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Coherent structures in 2D flows: Maximum entropy or maximum viscous mixing?

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Abstract

It is shown that the predictions of the Miller-Robert-Sommeria (MRS) statistical mechanical approach to 2D coherent structures can be derived also in a viscous context. From this perspective, what one maximizes is not the entropy but the viscous-induced mixing. Moreover, it is possible to give a validity criterium at high Reynolds number.

1 Introduction

Numerical simulations of the freely decaying, incompressible Navier-Stokes equations in two dimensions have shown that under appropriate conditions and after a relatively short period of chaotic mixing, the vorticity becomes strongly localized in a collection of vortices which move in a background of weak vorticity gradients [[38]]. As long as their sizes are much smaller than the extension of the domain, the collection of vortices may evolve self-similarly in time [[34] ,[13] ,[14] ,[5] ,[3]] until one large-scale structure remains. If the corresponding Reynolds number is large enough, the time evolution of these so-called coherent structures is usually given by a uniform translation or rotation and by relatively slow decay and diffusion, the last two are due to the presence of a non-vanishing viscosity. In other words, in a co-translating or co-rotating frame of reference, one has quasi-stationary structures (QSS) which are, to a good approximation, stationary solutions of the inviscid Euler equations. Accordingly, their corresponding vorticity[2] fields $\omega_S(x,y)$ and stream functions $\psi_S(x,y)$ are, to a good approximation, functionally related, i.e., $\omega_S(x,y)\approx\omega_S(\psi_S(x,y))$. Similar phenomena have been observed in the quasi two-dimensional flows studied in the laboratory [[18] ,[24]]. The only exception to this rule is provided by the large-scale, oscillatory states that occasionally result at the end of the chaotic mixing period [[35] ,[6]]. In many cases, e.g., when the initial vorticity field is randomly distributed in space, the formation of the QSS corresponds to the segregation of different-sign vorticity and the subsequent coalescence of equal-sign vorticity, i.e., to a spatial demixing of vorticity.
Besides the theoretical fluid-dynamics context, a good understanding of the above-described process has implications in many other physically interesting situations like: geophysical flows [[19]], plasmas in magnetic fields [[23]], galaxy structure [[17]], etc. For these reasons numerical and experimental studies are still being performed and have already led to a number of "scatter plots", i.e., to the determination of the ω_S-ψ_S functional relation as a characterization of

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[2]We use the subscript S in order to indicate that the field corresponds to an observed QSS.

the QSS which appear under different circumstances. Simultaneously, on the theoretical side, approaches have been proposed which attempt at, among other things, predicting the QSS directly from the initial vorticity field; if successful in this, such methods would alleviate the need of performing costly numerical and laboratory studies.

In this context, it may be worthwhile recalling that, at least in some cases, the agreement between an experiment and the statistical mechanical prediction can be greatly improved by taking as "initial condition" not the field at the start of the experiment but a later one, after some preliminary mixing has taken place but well before the QSS appears, see [[8]]. In other cases, a detailed consideration of the boundary is necessary and, sometimes, the satistical mechanics approach may be applicable in a well-chosen subdomain, see [[15] ,[7]].

The paper is structured as follows: In the following Section, we recall the MRS statistical-mechanics approach as developed by J. Miller et al. [[25] ,[26]] and by Robert and Sommeria [[30] ,[32]], moreover, we point out an important asymmetry in the characterization of initial and final states. In Section 3 we develop two toy models for the dynamics of the microscopic vorticity distribution; these models are such that, on a macroscopic level, they imply the viscous Navier-Stokes equations. The first model has, surprisingly, an infinite number of conservation laws: $m(\sigma, t)$, the fluid mass associated with any value, say σ, of the microscopic vorticity is independent of time t. This, and the presence of chaotic mixing, leads us to introduce in Section 4 a validity criterium for the MRS inviscid prediction: the prediction holds whenever the degree of mixing grows much faster than the relative change of the masses $m(\sigma, t)$. We also prove an H-theorem for the degree of mixing. In the last Section we summarize our results.

2 Review of the Miller-Robert-Sommeria (MRS) Theory

The pillars on which the statistical mechanical approach stands are the conserved quantities of the non-dissipative, inviscid fluid, i.e., of the Euler equations. These quantities are: 1) the energy per unit mass E, 2) the area, denoted by $G_o(\sigma)$, occupied by fluid with vorticity values between σ and $\sigma + d\sigma$

$$G_o(\sigma) := \int_A dxdy\, \delta(\sigma - \omega_o(x, y)), \tag{1}$$

where $\omega_o(x, y)$ is the initial vorticity field and A is the spatial domain occupied by the fluid, and 3) depending upon the symmetry of the domain A, the linear momentum and/or the angular momentum. For the sake of simplicity, we will ignore these last two quantities.

As usual, one derives a probability distribution for observing, on a microscopic level, a vorticity value σ by maximizing the entropy S under the constraints defined by the conserved quantities. The entropy used by MRS is

$$S := -\int_A dxdy \int d\sigma\, \rho(\sigma, \psi(x, y)) \ln \rho(\sigma, \psi(x, y)), \tag{2}$$

where $\rho(\sigma, \psi(x, y)) \geq 0$ is the probability of finding a vorticity value σ at position (x, y), therefore

$$\int d\sigma\, \rho(\sigma, \psi(x, y)) = 1, \;\; for\ all\ (x, y), \tag{3}$$

and $\psi(x,y)$ is the stream function of the most probable state[3], see also (6) below. The vorticity distribution $\rho(\sigma,\psi)$ one obtains is

$$\begin{aligned} \rho(\sigma,\psi(x,y)) \ &: \ = Z^{-1}\exp\left[-\beta\sigma\psi(x,y)+\mu(\sigma)\right], \\ \textit{with } Z(\psi(x,y)) \ &: \ = \int d\sigma\, \exp\left[-\beta\sigma\psi(x,y)+\mu(\sigma)\right]. \end{aligned} \tag{4}$$

In (4), β and $\mu(\sigma)$ are Lagrange multipliers such that the energy per unit mass E and the microscopic-vorticity area distribution $g(\sigma)$,

$$g(\sigma) := \int dxdy\, \rho(\sigma,\psi(x,y)), \tag{5}$$

have the same values as in the initial vorticity field, i.e., $g(\sigma) = G_o(\sigma)$. The macroscopic vorticity field in the most probable state, denoted by $\omega_S(x,y)$, is the average of σ with respect to $\rho(\sigma,\psi(x,y))$,

$$\omega_S(x,y) = \int d\sigma\, \sigma\rho(\sigma,\psi(x,y)).$$

The r.h.s. of this equation defines the ω-ψ relation

$$\Omega(\psi) := \int d\sigma\, \sigma\rho(\sigma,\psi),$$

which, in an experimental context, is often called the scatter-plot. The system of equations is closed by

$$\omega_S(x,y) = -\Delta\psi, \tag{6}$$

which embodies the mean-field approximation.

2.1 The asymmetry between initial and final states

There is a deep asymmetry in the characterization of the initial vorticity field and that of the corresponding most probable state. It is assumed that the initial state has zero entropy, i.e., that it is an "unmixed state" for which the microscopic and macroscopic description coincide. This allows to equate the microscopic vorticity distribution of the initial state with $G_o(\sigma)$ which is defined in terms of the (macroscopic) vorticity field $\omega_o(x,y)$, confer (1). By contrast, the microscopic vorticity-area density of the most probable state, $g(\sigma)$, is given by $\int dxdy\, \rho(\sigma,\psi(x,y))$, confer (5). Notice that $G_S(\sigma)$, the macroscopic vorticity density of the most probable state,

$$G_S(\sigma) := \int dxdy\, \delta(\sigma-\omega_S(x,y)),$$

is, in most cases, different from the microscopic vorticity-area density of the most probable state, i.e., $G_S(\sigma) \neq g(\sigma)$. For example, the even moments of $G_S(\sigma)$ are smaller than or equal to those of $g(\sigma)$, confer (17) below. Confusion about this fact has led to misinterpretations of the MRS theory [[21] ,[9]].

[3]This is a mean-field approximation, valid for this system [[26]].

3 Microscopic Viscous Models

In analogy to what is done in order to derive the Euler and the Navier-Stokes equations from a microscopic theory [[4] ,[37]] let us introduce $\phi(\sigma, x, y, t)d\sigma$, the probability of finding at time t a microscopic vorticity value in the range $(\sigma, \sigma + d\sigma)$ at a position (x, y); this ϕ must be non-negative and normalized

$$\int d\sigma \, \phi(\sigma, x, y, t) = 1. \tag{7}$$

The macroscopic vorticity field is given by

$$\omega(x, y, t) = \int d\sigma \, \sigma \phi(\sigma, x, y, t). \tag{8}$$

In the inviscid case, the time evolution of $\phi(\sigma, x, y, t)$ can be taken to be

$$\frac{\partial \phi(\sigma, x, y, t)}{\partial t} + \vec{v}(x, y, t) \cdot \nabla \phi(\sigma, x, y, t) = 0, \tag{9}$$

where the incompressible velocity field $\vec{v}(x, y, t)$ satisfies appropriate boundary conditions and is related to the macroscopic vorticity $\omega(x, y, t)$ by

$$\nabla \times \vec{v} = \omega \tilde{z}, \tag{10}$$

with $\tilde{z}$ a unit vector perpendicular to the (x, y)-plane. Consequently, the advective term in equation (9) is quadratic in ϕ, moreover, different values of σ are coupled by this term.

In this Section we shall consider some model evolution equations for the probability density $\phi(\sigma, x, y, t)$. All these models should be such that they imply, on a macroscopic level, the Navier-Stokes equations. The general form of these models is then

$$\frac{\partial \phi(\sigma, x, y, t)}{\partial t} + \vec{v}(x, y, t) \cdot \nabla \phi(\sigma, x, y, t) = \nu O, \tag{11}$$

with ν the fluid viscosity and O yet undefined but constrained by 1) the conservation of probabilty,

$$\int d\sigma \, O = 0,$$

2) the conservation of total circulation, i.e., we consider now either periodic boundary conditions or vorticity distributions of compact support that remain always away from the boundary,

$$\int dxdy \int d\sigma \, \sigma O = 0$$

and 3) compatible with the macroscopic Naviers-Stokes equation

$$\int d\sigma \, \sigma O = \Delta \omega(x, y, t).$$

3.1 Viscous model with an infinite number of conservation laws.

The simplest model satisfying the above requirements is

$$\frac{\partial \phi(\sigma,x,y,t)}{\partial t} + \vec{v}(x,y,t)\cdot\nabla\phi(\sigma,x,y,t) = \nu\Delta\phi(\sigma,x,y,t), \tag{12}$$

with the incompressible velocity field related to the vorticity as in (10). In fact, multiplying by σ, integrating over σ and making use of (8), one gets the Navier-Stokes equation

$$\frac{\partial \omega(x,y,t)}{\partial t} + \vec{v}(x,y,t)\cdot\nabla\omega(x,y,t) = \nu\Delta\omega(x,y,t). \tag{13}$$

This simple model is very instructive because, while it dissipates energy, it has an infinite number of conserved quantities. These quantities are the "masses" associated with each value of the microscopic vorticity: these "masses" are given by

$$m(\sigma,t) := \int dxdy\,\phi(\sigma,x,y,t), \tag{14}$$

and (12) implies that

$$\frac{\partial m(\sigma,t)}{\partial t} = 0. \tag{15}$$

In order to derive these conservation laws it is necessary that the total flux of $\phi(\sigma,x,y,t)$ through the boundary vanishes, i.e.,

$$\oint ds\,\vec{n}\cdot\left[\vec{v}(x,y,t)\phi(\sigma,x,y,t) - \nu\vec{\nabla}\phi(\sigma,x,y,t)\right] = 0,$$

where the path integral is taken over the boundary and $\vec{n}$ is the normal unit vector. In most applications one has that, on the boundary, $\vec{n}\cdot\vec{v}\equiv 0$, and the last condition reduces to

$$\oint ds\,\vec{n}\cdot\vec{\nabla}\phi(\sigma,x,y,t) = 0. \tag{16}$$

On a macroscopic level this implies that

$$\oint ds\,\vec{n}\cdot\vec{\nabla}\omega(x,y,t) = 0,$$

this condition is satisfied in the case of periodic boundary conditions as well as by vorticity fields of compact support that stay away from the boundary at all times.

One of the consequences of the conservation laws (15) is that all the microscopic-vorticity moments $\langle\sigma^n\rangle := \int d\sigma\,\sigma^n m(\sigma,t)$ are constants of the motion, i.e.,

$$\frac{d\,\langle\sigma^n\rangle}{dt} = 0.$$

On the other hand, the macroscopic enstrophy $\int dxdy\,\omega^2(x,y,t)$ as well as the higher even moments of the macroscopic vorticity, $\Gamma_{2n} := \int dxdy\,\omega^{2n}(x,y,t)$, are dissipated, as implied by the Navier-Stokes equation (13),

$$\frac{d\Gamma_{2n}}{dt} = -\nu 2n(2n-1)\int dxdy\,\omega^{2(n-1)}\left|\nabla\omega\right|^2 \leq 0. \tag{17}$$

In the inviscid case, incompressibility and the fact that the vorticity is just advected by the velocity field, allow for a complete identification between a vorticity value σ and the area that is occupied by such value. In this case often one talks of 'conservation of the area occupied by a vorticity value'. As soon as we introduce a diffusion process, as it is implied by (12) with $\nu \neq 0$, such an identification becomes problematic if not impossible. It is for this reason that we call $m(\sigma, t)$ a "mass" and, by doing so, we stress the obvious analogy with an advection-diffusion process of an infinite number of "chemical species", one species for each value σ.

In conclusion: The viscous Navier-Stokes equation (13) does not exclude the possibility of an infinite number of conserved quantities $m(\sigma, t)$ as defined in equation (14). Only in the inviscid case do these conserved quantities coincide with the areas occupied by a vorticity value σ.

3.2 Enslaved viscous model

One can define a vorticity probability density at all times as

$$\phi(\sigma, x, y, t) := \delta(\sigma - \omega(x, y, t)). \tag{18}$$

With this choice, the microscopic and macroscopic descriptions coincide at all times; it is for this reason that we shall denote the microscopic model built on this assumption as 'the enslaved model'. With (18), formal manipulations of the Navier-Stokes equation (13) fix the time evolution of this ϕ, to wit

$$\frac{\partial \phi(\sigma, x, y, t)}{\partial t} + \vec{v}(x, y, t) \cdot \nabla \phi(\sigma, x, y, t) = \nu \Delta \phi(\sigma, x, y, t) - \nu \left| \nabla \omega(x, y, t) \right|^2 \frac{\partial^2 \phi}{\partial \sigma^2}. \tag{19}$$

It should be stressed that this evolution equation is also a 'toy model' since it follows from the Navier-Stokes equations only under the assumption expressed by (18). In this model the "masses" $m(\sigma, t)$ are not conserved, one finds

$$\begin{aligned} \frac{\partial m(\sigma, t)}{\partial t} &= -\nu \frac{\partial}{\partial \sigma} \int d^2x \, \delta(\sigma - \omega(x, y, t)) \Delta \omega & (20) \\ &= -\nu \frac{\partial^2}{\partial \sigma^2} \int d^2x \, \delta(\sigma - \omega(x, y, t)) \left| \nabla \omega \right|^2. & (21) \end{aligned}$$

The last expression makes possible the introduction of a time and σ-dependent diffusion coefficient in σ-space, see [[11]].

4 Chaotic mixing

Considering once more the simple model defined by equation (12), we notice that it is an advection-diffusion equation for the non-passive scalar ϕ with the viscosity ν playing the role of a diffusion coefficient. This type of equations has been studied extensively, see e.g. [[29]] and the references therein. It is well-known that a time-dependent velocity field $\vec{v}(x, y, t)$ usually leads to chaotic trajectories, i.e., to the explosive growth of small-scale ϕ-gradients. These small-scale gradients are then efficiently smoothed out by diffusion, the net result being a very large effective diffusion coefficient, large in comparison to the molecular coefficient ν.

On the other hand, theoretical insight and experimental evidence tell us that, in high-Reynolds' number, two-dimensional flows, the energy is transported from the small to the large scales and, consequently, it is hardly affected by viscous dissipation.

Based on these considerations, we are led to the following conjecture: if mixing takes place much faster than[4] the changes in the 'masses' $m(\sigma,0)$ during a period of time ending with the formation of quasi-stationary structures (QSSs), then the evolution may be characterized by 1) an approximately constant energy value and an infinite number of quasi-conserved quantities $m(\sigma,t) \simeq m(\sigma,0)$ and 2) QSSs that maximize the spatial spreading (or mixing) of the 'masses' $m(\sigma,0)$. In order to express this in a more quantitative form, we need a definition of the degree of spreading or mixing of a solute's mass in a domain. This is done in the next Subsection.

4.1 Degree of mixing

The main idea is that a mass of solute m achieves the highest degree of mixing when it is homogeneously distributed over the area A, i.e., when its concentration is m/A. Let us introduce $\delta(\sigma,x,y,t)$ the (spatial) density of the σ-species at time t, i.e.,

$$\begin{aligned} \delta(\sigma,x,y,t) \quad &: \quad = \frac{\phi(\sigma,x,y,t)}{m(\sigma,t)} \geq 0, \\ \int dxdy\, \delta(\sigma,x,y,t) \quad &= \quad 1, \end{aligned}$$

In order to determine how well mixed is this σ-mass one has to determine 'how close' is the corresponding $\delta(\sigma,x,y,t)$ to the homogeneous distribution $1/A$. As it is known, given two spatial densities, called them $\delta_1(x,y)$ and $\delta_2(x,y)$, there is a non-negative, convex functionals $d(\delta_1,\delta_2)$ satisfying $d(\delta_1,\delta_2)=0 \longleftrightarrow \delta_1 \overset{a.e.}{=} \delta_2$ and measuring a sort of distance between them, namely

$$d(\delta_1,\delta_2) := -\int dxdy\, \delta_1(x,y) \ln\left(\delta_2(x,y)/\delta_1(x,y)\right),$$

under the assumption that both δ_1 and δ_2 vanish on sets of measure zero.

We can measure the mixing degree of the σ-mass by $-d(\delta(\sigma,x,y,t),1/A)$. Its weighted contribution to the total mixing degree will be denoted by $s(\sigma,t)$, i.e.,

$$s(\sigma,t) := -m(\sigma,t)\int dxdy\, \delta(\sigma,x,y,t) \ln A\delta(\sigma,x,y,t) \leq 0,$$

the corresponding total degree of mixing being

$$S^*(t) := -\int d\sigma dxdy\, \phi(\sigma,x,y,t) \ln\left[A\phi(\sigma,x,y,t)/m(\sigma,t)\right] \leq 0. \tag{22}$$

This degree of mixing satisfies a kind of H-theorem since, ignoring the time-dependence of $m(\sigma,t)$ and assuming that time derivation and space integration commute, one has that

$$\frac{\partial s(\sigma,t)}{\partial t} = +\nu \int dxdy\, \frac{1}{\phi} \left|\nabla\phi\right|^2 \geq 0, \tag{23}$$

[4]This statement will get a precise mathematical formulation Subsection VB, eqs (28) and (29).

4.2 Fast mixing

Now it is possible to express our conjecture in a more quantitative form: If mixing takes place much faster than the changes in the σ-masses, i.e., if the following inequalities hold,

$$\frac{\partial s(\sigma,t)}{\partial t} \gg \left|\frac{\partial m(\sigma,t)}{\partial t}\right|, \tag{24}$$

then the QSS' microscopic vorticity field $\phi_S(\sigma,x,y)$ maximizes the total degree of mixing S^* under the constraints of energy and σ-masses fixed at their initial values, i.e., $E_S = E_o$ and $m_S(\sigma) = m(\sigma,0)$. Introducing the corresponding Lagrange multipliers β and $\tilde{\mu}(\sigma)$, as well as a Lagrange multiplier $\gamma(x,y)$ associated with the normalization constraint (3), the constrained variation of (??) leads to

$$\begin{aligned} 0 &= \beta\sigma\psi(x,y) + \tilde{\mu}(\sigma) + \gamma(x,y) + \ln\frac{A\phi_S(\sigma,x,y)}{m(\sigma)}, \\ \textit{i.e.,}\ \phi_S(\sigma,x,y) &= A^{-1}m(\sigma)\exp(-\beta\sigma\psi(x,y) - \tilde{\mu}(\sigma) - \gamma(x,y)). \end{aligned}$$

Since $A^{-1}m(\sigma) \geq 0$, we can define $\mu(\sigma) := -\tilde{\mu}(\sigma) + \ln A^{-1}m(\sigma)$ and implementing the normalization constraint (3), one arrives at

$$\begin{aligned} \phi_S(\sigma,x,y) &= Z^{-1}\exp(-\beta\sigma\psi(x,y) + \mu(\sigma)), \\ \textit{with}\ Z(\psi(x,y)) \ &: \ = \int d\sigma\ \exp\left[-\beta\sigma\psi(x,y) + \mu(\sigma)\right]. \end{aligned}$$

These are the equations of the inviscid, statistical mechanics approach (4), i.e., $\phi_S(\sigma,x,y) \equiv \rho(\sigma,\psi(x,y))$. New are the conditions expressed by (24), i.e., a criterium of applicability in the case of viscous flows. Taking into account (23) and (20), these conditions read

$$\int dxdy\,\phi(\sigma,x,y,t)\left|\nabla\ln\phi\right|^2 \gg \left|\frac{\partial^2}{\partial\sigma^2}\int dxdy\,\delta(\sigma-\omega(x,y,t))\left|\nabla\omega\right|^2\right|.$$

We can address also the following question: Given a vorticity field $\omega(x,y,t)$, what is the corresponding microscopic distribution $\phi(\sigma,x,y,t)$ satisfying (8) and the last inequalities? We proceed as follows: if the last conditions are satisfied, then it also holds that

$$\int d\sigma\int dxdy\,\phi(\sigma,x,y,t)\left|\nabla\ln\phi\right|^2 \gg \int d\sigma\left|\frac{\partial^2}{\partial\sigma^2}\int dxdy\,\delta(\sigma-\omega(x,y,t))\left|\nabla\omega\right|^2\right|. \tag{25}$$

The rhs of this inequality is completely determined by the given $\omega(x,y,t)$. Let us maximize the lhs under the (x,y)-dependent constraints (8) and (7). The constrained variation is

$$\delta\left[\frac{1}{\phi}\left|\nabla\phi\right|^2\right] + \lambda(x,y)\sigma\delta\phi + \alpha(x,y)\delta\phi = \left[\lambda(x,y)\sigma + \alpha(x,y) + \frac{1}{\phi^2}\left|\nabla\phi\right|^2 - \frac{2}{\phi}\Delta\phi\right]\delta\phi,$$

where $\lambda(x,y)$ is the Lagrange multiplier corresponding to (8) and $\alpha(x,y)$ the one corresponding to (7). Setting this variation equal to zero, we get the differential equation that the maximizer has to satisfy:

$$\begin{aligned} \Delta\phi &= -\frac{1}{2}\left[\lambda(x,y)\sigma + \alpha(x,y) + \frac{1}{\phi^2}\left|\nabla\phi\right|^2\right]\phi, \\ \textit{i.e.,}\ \frac{\Delta\phi}{\phi} + \frac{1}{2}\frac{1}{\phi^2}\left|\nabla\phi\right|^2 &= -\frac{1}{2}\left[\lambda(x,y)\sigma + \alpha(x,y)\right], \end{aligned}$$

with appropriate boundary conditions. Introducing $e(\sigma, x, y, t) := \phi^{3/2}$ and noticing that

$$\Delta e = \frac{3}{2}\left[\frac{\Delta\phi}{\phi} + \frac{1}{2}\frac{1}{\phi^2}\left|\nabla\phi\right|^2\right] e,$$

we rewrite the last equation for ϕ as

$$\begin{aligned} \frac{2}{3}e^{-1}\Delta e &= -\frac{1}{2}\left[\lambda(x,y)\sigma + \alpha(x,y)\right], \\ i.e., \ \Delta e &= -\frac{3}{4}\left[\lambda(x,y)\sigma + \alpha(x,y)\right] e. \end{aligned}$$

This is a linear equation in e. In terms of e, the constraints (7) and (8) are nonlinear:

$$\begin{aligned} \int d\sigma\, e^{2/3}(\sigma, x, y, t) &= 1, \\ \int d\sigma\, \sigma e^{2/3}(\sigma, x, y, t) &= \omega(x, y, t). \end{aligned}$$

4.3 These models in contraposition to the Chavanis theory

In [[16]] coarse graining and energy conservation

If the monotonic time development predicted by Eq. (??) could be proved rapid compared to the decay of all the ideal invariants except that of the energy E, it would constitute an "H-theorem" for the system [[28]] .

5 Conclusions

The viscous Navier-Stokes equation (13) does not exclude the possibility of an infinite number of conserved quantities $m(\sigma, t)$ as defined in equation (14). Only in the inviscid case do these conserved quantities coincide with the areas occupied by a vorticity value σ. One expects that molecular viscosity and chaotic mixing in equations like (11) and (19) lead to an explosive growth of the degree of mixing $s(\sigma, t)$. By maximizing this degree of mixing under the constraints of fixed energy and masses $m(\sigma, t)$, one arrives at the same mean-field equations proposed by Miller, Robert and Sommeria but now one expects them to be valid for weakly viscous systems, ie., for large but finite Reynolds numbers. More precisely, one expects them to hold whenever the inequalities (24) hold.

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