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On the q-deformed Poisson distribution

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1. Introduction

This is the joint work with N. Saitoh at Ochanomizu University. A noncommutative or quantum probability space is a unital (possibly noncommutative) algebra, $\mathcal{A}$ together with a linear functional, $\phi : \mathcal{A} \to \mathbb{C}$, such that $\phi(1) = 1$. If $\mathcal{A}$ is a C^*-algebra and ϕ is a state then we call a noncommutative probability space $(\mathcal{A}, \phi)$ a C^*-probability space. $\mathcal{A}$ corresponds to the algebra of measurable functions and hence an element in $\mathcal{A}$ is regarded as a noncommutative random variable. The distribution of $x \in \mathcal{A}$ under ϕ is determined as the linear functional on $\mathbb{C}[X]$ (the polynomials in one variable) by $\mathbb{C}[X] \ni f \longmapsto \phi(f(X)) \in \mathbb{C}$. Considered in the C^*-probability context, the distribution of a self-adjoint element in $\mathcal{A}$ can be realized as the probability measure on $\mathbb{R}$.

In recent years the question has been considered in many papers, what distribution will be obtained in a noncommutative central limit, that is in the case where we replace the classical commutative notion of independence by some other type in a noncommutative probability space. For free independence which has been introduced by Voiculescu in [Vo], the Gaussian distribution is replaced by the Wigner's semicircle distribution, which is called the free central limit theorem (see, for instance, [VDN]). Bo.zejko, Kümmerer, and Speicher introduced q-analogues of Brownian motions and Gaussian processes in [BKS], [BS1] and [BS2], which is governed by classical independence for $q = 1$ and free independence for $q = 0$. van Leeuwen and Maassen also investigated a q-deformed Gaussian distribution in [LM], which takes the semicircle distribution for $q = 0$ and recovers the Gaussian distribution for $q = 1$. Their constructions were based on $\mathcal{F}_q(\mathcal{H})$, the q-deformation of the Fock space over a Hibert space $\mathcal{H}$. They regarded the distribution of the operator $a(\xi) + a(\xi)^*$ under the vacuum vector state ϕ as the q-deformed Gaussian distribution in a noncommutative probability space $(\Gamma_q(\mathcal{H}), \phi)$, where $a(\xi)$ and $a(\xi)^*$ are the annihilation and the creation operators associated with $\xi \in \mathcal{H}$ satisfying the q-commutation relation, respectively. Furthermore it is very worth to note that this q-deformed Gaussian distribution can be associated with the q-Hermit polynomials. By virtue of this, one can see that, for $||\xi|| = 1$, the q-deformed Gaussian distribution is supported on the interval $[-2/\sqrt{1-q}, 2/\sqrt{1-q}]$ with the density $f(t) = \frac{1}{\pi}\sqrt{1-q}\sin\theta\prod_{n=1}^{\infty}(1-q^n)\left|1-e^{i2\theta}q^n\right|^2$, where $\theta \in [0,\pi]$ is such that $t = (2/\sqrt{1-q})\cos\theta$ (for more details see [BKS]). We should mention here that Nica has found in [Ni] the nice q-analogue of the cumulants generating function $R_q(z)$ which takes Voiculescu's R-transform for the free convolution in the limit $q \to 0$ and recovers the classical cumulants generating function, the logarithm of Fourier transform, if one takes the limit $q \to 1$. He has also investigated the q-deformed convolution in terms of $R_q(z)$ and the central limit theorem, in which the q-deformed Gaussian distribution appears as its limit distribution.

It is natural to regard the distribution of a sum of a free family of projections as the free analogue of the binomial distribution because a projection corresponds to the Bernoulli distribution. In [AY], they have studied its combinatorial structure and derived the sequence of orthogonal polynomials associated with the free analogue of the binomial distribution, of which three terms recurrence relation is the constant coefficients type of Cohen-Trenholme [CT]. They have also showed the free Poisson limit and the free de Moivre-Laplace by using the recurrence relation of the orthogonal polynomials for the free analogue of the binomial distribution.

Inspired by the above, we would like to introduce new q-deformed binomial and Poisson distributions based on orthogonal polynomials in this note. We first introduce a q-deformed binomial distributions by virtue of a q-deformed sequence of orthogonal polynomials, which takes the free binomial distribution in the limit $q \to 0$ and reduces to the usual binomial distribution when $q \to 1$. Furthermore, we see that it is compatible with the q-deformed Gaussian distribution if we take the limiting procedure of de Moivre-Laplace. By taking the Poisson limit in our q-deformed binomial distribution, we obtain a new q-deformed Poisson distribution which is not discrete but has the absolutely continuous part, and also define its probability measure by using the formulas for the Al-Salam – Chihara polynomials of Askey and Ismail in [AI]. It will be also discussed the representation of this q-deformed Poisson random variable on the q-Fock space.

2. A q-deformed binomial distribution

Throughout this note, we make use of the terms of q-calculus, which is over a century old. Let us just remind of some basic notations here.

We put for $n \in \mathbb{N}_0$

$$[n]_q := \frac{1-q^n}{1-q} = 1 + q + \cdots + q^{n-1}, \qquad \text{with} \quad [0]_q := 0, \tag{2.1}$$

and call the q-integer. Then we have the q-factorial

$$[n]_q! := [1]_q[2]_q \cdots [n]_q, \qquad \text{with} \quad [0]_q! := 1. \tag{2.2}$$

Another frequently used symbol is the q-shifted factorial, the q-analogue of the Pochhammer symbol,

$$(a;q)_n := \prod_{j=0}^{n-1}(1-aq^j), \quad \text{in paticular} \quad (a;q)_\infty := \prod_{j=0}^{\infty}(1-aq^j), \tag{2.3}$$

where we use the convention that $(a;q)_0 := 1$. A product of these q-shifted factorials $(a_1;q)_n(a_2;q)_n \cdots (a_r;q)_n$ is denoted as $(a_1, a_2, ..., a_r; q)_n$.

There is a good q-deformation of the exponential function defined as

$$\exp_q(x) := \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!}, \tag{2.4}$$

which satisfies the relation that

$$\prod_{n=0}^{\infty}(1-(1-q)q^n x)^{-1} = \exp_q(x). \tag{2.5}$$

Now we shall recall the basic facts on the orthogonal polynomials. Let ν be a probability measure on $\mathbb{R}$ with finite moments of all orders. Then it is well-known [Sz] that there exists two sequences of real numbers $\alpha_m \in \mathbb{R}$ and $\beta_m \geq 0$, which shall be called the Jacobi parameters, such that the sequence of the orthogonal polynomials $\{P_m(X)\}$ with respect to the measure ν can be given by the recurrence relation,

$$\begin{aligned} &P_0(X) = 1, \quad P_1(X) = X - \alpha_0, \\ &P_{m+1}(X) = (X-\alpha_m)P_m(X) - \beta_m P_{m-1}(X) \quad (m \geq 1). \end{aligned} \tag{2.6}$$

Moreover they satisfy that

$$\int_{t\in\mathbb{R}} P_k(t)P_m(t)d\nu(t) = \delta_{k,m}\beta_1\beta_2\cdots\beta_m. \tag{2.7}$$

The Jacobi parameters are determined only by the sequence of the moments of ν. Conversely, given the parameters α_m and β_m, Favard theorem ensures the existence of the probability measure for which the sequence of the polynomials determined by the above recurrence relation are orthogonal. It also can be shown that the probability measure ν is supported only in finitely many points if and only if $\beta_m = 0$ for some m, thus the sequence of polynomials is essentially finite.

Proposition 2.1. *Let $\nu_{(n,p)}$ be the probability measure for the binomial distribution $B(n,p)$, that is*

$$\nu_{(n,p)}(dt) = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} \delta_k, \tag{2.8}$$

where dt is the Lebesgue measure and δ_k denotes the Dirac unit mass at $t = k$. Then the orthogonal polynomials for ν_n is determined by the following recurrence relation:

$$\begin{aligned} &P_0(X) = 1, \quad P_1(X) = X - np, \\ &P_{m+1}(X) = \left(X - \alpha_m^{(n)}\right) P_m(X) - \beta_m^{(n)} P_{m-1}(X) \end{aligned} \tag{2.9}$$

with the Jacobi parameters

$$\alpha_m^{(n)} = np + (1-2p)m, \quad \beta_m^{(n)} = m(n-m+1)p(1-p), \tag{2.10}$$

for $m = 1, 2, ..., n$, where $\beta_m^{(n)} = 0$ for $m \geq n+1$.

This orthogonal polynomials are well-known as classical orthogonal polynomials, namely the Krawtchouk polynomials (see, for example [Ch]).

Before making a deformation on the orthogonal polynomial for the binomial distribution, we would like to recall the recurrence relation of the orthogonal polynomial for the free binomial distribution $B_{free}(n,p)$, of which three terms recurrence relation is the constant coefficients type of Cohen-Trenholme [CT]. In [AY], it has been studied the combinatorial structure of the operator,

$$x = p_1 + p_2 + \cdots + p_n, \tag{2.11}$$

for a free family of projections $\{p_i\}_{i=1}^n$ with $\phi(p_i) = \alpha$ $(i = 1, 2, ..., n)$ in a C^*-probability space $(\mathcal{A}, \phi)$ and it has been obtained the three terms recurrence relation for the free binomial distribution $B_{free}(n,p)$ that

(2.12)

$$\begin{aligned} &P_0(X) = 1, \quad P_1(X) = X - np, \\ &P_2(X) = (X - np - (1-2p))\, P_1(X) - np(1-p)P_0(X), \\ &P_{m+1}(X) = (X - np - (1-2p))\, P_m(X) - (n-1)p(1-p)P_{m-1}(X) \quad (m \geq 2). \end{aligned}$$

Having this formula in mind, we shall make deformation on the Jacob parameters for the binomial distribution in (2.10) and define a q-deformed binomial distribution.

Definition 2.2. We call the probability measure $(\nu_{(n,p)})_q$ induced from the Jacobi parameters

$$(2.13)\quad \left(\alpha_m^{(n,p)}\right)_q = np + (1-2p)[m]_q, \quad \left(\beta_m^{(n,p)}\right)_q = [m]_q\left(n - [m-1]_q\right)p(1-p),$$

the q-deformed binomial distribution and denote $B_q(n,p)$. Here we shall consider $\left(\beta_m^{(n,p)}\right)_q = 0$ unless $n - [m-1]_q > 0$.

Example 2.3. In the limit $q \to 0$, it holds that

$$(2.14)\qquad \lim_{q\to 0}\left(\alpha_m^{(n,p)}\right)_q = np + (1-2p) \quad \text{for } m \geq 1,$$

$$(2.15)\qquad \lim_{q\to 0}\left(\beta_m^{(n,p)}\right)_q = \begin{cases} (n-1)p(1-p) & \text{if } m \geq 2, \\ np(1-p) & \text{if } m = 1. \end{cases}$$

Hence this q-deformed binomial distribution takes the free binomial distribution when $q = 0$. The probability measure $(\nu_{(n,p)})_0$ can be given as follows (see Section 2 in [AY]):

$$(2.16)\qquad (\nu_n)_0 = \frac{-n\sqrt{-(t-\gamma_-)(t-\gamma_+)}}{2\pi t(t-n)}\chi_{[\gamma_-,\gamma_+]}dt + \max(0, 1-np)\delta_0 + \max(0, 1-n(1-p))\delta_n,$$

where $\gamma_\pm = \left(\sqrt{(n-1)p} \pm \sqrt{1-p}\right)^2$ and χ_I means the characteristic function on the interval I. It is clear that one can recover the usual binomial distribution when $q = 1$.

It can be also performed the de Moivre-Laplace limiting procedure in our q-deformed binomials distribution (see, for details, [SY1]) .

3. A q-deformed Poisson distribution

In this section, we define a q-deformed Poisson distribution based on the orthogonal polynomials for $B_q(n,p)$. We shall give it by taking the Poisson limits in the Jacobi parameters (2.14) and (2.15) that is $n \to \infty$ and $p \to 0$ but $np = \lambda > 0$ remains finite, hence,

$$(3.1)\qquad \left(\alpha_m^{(n,p)}\right)_q = np + (1-2p)[m]_q \longrightarrow \lambda + [m]_q,$$

$$(3.2)\qquad \left(\beta_m^{(n,p)}\right)_q = [m]_q\left(n - [m-1]_q\right)p(1-p) \longrightarrow \lambda[m]_q.$$

Definition 3.1. We call the induced probability measure μ_q from the sequence of polynomials

(3.3)
$$P_0(X) = 1, \quad P_1(X) = X - \lambda,$$
$$P_{m+1}(X) = \left(X - (\lambda + [m]_q)\right)P_m(X) - \lambda[m]_q P_{m-1}(X) \quad (m \geq 1),$$

the q-deformed Poisson distribution of the parameter λ.

We can determine the probability measure of this q-deformed Poisson distribution by using the Al-Salam – Chihara polynomials. Al-Salam and Chihara in [AC] defined the orthogonal polynomials $P_m(X) \equiv P_m(X;q;a,b,c)$ which satisfy the three terms recurrence relation

(3.4)
$$P_0(X) = 1, \quad P_1(X) = X - a,$$
$$P_{m+1}(X) = (X - aq^m)\,P_m(X) - (c - bq^{m-1})(1-q^m)P_{m-1}(X) \quad (m \geq 1),$$

as the result for the characterization problem of some convolution formulas.

If we put $Q_m(X) = P_m(X + (\lambda + \frac{1}{1-q}))$ in (3.3) then we have the relation,

$$\begin{aligned}Q_0(X) &= 1, \quad Q_1(X) = X - \left(\frac{-1}{1-q}\right), \\ Q_{m+1}(X) &= \left(X - \left(\frac{-1}{1-q}\right)q^m\right) Q_m(X) \\ &\qquad - \frac{\lambda}{1-q}(1-q^n)Q_{m-1}(X) \quad (m \geq 1),\end{aligned} \tag{3.5}$$

which means that $\{Q_m\}_{m\geq 0}$ is the Al-Salam – Chihara polynomials of parameters

$$a = \frac{-1}{1-q}, \qquad b = 0, \qquad c = \frac{\lambda}{1-q}. \tag{3.6}$$

Askey and Ismail in [AI] succeeded to give the distribution function for the Al-Salam – Chihara polynomials in general cases. It enables us to obtain the probability measure induced from the orthogonal polynomials $\{Q_m\}_{m\geq 0}$. Then shift it $(\lambda + \frac{1}{1-q})$ in right, we have the probability measure for our q-deformed Poisson distribution.

Thorem 3.2. *The probability measure* μ_q *for the q-deformed Poisson distribution of parameter* λ *can be given as follows: We set the function* $f_q(t)$ *as*

(3.7)
$$\begin{aligned}f_q(t) = &\frac{(1-q)\sqrt{\frac{4\lambda}{1-q} - \left(t - \lambda - \frac{1}{1-q}\right)^2}}{2\pi t} \\ &\times \prod_{n=1}^{\infty}(1-q^n)\frac{\lambda(1+q^n)^2 - (1-q)q^n\left(t - \lambda - \frac{1}{1-q}\right)^2}{q^n\left(t - \lambda - \frac{1}{1-q}\right) + \lambda + \frac{q^{2n}}{1-q}}\chi_{I_q},\end{aligned}$$

with the characteristic function χ_{I_q} *on the interval*

$$\begin{aligned}I_q &= \left[-2\sqrt{\tfrac{\lambda}{1-q}} + \lambda + \tfrac{1}{1-q},\ 2\sqrt{\tfrac{\lambda}{1-q}} + \lambda + \tfrac{1}{1-q}\right] \\ &= \left[\left(\sqrt{\lambda} - \sqrt{\tfrac{1}{1-q}}\right)^2,\ \left(\sqrt{\lambda} + \sqrt{\tfrac{1}{1-q}}\right)^2\right],\end{aligned} \tag{3.8}$$

and put

$$(p_k)_q = (t_k)_q + \lambda + \frac{1}{1-q} = [k]_q + \lambda\left(1 - \frac{1}{q^k}\right). \tag{3.9}$$

Then we have

$$\mu_q(dt) = f_q(t)\chi_{I_q}dt + \sum_{k=0}^{K}(J_k)_q\delta_{(p_k)_q}, \tag{3.10}$$

where dt denotes the Lebesgue measure and $\delta_{(p_k)_q}$ is the Dirac nit mass at $t = (p_k)_q$. Here we set

$$K = \sup\{\, k \mid q^{2k} \geq \lambda(1-q)\} \tag{3.11}$$

and

$$(J_k)_q = \left(1 - \frac{\lambda(1-q)}{q^{2k}}\right)\frac{\left(\lambda q^{-k+1}\right)^k}{[k]_q!}\frac{1}{\exp_q\left(\lambda q^{-k+1}\right)}. \tag{3.12}$$

See for the proof in [SY1].

Exmaple 3.3. It is easy to see that in the case of $q \to 1$ the absolutely continuous part vanishes and the distribution is supported on infinite discrete points k ($k = 0, 1, 2, \ldots$) with the mass $(J_k)_1 = \mathrm{e}^{-\lambda}\frac{\lambda^k}{k!}$. Thus we can recover the usual Poisson distribution,

$$\mu_1(dt) = \sum_{k=0}^{\infty}\mathrm{e}^{-\lambda}\frac{\lambda^k}{k!}\delta_k. \tag{3.13}$$

On the other hand, taking the limit $q \to 0$ the density function of the absolutely continuous part becomes

$$f_0(t) = \frac{\sqrt{4\lambda - (t-\lambda-1)^2}}{2\pi t}\chi_{\left[(\sqrt{\lambda}-1)^2,(\sqrt{\lambda}+1)^2\right]}. \tag{3.14}$$

The point mass $(p_k)_0$ will survive for k such that $\frac{q^{2k}}{1-q} \geq \lambda$ holds, and as $q \to 0$ the left hand side of the inequality tends to 0 if $k \neq 0$, and to 1 if $k = 0$. Hence we have the probability measure,

$$\mu_0(dt) = f_0(t)dt + \max(1-\lambda, 0)\delta_0 \tag{3.15}$$

for the case $q = 0$, which is nothing but the free Poisson distribution (see, for instance, Section 3.7 in [VDN]).

4. The q-deformed Poisson random variable on the q-Fock space

In this section, we will give the operator on the q-Fock space, which has the q-deformed Poisson distribution with respect to the vacuum state. Here we shall recall the definition of the q-Fock space.

For a Hibert space $\mathcal{H}$ and $q \in [0,1)$, the q-Fock space $\mathcal{F}_q(\mathcal{H})$ can be defined as follows (see for instance [BKS]): Let $\mathcal{F}^{fin}(\mathcal{H})$ be the linear span of vectors of the form $\xi_1 \otimes \cdots \otimes \xi_n \in \mathcal{H}^{\otimes n}$, where n varies in $\mathbb{Z}_{\geq 0}$ and we put $\mathcal{H}^{\otimes 0} \cong \mathbb{C}\Omega$ for some distinguished vector Ω called vacuum. We consider the sesquilinear form $\langle \cdot | \cdot \rangle_q$ on $\mathcal{F}^{fin}(\mathcal{H})$ given by the sesquilinear extension of

$$(4.1) \qquad \langle \xi_1 \otimes \cdots \otimes \xi_n | \eta_1 \otimes \cdots \otimes \eta_m \rangle_q = \delta_{n,m} \sum_{\pi \in S_n} q^{i(\pi)} \langle \xi_1 | \eta_{\pi(1)} \rangle \cdots \langle \xi_n | \eta_{\pi(n)} \rangle,$$

where S_n denotes the symmetric group of permutations of n elements and $i(\pi)$ is the number of inversions of permutation $\pi \in S_n$ defined by

$$(4.2) \qquad i(\pi) = \#\{(i,j) | 1 \leq i < j \leq n, \pi(i) > \pi(j)\}.$$

The strict positivity of $\langle \cdot | \cdot \rangle_q$ allows the following definitions (see [BS1]):

Definition 4.1. The q-Fock space $\mathcal{F}_q(\mathcal{H})$ is the completion of $\mathcal{F}^{fin}(\mathcal{H})$ with respect to $\langle \cdot | \cdot \rangle_q$, and given the vector $\xi \in \mathcal{H}$, we define the creation operator $a^*(\xi)$ and the annihilation operator $a(\xi)$ on $\mathcal{F}_q(\mathcal{H})$ by

$$(4.3) \qquad \begin{aligned} a^*(\xi)\Omega &= \xi, \\ a^*(\xi)\xi_1 \otimes \cdots \otimes \xi_n &= \xi \otimes \xi_1 \otimes \cdots \otimes \xi_n \end{aligned}$$

and

$$(4.4) \qquad \begin{aligned} a(\xi)\Omega &= 0, \\ a(\xi)\xi_1 \otimes \cdots \otimes \xi_n &= \sum_{i=1}^{n} q^{i-1} \langle \xi | \xi_i \rangle \xi_1 \otimes \cdots \otimes \check{\xi}_i \otimes \cdots \otimes \xi_n, \end{aligned}$$

where the symbol $\check{\xi}_i$ means that ξ_i has to be deleted in the tensor product.

Remark 4.2. The operators $a(\xi)$ and $a^*(\xi)$ are bounded operators on $\mathcal{F}_q(\mathcal{H})$ with

$$(4.5) \qquad ||a(\xi)||_q = ||a^*(\xi)|| = \frac{||\xi||}{\sqrt{1-q}}$$

and they are adjoints of each other with respect to the scalar product $\langle\cdot|\cdot\rangle_q$. Furthermore, it is very important to note that they fulfill the q-commutation relations,

$$a(\xi)a^*(\eta) - a^*(\eta)a(\xi) = \langle\xi|\eta\rangle \cdot \mathbf{1} \qquad \xi, \eta \in \mathcal{H}. \tag{4.6}$$

For $\xi \in \mathcal{H}$ with $||\xi|| = 1$ and $\lambda > 0$, we consider the operator

$$\begin{aligned} x(\xi,\lambda) =& \left(a^*(\xi) + \sqrt{\lambda}\cdot\mathbf{1}\right)\left(a(\xi) + \sqrt{\lambda}\cdot\mathbf{1}\right) \\ =& a^*(\xi)a(\xi) + \sqrt{\lambda}\,(a^*(\xi) + a(\xi)) + \lambda\cdot\mathbf{1}, \end{aligned} \tag{4.7}$$

that is, $x(\xi,\lambda)$ is the sum of the q-number operator $a^*(\xi)a(\xi)$ (see Remark 4.4 below), the q-Gaussian random variable $\sqrt{\lambda}\,(a^*(\xi) + a(\xi))$, and the scalar operator $\lambda\cdot\mathbf{1}$.

Now we shall see that the operator $x(\xi,\lambda)$ is the q-deformed Poisson random variable of the parameter λ with respect to the vacuum state (cf. [HuP] and [Sp]). First we give the basic relations for the q-creation and the q-annihilation operators with vectors of the special forms in the q-Fock space, which are direct consequences of the definitions.

Lemma 4.3. *For $\xi \in \mathcal{H}$ with $||\xi|| = 1$, we have*

$$a^*(\xi)\xi^{\otimes n} = \xi^{\otimes(n+1)}, \quad (n \geq 0) \tag{4.8}$$

$$a(\xi)\xi^{\otimes n} = [n]_q\xi^{\otimes(n-1)}, \quad (n \geq 1) \tag{4.9}$$

where we use the convention that $\xi^{\otimes 0} = \Omega$.

Remark 4.4. Combining the relations (4.8) and (4.9) in Lemma 4.3, we have that for $\xi \in \mathcal{H}$ with $||\xi|| = 1$ and $n \geq 1$,

$$a^*(\xi)a(\xi)\xi^{\otimes n} = [n]_q\xi^{\otimes n}. \tag{4.10}$$

Hence we may regard $a^*(\xi)a(\xi)$ as *the q-number operator.*

Theorem 4.5. *Let $\xi \in \mathcal{H}$ with $||\xi|| = 1$ and $\lambda > 0$ and we simply denote $x(\xi,\lambda)$ by x. Then we have*

$$P_n(x)\Omega = \sqrt{\lambda^n}\xi^{\otimes n}, \quad (n \geq 0) \tag{4.11}$$

where P_n is the monic polynomial of degree n defined by the recurrence relation (3.3) in the definition of the q-deformed Poisson distribution.

Proof. We show this by induction on n. It is clear that

(4.12)
$$P_0(x)\Omega = \mathbf{1}\Omega = \Omega,$$

(4.13)
$$\begin{aligned} P_1(x)\Omega &= (x - \lambda \cdot \mathbf{1})\,\Omega = \Big(a^*(\xi)a(\xi) + \sqrt{\lambda}\,(a^*(\xi) + a(\xi))\Big)\,\Omega \\ &= \sqrt{\lambda}a^*(\xi)\Omega = \sqrt{\lambda}\xi. \end{aligned}$$

With the helps of Lemma 4.3 and the assumptions of induction, we obtain that

(4.14)
$$\begin{aligned} P_{n+1}(x)\Omega =& \Big(\,(x - (\lambda + [n]_q))\,P_n(x) - \lambda[n]_q P_{n-1}(x)\Big)\Omega \\ =& x\left(\sqrt{\lambda^n}\xi^{\otimes n}\right) - (\lambda + [n]_q)\left(\sqrt{\lambda^n}\xi^{\otimes n}\right) - \lambda[n]_q\left(\sqrt{\lambda^{n-1}}\xi^{\otimes(n-1)}\right) \\ =& \left(a^*(\xi)a(\xi) + \sqrt{\lambda}\,(a^*(\xi) + a(\xi)) + \lambda \cdot \mathbf{1}\right)\left(\sqrt{\lambda^n}\xi^{\otimes n}\right) \\ & \quad - (\lambda + [n]_q)\left(\sqrt{\lambda^n}\xi^{\otimes n}\right) - \lambda[n]_q\left(\sqrt{\lambda^{n-1}}\xi^{\otimes(n-1)}\right) \\ =& [n]_q\sqrt{\lambda^n}\xi^{\otimes n} + \sqrt{\lambda^{n+1}}\xi^{\otimes(n+1)} + [n]_q\sqrt{\lambda^{n+1}}\xi^{\otimes(n-1)} + \lambda\sqrt{\lambda^n}\xi^{\otimes n} \\ & \quad - (\lambda + [n]_q)\left(\sqrt{\lambda^n}\xi^{\otimes n}\right) - \lambda[n]_q\left(\sqrt{\lambda^{n-1}}\xi^{\otimes(n-1)}\right) \\ =& \sqrt{\lambda^{n+1}}\xi^{\otimes(n+1)}. \end{aligned}$$

□

The above theorem says that

(4.15) $$\langle P_n(x)P_m(x)\Omega|\Omega\rangle_q = \langle P_m(x)\Omega|P_n(x)\Omega\rangle_q = 0 \quad \text{if } m \neq n,$$

because the element x is self-adjoint with respect to $\langle\,\cdot\,|\,\cdot\,\rangle_q$. This means that the distribution ν of the random variable x with respect to the vacuum state $\langle\,\cdot\,\Omega|\Omega\rangle_q$ can be extended by

(4.16) $$\langle f(x)\Omega|\Omega\rangle_q = \int_{t\in\mathbb{R}} f(t)\,d\nu(t) \quad \text{for all polynomial } f,$$

to the probability measure $d\nu$ on $\mathbb{R}$, of which the sequence of orthogonal polynomials is determined by the recurrence relation (3.3), which is nothing but the q-deformed Poisson distribution. For more about the q-deformed Poisson random variables on the q-Fock spaces see [SY2].

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