

Theorem 2.1 [12] *Let $\kappa_{l,m} \in \mathcal{L}(E_{\mathbb{C}}^{\otimes m}, (E_{\mathbb{C}}^{\otimes l})^*)$. Then the followings hold:*

(i) *if $\kappa_{l,m} \in \mathcal{L}(K^{\otimes m}, K^{\otimes l})$, then $\Xi_{l,m}(\kappa_{l,m}) \in \mathcal{L}(\mathcal{G})$. In this case, for any $p \in \mathbb{R}$ and $q > 0$, we obtain that for any $\phi \in \mathcal{G}$*

$$\|\Xi_{l,m}(\kappa_{l,m})\phi\|_p \leq C e^{pl-(p+q)m+q/2} \left(l^l m^m D_q^{(l+m)} \right)^{1/2} \|\phi\|_{p+q},$$

where $C \geq 0$ with $|\kappa_{l,m} f| \leq C|f|$ for any $f \in H_{\mathbb{C}}^{\otimes m}$ and $D_q = e^{q/2}/(eq)$.

(ii) *if $\Xi_{l,m}(\kappa_{l,m}) \in \mathcal{L}(\mathcal{G}, \mathcal{G}^\star)$, then $\kappa_{l,m} \in \mathcal{L}(K^{\otimes m}, K^{\otimes l})$.*

Theorem 2.2 [12] *Let $\kappa_{l,m} \in \mathcal{L}(K^{\otimes m}, K^{\otimes l})$. Then $\Xi_{l,m}(\kappa_{l,m}) \in \mathcal{L}(\mathcal{G})$ has a unique extension to a continuous linear operator $\overline{\Xi}_{l,m}(\kappa_{l,m})$ from $\mathcal{G}^\star$ into itself. We denote $\overline{\Xi}_{l,m}(\kappa_{l,m})$ by the same notation $\Xi_{l,m}(\kappa_{l,m})$.*

Example 2.3 Let $\kappa \in (K^{\otimes m})^*$ (with respect to $H_{\mathbb{C}}^{\otimes m}$). Then by Theorem 2.2, $\Xi_{0,m}(\kappa) \in \mathcal{L}(\mathcal{G})$ can be extended to a continuous linear operator from $\mathcal{G}^\star$ into itself. Moreover, for any $\Phi \sim (F_n) \in \mathcal{G}^\star$,

$$\Xi_{0,m}(\kappa)\Phi \sim \left(\frac{(n+m)!}{n!} (\kappa \otimes I^{\otimes n} F_{n+m}) \right) \in \mathcal{G}^\star.$$

From now on, we use $\kappa\widehat{\otimes}_m F_{n+m}$ instead of $\kappa \otimes I^{\otimes n} F_{n+m}$. Also, we write $D_\zeta = \Xi_{0,1}(\zeta)$ and $D_\zeta^* = \Xi_{1,0}(\zeta)$, $\zeta \in K$. If the imbedding $K \hookrightarrow H_{\mathbb{C}}$ be the Hilbert-Schmidt operator, then the trace $\tau \in (K^{\otimes 2})^*$ is defined by

$$\langle \tau, \xi \otimes \eta \rangle = \langle \xi, \eta \rangle, \qquad \xi, \eta \in K.$$

Hence, by Theorem 2.2, the *Gross Laplacian* $\Delta_{\mathrm{G}} = \Xi_{0,2}(\tau) \in \mathcal{L}(\mathcal{G})$ can be extended to a continuous linear operator from $\mathcal{G}^\star$ into itself.

Example 2.4 For each $B \in \mathcal{L}(K)$, the *differential second quantization operator* $d\Gamma(B)$ of B is defined by $d\Gamma(B) = \Xi_{1,1}(B)$ and hence $d\Gamma(B)$ can be extended to a continuous linear operator from $\mathcal{G}^\star$ into itself. Moreover, we have

$$d\Gamma(B)\Phi \sim \left((n+1)(B \otimes I^{\otimes n} F_{n+1})_{\mathrm{sym}}\right), \qquad \phi \sim (F_n) \in \mathcal{G}^\star.$$

The *number operator* N is defined by $N = d\Gamma(I)$.

3 Transformation groups

Lemma 3.1 *Let $\kappa \in (K^{\otimes 2})^*$, $B \in \mathcal{L}(K)$ and $\zeta, \eta \in K$. Then for any $p \in \mathbb{R}$ and $\phi \sim (f_n) \in \mathcal{G}$ the series*

$$M \equiv \sum_{n=0}^{\infty} n! e^{2pn} \left| \sum_{l+k_3=n} \sum_{k_1,k_2=0}^{\infty} \frac{(l+k_1+2k_2)!}{l!k_1!k_2!k_3!} \zeta^{\otimes k_3} \widehat{\otimes} \left(B^{\otimes l}\left(\kappa^{\otimes k_2} \widehat{\otimes}_{2k_2} \left(\eta^{\otimes k_1} \widehat{\otimes}_{k_1} f_{l+k_1+2k_2}\right)\right)\right)\right|^2$$

converges. In this case, for any $p \in \mathbb{R}$ there exist $q \geq 0$ and $\epsilon > 0$ with $4\|\kappa\|e^{-2(p+q)} < 1$ and $\epsilon e^{2p}|\zeta|^2 + (2e^{-q}\|B\|)^2 < 1$ such that

$$M \leq M_{1,p+q}(\eta)^2 C_{p;q}(\kappa)^2 e^{1/\epsilon} \left(\sum_{n=0}^{\infty} \left(\epsilon e^{2p}|\zeta|^2 + \left(2e^{-q}\|B\|\right)^2\right)^n\right) \|\phi\|_{p+q}^2 < \infty, \tag{3.1}$$

where $M_{1,p}(\eta)$ is given as in (3.2) and $C_{p;q}(\kappa)^2 = (1-(4\|\kappa\|e^{-2(p+q)})^2)^{-1}$ and $\|\cdot\|$ denotes the operator or functional norm.

PROOF. By the Schwartz's inequality and the fact:

$$(l+k_1+2k_2)! \leq l!k_1!\,(k_2!)^2 4^{(l+k_1+2k_2)},$$

we can prove that for some $q \geq 0$ with $4\|\kappa\|e^{-2(p+q)} < 1$

$$\sum_{k_2=0}^{\infty} \frac{(l+k_1+2k_2)!}{k_2!} \|\kappa\|^{k_2} |f_{l+k_1+2k_2}| \leq \left(2e^{-(p+q)}\right)^{(l+k_1)} \sqrt{l!k_1!}\, C_{p;q}(\kappa) \|\phi\|_{p+q}.$$

Since for any $p \in \mathbb{R}$

$$M_{1,p}(\eta) \equiv \sum_{k_1=0}^{\infty} \frac{1}{\sqrt{k_1!}} \left(2e^{-p}\|\eta\|\right)^{k_1} < \infty, \tag{3.2}$$

we have

$$\sum_{k_1,k_2=0}^{\infty} \frac{(l+k_1+2k_2)!}{k_1!k_2!} \|\kappa\|^{k_2} \|\eta\|^{k_1} |f_{l+k_1+2k_2}| \leq M_{1,p+q}(\eta) \left(2e^{-(p+q)}\right)^l \sqrt{l!}\, C_{p;q}(\kappa) \|\phi\|_{p+q}.$$

Note that

$$\left|\zeta^{\otimes k_3} \widehat{\otimes} \left(B^{\otimes l} \left(\kappa^{\otimes k_2} \widehat{\otimes}_{2k_2} \left(\eta^{\otimes k_1} \widehat{\otimes}_{k_1} f_{l+k_1+2k_2}\right)\right)\right)\right| \leq |\zeta|^{k_3} \|B\|^l \|\kappa\|^{k_2} \|\eta\|^{k_1} |f_{l+k_1+2k_2}|.$$

Therefore, for some $q \geq 0$ with $4\|\kappa\| e^{-2(p+q)} < 1$ and any $\epsilon > 0$

$$\begin{aligned} M &\leq M_{1,p+q}(\eta)^2 C_{p;q}(\kappa)^2 \sum_{n=0}^{\infty} n! e^{2pn} \left(\sum_{l+k_3=n} \frac{1}{\sqrt{l!}k_3!} |\zeta|^{k_3} \left(2e^{-(p+q)}\|B\|\right)^l \right)^2 \|\phi\|_{p+q}^2 \\ &\leq M_{1,p+q}(\eta)^2 C_{p;q}(\kappa)^2 e^{1/\epsilon} \sum_{n=0}^{\infty} \sum_{l+k_3=n} \frac{(l+k_3)!}{l!k_3!} (\epsilon e^{2p}|\zeta|^2)^{k_3} \left(2e^{-q}\|B\|\right)^{2l} \|\phi\|_{p+q}^2 \\ &= M_{1,p+q}(\eta)^2 C_{p;q}(\kappa)^2 e^{1/\epsilon} \left(\sum_{n=0}^{\infty} \left(\epsilon e^{2p}|\zeta|^2 + \left(2e^{-q}\|B\|\right)^2\right)^n \right) \|\phi\|_{p+q}^2. \end{aligned}$$

Hence, for any $p \in \mathbb{R}$ there exist $q \geq 0$ and $\epsilon > 0$ with $4\|\kappa\| e^{-2(p+q)} < 1$ and $\epsilon e^{2p}|\zeta|^2 + (2e^{-q}\|B\|)^2 < 1$ such that (3.1) holds. It follows the proof. ∎

By Lemma 3.1, we can define a transform $G_{\eta,\kappa,B,\zeta}$ acting on $\mathcal{G}$ by

$$G_{\eta,\kappa,B,\zeta}\phi \sim \left(\sum_{l+k_3=n} \sum_{k_1,k_2=0}^{\infty} \frac{(l+k_1+2k_2)!}{l!k_1!k_2!k_3!} \zeta^{\otimes k_3} \widehat{\otimes} \left(B^{\otimes l} \left(\kappa^{\otimes k_2} \widehat{\otimes}_{2k_2} \left(\eta^{\otimes k_1} \widehat{\otimes}_{k_1} f_{l+k_1+2k_2}\right)\right)\right) \right)$$

for any $\phi \sim (f_n) \in \mathcal{G}$. Then the following theorem is obvious

Theorem 3.2 *For each $\kappa \in (K^{\otimes 2})^*$, $B \in \mathcal{L}(K)$ and $\zeta, \eta \in K$, $G_{\eta,\kappa,B,\zeta}$ is a continuous linear operator from $\mathcal{G}$ into itself. Moreover, for any $p \in \mathbb{R}$ there exist $q \geq 0$ and $\epsilon > 0$ with $4\|\kappa\| e^{-2(p+q)} < 1$ and $\epsilon e^{2p}|\zeta|^2 + (2e^{-q}\|B\|)^2 < 1$ such that*

$$\|G_{\eta,\kappa,B,\zeta}\phi\|_p \leq M_{1,p+q}(\eta) C_{p;q}(\kappa) e^{1/(2\epsilon)} \left(\sum_{n=0}^{\infty} \left(\epsilon e^{2p}|\zeta|^2 + \left(2e^{-q}\|B\|\right)^2\right)^n \right)^{1/2} \|\phi\|_{p+q},$$

where $M_{1,p}(\eta)$ is given as in (3.2) and $C_{p;q}(\kappa)$ is given as in Lemma 3.1.

For each $\xi \in K$, we can easily see that

$$G_{\eta,\kappa,B,\zeta}\phi_\xi = \exp\{\langle \kappa, \xi^{\otimes 2}\rangle + \langle \eta, \xi\rangle\} \phi_{B\xi+\zeta}. \tag{3.3}$$

From now on, we assume that the imbedding $K \hookrightarrow H_{\mathbb{C}}$ is the Hilbert-Schmidt operator.

Definition 3.3 A one-parameter group $\{\Omega_\theta\}_{\theta \in \mathbb{R}} \subset \mathcal{L}(\mathcal{G})$ is called *differentiable* if there exists a $\Xi \in \mathcal{L}(\mathcal{G})$ such that for each $\phi \in \mathcal{G}$,

$$\lim_{\theta \to 0} \left\| \frac{\Omega_\theta \phi - \phi}{\theta} - \Xi\phi \right\|_p = 0 \quad \text{for all } p \geq 0.$$

Such a Ξ is called the *infinitesimal generator* of $\{\Omega_\theta\}$. Note that Ξ is unique.

Now, we consider a differentiable one-parameter group with infinitesimal generator $a_1 I + a_2 D_\zeta + a_3 \Delta_G + a_4 N + a_5 D_\zeta^*$ for arbitrary $a_1, a_2, a_3, a_4, a_5 \in \mathbb{C}$ and $\zeta \in K$ with $\langle \zeta, \zeta \rangle \neq 0$ (see [3]). For each $\mathbf{a} = (a_1, a_2, a_3, a_4, a_5) \in \mathbb{C}^5$, we define the functions α_1, α_2, α_3, α_4 and α_5 by

$$\begin{cases} \alpha_1(\theta) = \exp\{a_1\theta + \frac{a_3 {a_5}^2 \langle\zeta,\zeta\rangle}{{a_4}^2}[\frac{1}{2a_4}(e^{2a_4\theta} - 1) - \frac{2}{a_4}(e^{a_4\theta} - 1) + \theta]\} \\ \qquad\qquad \times \exp\{\frac{a_2 a_5}{a_4}\langle\zeta,\zeta\rangle[\frac{1}{a_4}(e^{a_4\theta} - 1) - \theta]\}, \\ \alpha_2(\theta) = \frac{2a_3 a_5}{a_4}[\frac{1}{2a_4}(e^{2a_4\theta} - 1) - \frac{1}{a_4}(e^{a_4\theta} - 1)] + \frac{a_2}{a_4}(e^{a_4\theta} - 1), \\ \alpha_3(\theta) = \frac{a_3}{2a_4}(e^{2a_4\theta} - 1), \\ \alpha_4(\theta) = e^{a_4\theta}, \\ \alpha_5(\theta) = \frac{a_5}{a_4}(e^{a_4\theta} - 1) \end{cases} \tag{3.4}$$

if $a_4 \neq 0$;

$$\begin{cases} \alpha_1(\theta) = \exp\{a_1\theta + (\frac{{a_3}^2 a_5}{3}\theta^3 + \frac{a_2 a_5}{2}\theta^2)\langle\zeta,\zeta\rangle\}, \\ \alpha_2(\theta) = a_3 a_5 \theta^2 + a_2\theta, \\ \alpha_3(\theta) = a_3\theta, \\ \alpha_4(\theta) = 1 \ (\alpha_4(\theta) = e^{a_4\theta}), \\ \alpha_5(\theta) = a_5\theta \end{cases} \tag{3.5}$$

if $a_4 = 0$.

For each $\mathbf{a} = (a_1, a_2, \cdots, a_5) \in \mathbb{C}^5$, we define a family of transforms $\{\mathcal{H}_{\mathbf{a},\zeta;\theta}\}_{\theta\in\mathbb{R}}$ by

$$\mathcal{H}_{\mathbf{a},\zeta;\theta} = \alpha_1(\theta)\mathcal{I}_{\tilde{\mathbf{a}},\zeta;\theta} = \alpha_1(\theta)G_{\alpha_2(\theta)\zeta,\alpha_3(\theta)\tau,\alpha_4(\theta)I,\alpha_5(\theta)\zeta}, \qquad \theta \in \mathbb{R},$$

where $\tilde{\mathbf{a}} = (a_2, \cdots, a_5) \in \mathbb{C}^4$ and the functions $\alpha_1, \cdots, \alpha_5$ are given as in (3.4) or (3.5). Then, by direct computations using (3.3), $\{\mathcal{H}_{\mathbf{a},\zeta;\theta}\}_{\theta\in\mathbb{R}}$ is a one-parameter transformation group.

Lemma 3.4 *For each $\tilde{\mathbf{a}} = (a_2, \cdots, a_5) \in \mathbb{C}^4$ and for any $\phi \in \mathcal{G}$, we have*

$$\lim_{\theta\to 0}\left\|\frac{\mathcal{I}_{\tilde{\mathbf{a}},\zeta;\theta}\phi - \phi}{\theta} - (a_2 D_\zeta + a_3\Delta_G + a_4 N + a_5 D_\zeta^*)\phi\right\|_p = 0, \qquad p \in \mathbb{R}.$$

PROOF. Let $p \in \mathbb{R}$ and $\phi \sim (f_n) \in \mathcal{G}$ be given. Then by definition of $\mathcal{I}_{\tilde{\mathbf{a}},\zeta;\theta}$, we have

$$\begin{aligned} &\frac{\mathcal{I}_{\tilde{\mathbf{a}},\zeta;\theta}\phi - \phi}{\theta} - (a_2 D_\zeta + a_3\Delta_G + a_4 N + a_5 D_\zeta^*)\phi \\ &\quad \sim \left(\left[\frac{\alpha_4^n(\theta) - 1}{\theta} - a_4 n\right] f_n\right) \\ &\qquad + \left(\left[\alpha_4^n(\theta)\frac{\alpha_5(\theta)}{\theta} - a_5\right]\zeta\widehat{\otimes} f_n\right) \\ &\qquad + \left(\frac{(n+1)!}{n!}\left[\alpha_4^n(\theta)\frac{\alpha_2(\theta)}{\theta} - a_2\right]\zeta\widehat{\otimes}_1 f_{n+1}\right) \\ &\qquad + \left(\frac{(n+2)!}{n!}\left[\alpha_4^n(\theta)\frac{\alpha_3(\theta)}{\theta} - a_3\right]\tau\widehat{\otimes}_2 f_{n+2}\right) \\ &\qquad + \left(\frac{1}{\theta}g_n\right), \end{aligned}$$

where

$$g_n = \sum_{l+k_3=n} \sum_{k_1+k_2+k_3\geq 2} \frac{(l+k_1+2k_2)!}{l!k_1!k_2!k_3!} \alpha_5^{k_3}(\theta)\alpha_4^{l}(\theta)\alpha_3^{k_2}(\theta)\alpha_2^{k_1}(\theta) \times \zeta^{\otimes k_3} \widehat{\otimes} \left(\tau^{\otimes k_2} \widehat{\otimes}_{2k_2} \left(\zeta^{\otimes k_1} \widehat{\otimes}_{k_1} f_{l+k_1+2k_2}\right)\right).$$

Therefore, we obtain that

$$\left\| \frac{\mathcal{I}_{\widetilde{a},\zeta;\theta}\phi - \phi}{\theta} - (a_2 D_\zeta + a_3 \Delta_G + a_4 N + a_5 D_\zeta^*)\phi \right\|_p^2 \leq 5 \sum_{j=1}^{5} I_j(\theta),$$

where

$$I_1(\theta) = \sum_{n=0}^{\infty} n! e^{2pn} \left| \left[\frac{\alpha_4^n(\theta) - 1}{\theta} - a_4 n \right] f_n \right|^2,$$

$$I_2(\theta) = \sum_{n=0}^{\infty} n! e^{2pn} \left| \frac{(n+1)!}{n!} \left[\alpha_4^n(\theta) \frac{\alpha_2(\theta)}{\theta} - a_2 \right] \zeta \widehat{\otimes}_1 f_{n+1} \right|^2,$$

$$I_3(\theta) = \sum_{n=0}^{\infty} n! e^{2pn} \left| \frac{(n+2)!}{n!} \left[\alpha_4^n(\theta) \frac{\alpha_3(\theta)}{\theta} - a_3 \right] \tau \widehat{\otimes}_2 f_{n+2} \right|^2,$$

$$I_5(\theta) = \sum_{n=0}^{\infty} (n+1)! e^{2p(n+1)} \left| \left[\alpha_4^n(\theta) \frac{\alpha_5(\theta)}{\theta} - a_5 \right] \zeta \widehat{\otimes} f_n \right|^2$$

and

$$I_4(\theta) = \sum_{n=0}^{\infty} n! e^{2pn} \left| \frac{1}{\theta} g_n \right|^2.$$

The proof of the case $a_4 = 0$ is similar to the proof of the case $a_4 \neq 0$. We now prove only the case $a_4 \neq 0$. Then by direct computation, we obtain that for any $\epsilon > 0$ there exists $C_{1,\epsilon} \geq 0$ such that

$$\left| \frac{\alpha_4^n(\theta) - 1}{\theta} - a_4 n \right|^2 \leq |\theta|^2 C_{1,\epsilon}^2 e^{2|a_4|(\epsilon+|\theta|)n}, \qquad n \geq 0.$$

Therefore, we have

$$I_1(\theta) \leq |\theta|^2 C_{1,\epsilon}^2 \|\phi\|_{p+|a_4|(\epsilon+|\theta|)}^2. \tag{3.6}$$

Also, for any $\epsilon > 0$ there exists $C_\epsilon \geq 0$ such that

$$\left| \alpha_4^n(\theta) \frac{\alpha_i(\theta)}{\theta} - a_i \right|^2 \leq \left(\left| \frac{\alpha_i(\theta)}{\theta} - a_i \right| + |a_i| C_\epsilon |\theta| \right)^2 e^{2|a_4|(\epsilon+|\theta|)n}, \qquad n \geq 0, \quad i = 2, 3, 5.$$

Note that for any $\epsilon > 0$ there exists $C_{2,\epsilon} \geq 0$ such that $(n+1)(n+2) \leq C_{2,\epsilon} e^{2\epsilon n}$ for all $n \geq 0$. Hence we prove that

$$I_i(\theta) \leq C C_{2,\epsilon} \left(\left| \frac{\alpha_i(\theta)}{\theta} - a_i \right| + |a_i| C_\epsilon |\theta| \right)^2 \|\phi\|_{p+\epsilon+|a_4|(\epsilon+|\theta|)}^2, \qquad i = 2, 3, 5, \tag{3.7}$$

where $C = \max\{e^{2p}|\zeta|^2, e^{-2p}\|\zeta\|^2, e^{-4p}\|\tau\|^2\}$. By similar arguments in the proof of Lemma 3.1, we can prove that for any $p \in \mathbb{R}$ there exist $q \geq 0$ and $\epsilon > 0$ with $4\|\alpha_3(\theta)\tau\|e^{-2(p+q)} < 1$ and $\epsilon e^{2p}|\alpha_5(\theta)\zeta|^2 + (2e^{-q}|\alpha_4(\theta)|)^2 < 1$ such that

$$\begin{aligned} I_4(\theta) &\leq C(\theta)M_{1,p+q}(\alpha_2(\theta)\zeta)^2 C_{p;q}^2(\alpha_3(\theta)\tau)e^{1/\epsilon} \\ &\quad \times \left(\sum_{n=0}^{\infty}\left(\epsilon e^{2p}|\alpha_5(\theta)\zeta|^2 + \left(2e^{-q}|\alpha_4(\theta)|\right)^2\right)^n\right)\|\!|\phi|\!\|_{p+q}^2, \end{aligned} \tag{3.8}$$

where $M_1(\alpha_2(\theta)\zeta)$ is given as in (3.2) and $C_{p;q}(\alpha_3(\theta)\tau)$ is given as in Lemma 3.1, and

$$C(\theta) = 3\left(\left|\frac{1}{\theta}\alpha_5(\theta)\alpha_3(\theta)|\zeta|\|\tau\|\right|^2 + \left|\frac{1}{\theta}\alpha_5(\theta)\alpha_2(\theta)|\zeta|\|\zeta\|\right|^2 + \left|\frac{1}{\theta}\alpha_3(\theta)\alpha_2(\theta)\|\tau\|\|\zeta\|\right|^2\right).$$

Therefore, by (3.6), (3.7) and (3.8), we prove that $\lim_{\theta\to 0}(I_1(\theta) + I_2(\theta) + I_3(\theta) + I_4(\theta) + I_5(\theta)) = 0$. It follows the proof. ∎

Theorem 3.5 *$\{\mathcal{H}_{\mathbf{a},\zeta;\theta}\}_{\theta\in\mathbb{R}}$ is a differentiable one-parameter transformation group with the infinitesimal generator $a_1I + a_2D_\zeta + a_3\Delta_G + a_4N + a_5D_\zeta^*$.*

PROOF. Let $p \in \mathbb{R}$ and $\phi \in \mathcal{G}$ be given. Then we have

$$\begin{aligned} &\left\|\frac{\mathcal{H}_{\mathbf{a},\zeta;\theta}\phi - \phi}{\theta} - (a_1I + a_2D_\zeta + a_3\Delta_G + a_4N + a_5D_\zeta^*)\phi\right\|_p \\ &\leq \left|\frac{\alpha_1(\theta) - 1}{\theta} - a_1\right| \left\|\mathcal{I}_{\widetilde{\mathbf{a}},\zeta;\theta}\phi\right\|_p + \left\|a_1\left(\mathcal{I}_{\widetilde{\mathbf{a}},\zeta;\theta} - I\right)\phi\right\|_p \\ &\quad + \left\|\frac{(\mathcal{I}_{\widetilde{\mathbf{a}},\zeta;\theta} - I)\phi}{\theta} - (a_2D_\zeta + a_3\Delta_G + a_4N + a_5D_\zeta^*)\phi\right\|_p. \end{aligned}$$

By Lemma 3.4, we complete the proof. ∎

4 Cauchy problems

In this section, we consider the Cauchy problem of the following type:

$$\frac{du}{d\theta} = \Xi u, \qquad u(0) = \phi \in \mathcal{G}^*, \qquad -\delta < \theta < \delta, \tag{4.1}$$

where $\delta > 0$ (depend on ϕ) and $\Xi = a_1I + a_2D_\zeta + a_3\Delta_G + a_4N + a_5D_\zeta^*$, $\mathbf{a} = (a_1, \cdots, a_5) \in \mathbb{C}^5$. Let $\mathbf{I}$ be a given bounded open interval containing 0. For each $\mathbf{a} = (a_1, \cdots, a_5) \in \mathbb{C}^5$ and $p \geq 0$, we put

$$S_p = \{\theta \in \mathbb{R}; 4|\alpha_3(\theta)|\|\tau\|e^{2p} < 1\} \cap \mathbf{I}, \qquad M_p = \sup\{2e^{|a_4||\theta|}; \theta \in S_p\},$$

where $\alpha_3(\theta)$ is given as in (3.4) or (3.5). Obviously, S_p is a bounded open subset of $\mathbb{R}$ for any $p \geq 0$ and there exists an open ball $B_\epsilon(0) = (-\epsilon, \epsilon)$ such that $B_\epsilon(0) \subset S_p$. For any $q \geq p$, $S_q \subset S_p$ and $0 < M_q \leq M_p$.

Proposition 4.1 *Let $p \geq 0$ be given. Then for any $\theta \in S_p$ and $q > \log(M_p)$, $\mathcal{H}_{\mathbf{a},\zeta;\theta} \in \mathcal{L}(\mathcal{G}_{-p}, \mathcal{G}_{-(p+q)})$. Moreover, for any $\phi \in \mathcal{G}_{-p}$ there exists $\epsilon > 0$ with $\epsilon e^{-2(p+q)}|\alpha_5(\theta)\zeta|^2 + (2e^{-q}|\alpha_4(\theta)|)^2 < 1$ such that*

$$\|\mathcal{H}_{\mathbf{a},\zeta;\theta}\phi\|_{-(p+q)} \leq |\alpha_1(\theta)| M_{1,-p}(\alpha_2(\theta)\zeta) C_{-p;0}(\alpha_3(\theta)\tau) e^{1/(2\epsilon)} \times \left(\sum_{n=0}^{\infty} \left(\epsilon e^{-2(p+q)} |\alpha_5(\theta)\zeta|^2 + (2e^{-q}|\alpha_4(\theta)|)^2 \right)^n \right)^{1/2} \|\phi\|_{-p},$$

where $M_{1,-p}(\alpha_2(\theta)\zeta)$ is given as in (3.2) and $C_{-p;0}(\alpha_3(\theta)\tau)$ is given as in Lemma 3.1.

PROOF. Since $4\|\alpha_3(\theta)\tau\| e^{2p} < 1$ and $(2e^{-q}\|e^{a_4\theta}\|)^2 < 1$ for any $\theta \in S_p$ and $q > \log(M_p)$, there exists $\epsilon > 0$ such that $\epsilon e^{-2(p+q)}|\alpha_5(\theta)\zeta|^2 + (2e^{-q}|\alpha_4(\theta)|)^2 < 1$. Therefore, by the similar arguments in the proof of Lemma 3.1, we can complete the proof. ∎

Let $p \geq 0$ be given. By Proposition 4.1, for any $q > \log(M_p)$ and $\theta_1 \in S_p$, $\theta_2 \in S_{p+q}$ we have

$$\mathcal{H}_{\mathbf{a},\zeta;\theta_2}\mathcal{H}_{\mathbf{a},\zeta;\theta_1} \in \mathcal{L}(\mathcal{G}_{-p}, \mathcal{G}_{-(p+2q)}).$$

Therefore, for any $\theta \in S_p$ and $h \in S_{p+q}$ such that $\theta + h \in S_p$, $\mathcal{H}_{\mathbf{a},\zeta;\theta+h}$ coincide with $\mathcal{H}_{\mathbf{a},\zeta;h}\mathcal{H}_{\mathbf{a},\zeta;\theta}$ as a operator in $\mathcal{L}(\mathcal{G}_{-p}, \mathcal{G}_{-(p+2q)})$. Hence, by using the similar arguments in the proof of Theorem 3.5, we can prove that for each $\phi \in \mathcal{G}_{-p}$ and $\theta \in S_p$,

$$\lim_{h \to 0} \left\| \frac{\mathcal{H}_{\mathbf{a},\zeta;\theta+h}\phi - \mathcal{H}_{\mathbf{a},\zeta;\theta}\phi}{h} - \Xi \mathcal{H}_{\mathbf{a},\zeta;\theta}\phi \right\|_{-(p+2q)} = 0,$$

where $\Xi = a_1 I + a_2 D_\zeta + a_3 \Delta_G + a_4 N + a_5 D_\zeta^*$. It follows that $u(\theta) = \mathcal{H}_{\mathbf{a},\zeta;\theta}\phi \in \mathcal{G}_{-(p+q)}$ satisfies the initial-value problem:

$$\frac{du}{d\theta} = (a_1 I + a_2 D_\zeta + a_3 \Delta_G + a_4 N + a_5 D_\zeta^*) u, \quad u(0) = \phi \in \mathcal{G}_{-p}, \qquad \theta \in S_p. \tag{4.2}$$

Now, we consider the uniqueness of solution of (4.2). Suppose that $v(\theta) \in \mathcal{G}_{-(p+q)}$, $\theta \in S_p$ is another solution of (4.2) satisfying that

$$\lim_{h \to 0} \left\| \frac{v(\theta + h) - v(\theta)}{h} - \Xi v(\theta) \right\|_{-(p+2q)} = 0, \qquad \theta \in S_p, \tag{4.3}$$

where $\Xi = a_1 I + a_2 D_\zeta + a_3 \Delta_G + a_4 N + a_5 D_\zeta^*$. Take $\delta > 0$ such that $B_\delta(0) \subset S_{p+2q}$, where $q > \log(M_p)$. Then for any $\theta, \epsilon \in B_{\delta/2}(0)$, $\theta - \epsilon \in B_\delta(0) \subset S_p \cap S_{p+q}$. Therefore, by Proposition 4.1, for each given $\theta \in B_{\delta/2}(0)$ we can define a $\mathcal{G}_{-(p+2q)}$-valued function on $B_{\delta/2}(0)$ by $w(\epsilon) = \mathcal{H}_{\mathbf{a},\zeta;\theta-\epsilon} v(\epsilon)$. Moreover, by the similar arguments in the proof of Theorem 3.5, we obtain that $\frac{d}{d\epsilon} w(\epsilon) = 0$. This implies that $v(\theta) = \mathcal{H}_{\mathbf{a},\zeta;\theta}\phi$ for any $\theta \in B_{\delta/2}(0)$.

Thus the following theorem is obvious

Theorem 4.2 *Let $p \geq 0$ and $\phi \in \mathcal{G}_{-p}$ be given. Then $u(\theta) = \mathcal{H}_{\mathbf{a},\zeta;\theta}\phi \in \mathcal{G}_{-(p+q)}$ defined on S_p satisfies the initial-value problem (4.2), where $q > \log(M_p)$, Moreover, if $v(\theta) \in \mathcal{G}_{-(p+q)}$, $\theta \in S_p$ is another solution of (4.2) satisfying (4.3), then there exists $\delta > 0$ such that v is equal to u on $B_\delta(0)$.*

5 Finite dimensional partial differential equations

In this section, as an application, we discuss finite dimensional partial differential equations. For the details in finite dimensional white noise calculus, we refer [13] and Section 6.2 in [18].

Let $T = \{1, 2, \cdots, n\}$ be a finite set with discrete topology and counting measure ν. Then the Hilbert space $H \equiv L^2(T, \nu; \mathbb{R})$ is isomorphic to $\mathbb{R}^n$. By using a symmetric matrix A with eigenvalues $1 \leq \lambda_1 \leq \cdots \leq \lambda_n$, we construct the Gel'fand triple which becomes $E = H = E^* = \mathbb{R}^n$. Let $(L^2) \equiv L^2(\mathbb{R}^n, \mu; \mathbb{C})$, where μ is the standard Gaussian measure on $\mathbb{R}^n$. By using similar method in Section2, we construct a Gel'fand triple:

$$\mathcal{G} = \mathcal{G}(\mathbb{R}^n) \subset (L^2) \subset \mathcal{G}(\mathbb{R}^n)^* = \mathcal{G}^*.$$

Since there is a natural unitary isomorphism U from $L^2(\mathbb{R}^n, \mu)$ onto $L^2(\mathbb{R}^n, dx)$ defined by

$$U\phi(x) = \left(\frac{1}{\sqrt{2\pi}}\right)^{n/2} e^{-|x|^2/4}\phi(x), \qquad \phi \in L^2(\mathbb{R}^n, \mu),$$

we obtain a new Gel'fand triple:

$$\mathcal{N} \equiv U(\mathcal{G}) \subset L^2(\mathbb{R}^n, dx) \subset \mathcal{N}^* \equiv U(\mathcal{G}^*).$$

Therefore, for any $\Xi \in \mathcal{L}(\mathcal{G}, \mathcal{G}^*)$, $U\Xi U^{-1} \in \mathcal{L}(\mathcal{N}, \mathcal{N}^*)$. In particular, for the Hida's differential operator $\partial_j = \Xi_{0,1}(\delta_j)$ and its adjoint operator $\partial_j^* = -\partial_j + x_j$, we have

$$U\partial_j U^{-1} = \frac{\partial}{\partial x_j} + \frac{x_j}{2}, \qquad U\partial_j^* U^{-1} = -\frac{\partial}{\partial x_j} + \frac{x_j}{2}, \tag{5.1}$$

where δ_j is the evaluation map. Hence, the integral kernel operator $\Xi_{l,m}(\kappa)$ with kernel $\kappa \in \mathcal{L}((\mathbb{R}^n)^{\otimes m}, (\mathbb{R}^n)^{\otimes l}) \cong (\mathbb{R}^n)^{\otimes l+m}$ is translated into a partial differential operator with polynomial coefficients. Put

$$D_{l,m}(\kappa) = U\Xi_{l,m}(\kappa)U^{-1}.$$

Then we have

Proposition 5.1 *For each $l, m \geq 0$ and $\kappa \in (\mathbb{R}^n)^{\otimes l+m}$, the partial differential operator $D_{l,m}(\kappa)$ is a continuous linear operator from $\mathcal{N}$ into itself. Moreover, $D_{l,m}(\kappa)$ can be extended to a continuous linear operator from $\mathcal{N}^*$ into itself.*

The Gross Laplacian Δ_{G} and the number operator N are defined by

$$\Delta_{\mathrm{G}} = \sum_{j=1}^{n} \partial_j^2, \qquad N = \sum_{j=1}^{n} \partial_j^* \partial_j,$$

respectively. Then by direct computation with (5.1), we obtain that

$$U\Delta_{\mathrm{G}}U^{-1} = \sum_{j=1}^{n} \left(\frac{\partial^2}{\partial x_j^2} + x_j\frac{\partial}{\partial x_j} + \frac{x_j^2}{4} + \frac{1}{2}\right)$$

and

$$UNU^{-1} = \sum_{j=1}^{n} \left(-\frac{\partial^2}{\partial x_j^2} + \frac{x_j^2}{4} - \frac{1}{2} \right)$$

(see [18]). Also, we have

$$UD_{\mathbf{1}}U^{-1} = \sum_{j=1}^{n} \left(\frac{\partial}{\partial x_j} + \frac{x_j}{2} \right), \qquad UD_{\mathbf{1}}^{*}U^{-1} = \sum_{j=1}^{n} \left(-\frac{\partial}{\partial x_j} + \frac{x_j}{2} \right),$$

where $\mathbf{1} = (1, \cdots, 1) \in \mathbb{R}^n$. Therefore, for each $(a_1, \cdots, a_5) \in \mathbb{C}^5$ the operator

$$n \left(a_5 - \frac{1}{2} a_1 \right) I + \frac{a_2 + 2a_4}{2} D_{\mathbf{1}} + \frac{a_1 + 4a_3}{2} \Delta_G - \frac{a_1 - 4a_3}{2} N - \frac{a_2 - 2a_4}{2} D_{\mathbf{1}}^{*}$$

is translated into the partial differential operator:

$$\mathbf{D} = \sum_{j=1}^{n} \left(a_1 \frac{\partial^2}{\partial x_j^2} + \frac{a_1 + 4a_3}{2} x_j \frac{\partial}{\partial x_j} + a_2 \frac{\partial}{\partial x_j} + a_3 x_j^2 + a_4 x_j + a_5 \right). \tag{5.2}$$

For each $(a_1, \cdots, a_5) \in \mathbb{C}^5$, we put

$$\mathbf{a} = \left(n \left(a_5 - \frac{1}{2} a_1 \right), \frac{a_2 + 2a_4}{2}, \frac{a_1 + 4a_3}{2}, -\frac{a_1 - 4a_3}{2}, -\frac{a_2 - 2a_4}{2} \right) \in \mathbb{C}^5.$$

Then, by Theorem 4.2, we have the following theorem.

Theorem 5.2 *Let $(a_1, \cdots, a_5) \in \mathbb{C}^5$. Then for each $\phi \in \mathcal{N}^*$ the unique solution of the partial differential equation:*

$$\frac{\partial u(\theta, x)}{\partial \theta} = \mathbf{D} u(\theta, x), \qquad u(0, x) = \phi(x), \quad \theta \in B_\epsilon(0)$$

is given by $u(\theta, x) = U \mathcal{H}_{\mathbf{a},\mathbf{1};\theta} U^{-1} \phi(x) \in \mathcal{N}^*$, $\theta \in B_\epsilon(0)$, where $\mathbf{D}$ is given as in (5.2) and $\epsilon > 0$ depends on ϕ.

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