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# On the principal blocks of finite general linear groups in non-defining characteristic

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## 1 Introduction

Let $k$ be a field of characteristic $\ell > 0$. In this note, we consider the $\ell$-modular representation of a finite general linear group $\mathrm{GL}_n(q)$ with abelian Sylow $\ell$-subgroup of rank 2 where $q$ is a prime power which is not divided by $\ell$. We fix a positive integer $e$ such that $1 < e < \ell$. Let $e(q)$ be the minimal $a > 0$ such that $\ell \mid q^a - 1$. Let $r(q)$ be the maximal $r > 0$ such that $\ell^r \mid q^{e(q)} - 1$. We study the principal block of the group algebra $k\,\mathrm{GL}_{2e}(q)$ where $e = e(q)$. Note that the Sylow $\ell$-subgroup of $\mathrm{GL}_{2e}(q)$ is isomorphic to $C_{\ell^r} \times C_{\ell^r}$ where $r = r(q)$ and $C_{\ell^r}$ is a cyclic group of order $\ell^r$. On the other hand, the Sylow $\ell$-subgroup of $\mathrm{GL}_{2e-1}(q)$ is isomorphic to $C_{\ell^r}$ and the structure of $k\,\mathrm{GL}_{2e-1}(q)$ is well-known. Our main result is the following:

**Theorem 1.1.** *Let $q_i$ be a prime power which is not divided by $\ell$ for $i = 1, 2$. Let $B_i$ be the principal block of $k\,\mathrm{GL}_{2e}(q_i)$ where $e = e(q_1) = e(q_2)$. If $r(q_1) = r(q_2)$, then $B_1$ and $B_2$ are Morita equivalent.*

*Remark* The case $\ell = 3, e = 2, r(q_i) = 1$ is treated in [5]. The proof is essentially same as in [5],[9]. See [5],[9] for the details.

## 2 Stable equivalence

In this section, we state the outline of the proof of the main theorem. We keep the notation as in §1. First, we define some subgroups.

*Definition*

$$L(q_i) := \left\{ \left( \begin{array}{cc} X & O \\ O & Y \end{array} \right) \middle| \, X, Y \in \mathrm{GL}_e(q_i) \right\}, \ H(q_i) := L(q_i)\langle w_i \rangle \text{ where } w_i = \left( \begin{array}{cc} O & I \\ I & O \end{array} \right).$$

Note that $H(q_i)$ is the normalizer of $L(q_i)$ in $\mathrm{GL}_{2e}(q_i)$. By Broué's theorem ([1]), $B_i$ and the principal block $B_0(kH(q_i))$ of $kH(q_i)$ are stable equivalent of Morita type. Since $B_0(kH(q_1))$ and $B_0(kH(q_2))$ are Morita equivalent, there exists a $(B_1, B_2)$-bimodule $\mathcal{M}$ such that

$$- \otimes \mathcal{M} : \mathrm{mod}\ B_1 \longrightarrow \mathrm{mod}\ B_2$$

induces a stable equivalence.

In order to show that $\mathcal{M}$ induces a Morita equivalence, it suffices to show that $S \otimes_{B_1} \mathcal{M}$ is a simple $B_2$-module for every simple $B_1$-module $S$ by Linckelmann's theorem [6]. We construct (Corollary 4.3) $B_1$-module $Y$ such that,

(1) $Y/\operatorname{rad} Y$ and $\operatorname{soc} Y$ are isomorphic simple modules.
(2) $\operatorname{rad} Y/\operatorname{soc} Y$ is semisimple.
(3) $Y \otimes_{B_1} \mathcal{M}$ satisfies (1) and (2).
(4) $T \otimes \mathcal{M}$ is known (and simple) for every composition factor $T$ of $Y$ which is not isomorphic to $S$.
(5) The multiplicity of $S$ as a composition factor of $Y$ is one.

Using these properties of $Y$, we can show that $S \otimes \mathcal{M}$ is simple.

# 3 Representation theory of $\mathrm{GL}_n(q)$

In this section, we state some preliminary results on the representation theory of $\mathrm{GL}_n(q)$. First we recall some terminologies on partitions. If $\lambda$ is a partition of $n$, then we write $\lambda \vdash n$.

*Definition* Let $\lambda = (\lambda_1, \lambda_2, ...), \mu = (\mu_1, \mu_2, ...) \vdash n$.

1. $\lambda > \mu$ if there exists $k$ such that $\lambda_i = \mu_i$ $(i < k)$ and $\lambda_k > \mu_k$.

2. $\lambda' \vdash n$ where $(\lambda')_n := \operatorname{Card}\{\, j \mid \lambda_j \geq i \,\}$.

3. By removing $e$-rim hooks from $\lambda$ as possible, we obtain a partition, which has no hook of length $e$. This partion is uniquely determined by $\lambda$ and $e$, and called the $e$-core of $\lambda$.

4. (*Littelwood-Richardson coefficient* $a_{\alpha(1)\lambda}$)
If $\alpha = (\alpha_1, \alpha_2, \dots) \vdash n-1$, then $a_{\alpha(1)\lambda} = \begin{cases} 1 & \text{if } \lambda_i = \alpha_i + 1 \text{ for some } i \\ 0 & \text{otherwise.} \end{cases}$

Let $k$ be a field of characteristic $\ell > 0$, $\ell \nmid q$. For each $\lambda \vdash n$, James defines some $k\,\mathrm{GL}_n(q)$-modules, namely $S(\lambda) := S_k(1, \lambda), D(\lambda) := S(\lambda)/\operatorname{rad} S(\lambda)$ ([3]), and Dipper and James define Young module $X(\lambda) := X(1, \lambda)$ ([2]). For every $\lambda \vdash n$, $D(\lambda)$ is a simple module and every composition factor of $S(\lambda)$ is isomorphic to $D(\mu)$ for some $\mu \vdash n$. We denote the multiplicity of $D(\mu)$ in $S(\lambda)$ as compositon factors by $d_{\lambda\mu}$.

Let $U$ be a $k\,\mathrm{GL}_{n-1}(q)$-module. We may regard $U$ as a module for a parabolic subgroup $P$, where

$$P := \left\{ \begin{pmatrix} X & 0 \\ * & * \end{pmatrix} \in \mathrm{GL}_n(q) \;\middle|\; X \in \mathrm{GL}_{n-1}(q) \right\}.$$

We define $U \uparrow$ to be the induced module $\operatorname{Ind}_P^{\mathrm{GL}_n(q)}(\mathrm{U})$. If $k\,\mathrm{GL}_n(q)$-module $V$ has the same composition factors as $\bigoplus_{\lambda \vdash n} b_\lambda S(\lambda)$, then we write $V \downarrow$ for $\bigoplus_{\lambda \vdash n\ \alpha \vdash n-1} b_\lambda a_{\alpha(1)\lambda} S(\alpha)$.

Let $\Delta_n := (d_{\lambda\mu})_{\lambda,\mu}$, $T_n := (a_{\alpha(1)\lambda})_{\alpha,\lambda}$, $(u_{\alpha\lambda})_{\alpha,\lambda} := \Delta_{n-1}^{-1} T_n \Delta_n$. Then the following holds.

**Theorem 3.1 (Dipper-James).** ([2]) *If $\mu \vdash n$, then $X(\mu')$ has the same composition factors as $\bigoplus_{\lambda \vdash n} d_{\lambda\mu} S(\lambda')$.*

**Theorem 3.2 (James).** ([4])

1. *If $\lambda \vdash n$, then $X(\lambda') \downarrow$ has the same composition factors as $\bigoplus_{\alpha \vdash n-1} u_{\alpha\lambda} X(\alpha')$.*
2. *If $\alpha \vdash n-1$, then $D(\alpha) \uparrow$ has the same composition factors as $\bigoplus_{\lambda \vdash n} u_{\alpha\lambda} D(\lambda)$.*

# 4 Inductions of Young modules

Let $B$ be the principal block of $k\,\mathrm{GL}_{2e}(q)$ where $e = e(q)$, char $k = \ell, 1 < e < \ell$. In this section, we determine the decomposition matrix $\Delta_{2e}$ and construct the modules mentioned in the last part of §2.

*Definition*

1. $\Lambda := \{\lambda \vdash 2e \mid (e\text{-core of } \lambda) = \emptyset\}$, $\Gamma := \{\alpha \vdash 2e-1 \mid a_{\alpha(1)\lambda} \neq 0 \text{ for some } \lambda \in \Lambda\}$.
2. $\alpha^- := \min\{\lambda \in \Lambda \mid a_{\alpha(1)\lambda} \neq 0\}$, $\alpha^+ := \max\{\lambda \in \Lambda \mid a_{\alpha(1)\lambda} \neq 0\}$ for $\alpha \in \Gamma$.
3. $\lambda_+ := \max\{\alpha \in \Gamma \mid a_{\alpha(1)\lambda} \neq 0\}$ for $\lambda \in \Lambda$.

Then $\{D(\lambda) | \lambda \in \Lambda\}$ is a complete set of isomorphism classes of simple $B$-modules. Using these notation, we can describe Young module $X(\lambda)$ for $\lambda \in \Lambda$.

**Theorem 4.1.** *If $\alpha \in \Gamma$, then $X(\alpha) \uparrow \cdot 1_B \cong X(\alpha^-)$.*

If $\lambda \in \Lambda, \lambda \neq (2e), (e^2)$, then $\lambda = \alpha^-$ for some $\alpha \in \Gamma$. Since $\mathrm{GL}_{2e-1}(q)$ has a cyclic Sylow $\ell$-subgroup and the structure of the Young module $X(\alpha)$ $(\alpha \in \Gamma)$ is known, we obtain the decompositon number $d_{\lambda\mu}(\lambda, \mu \in \Lambda)$ by Theorem 3.2. Since $d_{\lambda\mu}(\lambda \notin \Lambda \text{ or } \mu \notin \Lambda)$ is well known, we can know all the decomposition numbers.

**Corollary 4.2.** *We can determine $\Delta_{2e}$.*

( This means that by [2] we can determine the $\ell$-modular decomposition matrix of $\mathrm{GL}_{2e}(q)$.) Using this result, we have the following result.

**Corollary 4.3.** *Assume that $\lambda \in \Lambda, \lambda \neq (2e), (e^2), (e, 1^e)$. Then the Loewy series of $D(\lambda_+) \uparrow \cdot 1_B$ is as follows:*

$$D(\lambda_+) \uparrow \cdot 1_B = \begin{pmatrix} & D((\lambda_+)^+) & \\ C & \oplus & D(\lambda) \\ & D((\lambda_+)^+) & \end{pmatrix}.$$

*Here, $C$ is a direct sum of some $D(\mu)$ where $\mu \in \Lambda, \mu > \lambda$.*

*Example*

1. Let $\lambda = (2e-1, 1) \in \Lambda$. Then, $\lambda_+ = (2e-1), (\lambda_+)^+ = (2e)$, and,

$$D(2e-1) \uparrow \cdot 1_B = \begin{pmatrix} D(2e) \\ D(2e-1, 1) \\ D(2e) \end{pmatrix}.$$

2. Let $e = 4$ and $\lambda = (4, 2, 1^2) \in \Lambda$. Then $\lambda_+ = (4, 2, 1), (\lambda_+)^+ = (4, 3, 1)$ and

$$D(4,2,1) \uparrow \cdot 1_B = \begin{pmatrix} & D(4,3,1) & \\ D(8) & D(4^2) & D(4,2,1^2) \\ & D(4,3,1) & \end{pmatrix}.$$

*Remark* (1) Let $G_n(q)$ be a finite group of Lie type over $\mathbb{F}_q$ whose rank is $n$. Suppose that $e = e(q) = e(q'), r(q) = r(q')$. By Theorem 1.1, the unipotent blocks of $\mathrm{GL}_{2e}(q)$ and $\mathrm{GL}_{2e}(q')$ are Morita equivalent. We believe that the unipotent blocks of $G_n(q)$ and $G_n(q')$ are Morita equivalet if the types of $G_n(q)$ and $G_n(q')$ are the same. ([10])

(2) After the meeting, we found the paper by M.J.Richards [8]. It seems that some results of this section are contained in his results [8](see also [7, p.126]).

# References


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