数理解析研究所講究録
1140 巻 2000 年 22-24

p-good groups and P-good modules

Tokyo Medical and Dental University, College of Arts and Sciences

Masao KIYOTA
(清田正夫)

This is a preliminary report of my joint work with Professors T.Wada, M.Murai and A.Hanaki. Let us fix our notation.

$G =$ a finite group
$k =$ an algebraically closed field of characteristic $p > 0$
$P_G(S) =$ the projective cover of a simple kG-module S
$1_G =$ the trivial kG-module
$P(G) = P_G(1_G)$, the projective cover of 1_G

Definition 1. A finite group G is called p-good if for every simple kG-module S the following condition holds;

$$P(G) \otimes S \cong m_S P_G(S)$$

with some positive integer m_S.

We are interested in group structures of p-good groups. If G is a p-group or a p'-group, then G is clearly p-good. G is p-good if and only if $G/O_p(G)$ is p-good. S_4, symmetric group of degree 4, is 2-good but not 3-good. We have two fudamental questions on group structures of p-good groups.

Question 1. If G is p-good, then is it true that G is p-solvable?

Question 2. Let G be p-solvable and p-good. Is it true that the p-length of G is bounded?

Now we will define good kG-modules.

Definition 2. Let M be a kG-module and P a projective kG-module. We call M P-good if a simple kG-module S is a compositon factor of M whenever $P_G(S)$ is a direct factor of $P \otimes M$. M is called good if M is $P(G)$-good.

By definition, M is $P_1 \oplus P_2$-good if and only if M is P_i-good ($i = 1, 2$). Note that the following statements are equivalent to each other;

(1) G is p-good,

(2) M is good for every (finitely generated) kG-module M,

(3) S is good for every simple kG-module S.

The following lemma is well-known.

Lemma 1. *Let $H \lhd G$, S be a simple kG-module and X a simple kH-module. Then X is a composition factor of S_H if and only if S is a composition factor of X^G. Here S_H is the restriction of S to H, and X^G is the induction of X to G.*

Using Lemma 1 we can prove the following propositions.

Proposition 1. *Let $H \lhd G$, M be a kG-module and P a projective kG-module. If M is P-good, then M_H is P_H-good.*

Proposition 2. *Let $H \lhd G$, N be a kH-module and Q a projective kH-module. If N is Q^x-good for all $x \in G$, then N^G is Q^G-good.*

Since $P(H)$ is a direct summand of $P(G)_H$ and $P(G)$ is a direct summand of $P(H)^G$, we have the following

Corollary 3. *Let $H \lhd G$, M be a kG-module and N a kH-module.*

(1) If M is good, then M_H is good.

(2) If N is good, then N^G is good.

When the index $|G : H|$ is a power of p, the converse of Proposition 1 also holds.

Proposition 4. *Let H be a normal subgruop of G with p-power index. Let M be a kG-module and P a projective kG-module. Then M is P-good, if and only if M_H is P_H-good.*

Corollary 5. *Let H be a normal subgroup of G with p-power index. Let M be a kG-module. Then M is good, if and only if M_H is good.*

Corollary 5 yields the following two results on p-good groups.

Corollary 6. *Let H be a normal subgroup of G with p-power index. If H is p-good then G is p-good.*

Corollary 7. *If G is a p-solvable group with $G = O_{p,p',p}(G)$, then G is p-good.*

Question 3. Does the converse of Corollary 6 hold?

Question 4. When G is p-good and p-solvable with p-length 1, describe the group structure of G. Recall that S_4 is 3-solvable with 3-length 1, but not 3-good.

For p-solvable groups G, we have the following criterion for a simple kG-module S to be good.

Proposition 8. *Let G be a p-solvable group with a p-complement L. Let S be a simple kG-module. Then the following inequality holds;*

$$dim_k End_{kL}(S_L) \leq (dim_k S)_p \, .$$

Moreover, the following statements are equivalent;
(1) S is good,
(2) $dim_k End_{kL}(S_L) = (dim_k S)_p$,
(3) $Hom_{kL}(S_L, T_L) = 0$ for every simple kG-module T with $T \ncong S$.

Corollary 9. *Let G be a p-solvable group. Then the following statements are equivalent;*
(1) G is p-good,
(2) $dim_k End_{kL}(S_L) = (dim_k S)_p$ for every simple kG-module S,
(3) $Hom_{kL}(S_L, T_L) = 0$ for all simple kG-modules S, T with $S \ncong T$.