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Information-theoretic Wehrl entropy, its density and Husimi function of optical fields

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A density of the information-theoretic Wehrl entropy in terms of the Husimi Q-function is defined. The entropic density is then applied to describe the quantum properties of some optical fields including Fock states, coherent and squeezed states, superposition of chaotic and coherent states, and Schrödinger cat and cat-like states. The entropic density is compared with both the conventional Wehrl entropy and the Husimi phase distribution. It is shown that the entropic density is a good measure of the phase-space uncertainty (noise), and the phase locking or phase bifurcation effects. The advantages of the entropic description of the superposition principle are presented. It is also demonstrated that the entropic density properly describes phase randomization processes, thus can be used as a measure of decoherence.

I. INTRODUCTION

Quantum entropy is one of the most fundamental concepts of quantum physics [1], particularly useful in quantum information [2]. Quantum entropy has been applied, e.g., as a measure of quantum entanglement, quantum decoherence, photocount statistics, quantum optical correlations, purity of states, quantum noise or accessible information in quantum measurement (capacity of quantum channel).

Quantum entropy, as a natural generalization of classical Boltzmann entropy, was proposed by von Neumann [3]. A classical-like entropy associated with quantum fields was introduced by Wehrl [4] in terms of the Glauber coherent states and the Husimi Q-function [5]. A rigorous proof that the von Neumann entropy tends to the Wehrl entropy in the classical limit $\hbar \to 0$ was given by Beretta [6]. The quantum von Neumann entropy can be expanded in power series of classical entropies. As was shown explicitly by Peřinová et al. [7], the first term of this expansion is the Wehrl entropy. Bužek et al. [8] related the von Neumann entropy with the sampling entropies based on operational approach to a phase-space measurement. They also identified the Wehrl entropy as a particular example of the sampling entropy, when the quantum ruler is represented by coherent states. Other aspects of the relation between classical and quantum-mechanical entropies have also been extensively studied (see, e.g., excellent surveys by Wehrl [9], and Ohya and Petz [1] with references included therein).

The Wehrl entropy has been successfully applied in a description of different properties of quantum optical fields. In particular, it has been shown explicitly that the Wehrl entropy is a useful measure of phase-space uncertainty (quantum noise, phase-space localization, wave-packet spreading) [8,10–14], quantum interference [10], decoherence [11,15,16], ionization [14], squeezing [16–18], Schrödinger cat formation [13,19] or, in general, splitting of Q-function [15].

The von Neumann entropy becomes zero for all pure states, thus cannot be used in discriminating them. Paradoxically, the classical Wehrl entropy is more sensitive in distinguishing states than the quantum von Neumann entropy, since it is dependent on the choice of pure states. However, there are properties of quantum fields (including phase decoherence or unique description of superposition states), which are not enough precisely described by the conventional Wehrl entropy. Therefore, we propose a new entropic measure – the phase-density of the Wehrl entropy. We analyze several quantum and classical optical states of light to show the usefulness and advantages of the new entropic measure in comparison to the conventional Wehrl entropy.

II. DEFINITIONS

The density matrix $\widehat{\rho}$ for an arbitrary (pure or mixed) state of light can be represented by the classical-like Husimi $Q(\alpha)$-function [5],

$$Q(\alpha) = \frac{1}{\pi}\mathrm{Tr}(\widehat{\rho}|\alpha\rangle\langle\alpha|) = \frac{1}{\pi}\langle\alpha|\widehat{\rho}|\alpha\rangle, \tag{1}$$

in terms of coherent states $|\alpha\rangle$. Eq. (1) is normalized to unity,

$$\int Q(\alpha)\mathrm{d}^2\alpha = 1, \tag{2}$$

where $\mathrm{d}^2\alpha \equiv \mathrm{d}Re\alpha\,\mathrm{d}Im\alpha = |\alpha|\,\mathrm{d}|\alpha|\,\mathrm{dArg}\alpha$. The Husimi representation (1) provides, equivalently to the Glauber-Sudarshan or Wigner representations, a basis for a formal equivalence between the quantum and classical descriptions of optical coherence [20].

The classical information-theoretic Wehrl entropy [4,21]:

$$S_\mathrm{W} = -\int Q(\alpha)\ln Q(\alpha)\mathrm{d}^2\alpha \tag{3}$$

is defined via the Husimi Q-function (1). The Wehrl entropy (3) is also called the Shannon information of Q-function [11]. We define the following entropic measure:

$$S_\theta = -\int Q(\alpha)\ln Q(\alpha)|\alpha|\mathrm{d}|\alpha|, \tag{4}$$

which might be interpreted as the Wehrl phase distribution or the phase density of the Wehrl entropy. In fact, the Wehrl entropy (3) can be obtained from Eq. (4) by simple integration,

$$S_\mathrm{W} \equiv -\int\int Q(\alpha)\ln Q(\alpha)|\alpha|\mathrm{d}|\alpha|\mathrm{d}\theta = \int S_\theta \mathrm{d}\theta, \tag{5}$$

where $\theta = \mathrm{Arg}\alpha$. For brevity, the terms *entropic density* and *Wehrl phase distribution* will be used for the function (4). We will show some formal similarities but also essential differences between the entropic density (4) and phase distributions, including the so-called Husimi phase distribution [23]

$$P_\theta = \int Q(\alpha)|\alpha|\mathrm{d}|\alpha|, \tag{6}$$

defined as the marginal function of the Husimi Q-function.

In the next Sections, we will calculate the Husimi Q-function, the Wehrl entropy and its phase density for some common states of light. We will show the advantages of the entropic density over the standard Wehrl entropy and phase distributions in a description of quantum properties of radiation.

III. ENTROPIC DESCRIPTION OF COMPLETE DECOHERENCE

We show, by analyzing states with random phase, that the entropic density properly describes fields, which are completely decoherent. States with random phase are usually defined by uniform classical phase distributions [22,23], uniform quantum Pegg-Barnett phase distribution [24,23] or, equivalently, by rotated-quadrature distributions independent of the reference phase [25]. These definitions can also be formulated in terms of the rotationally symmetric quasiprobability distributions, including Wigner or Husimi functions [25,23]. Consequently, the states with random phase, described by the phase-independent Husimi function $Q(\alpha) = f(|\alpha|)$, have also the phase-independent entropic density S_θ. Thus, the Wehrl entropy S_{W} is simply related to its density S_θ by the formula $S_\theta = S_{\mathrm{W}}/(2\pi)$. We briefly discuss two examples of such states.

A. Fock states

The Fock state $|n\rangle$ is described by the phase-independent Husimi Q-function in the form of the Poissonian distribution

$$Q(\alpha) = \frac{1}{\pi}\frac{|\alpha|^{2n}}{n!}\exp(-|\alpha|^2). \tag{7}$$

Consequently, the entropic density is given by:

$$S_\theta = \frac{1}{2\pi}S_{\rm W} = \frac{1}{2\pi}\left[1 + n - n\psi(n+1) + \ln(\pi n!)\right], \tag{8}$$

where $\psi(n+1) = -\gamma + \sum_{k=1}^{n}\frac{1}{k}$ is the digamma function defined by Euler's constant $\gamma = 0.57721566\cdots$. Eq. (8) instantly comes from the well-known expression for the Fock-state Wehrl entropy (see, e.g., Ref. [18,7]). The entropic densities for Fock states $|n\rangle$ with $n = 0, 10, \cdots, 50$ are depicted in polar coordinates in Fig. 1(a).

B. Chaotic field

The chaotic (Gaussian) field is defined as a state with the maximal von Neumann entropy. Its Husimi Q-function reads as [20]:

$$Q(\alpha) = \frac{1}{\pi(\langle n_{\rm ch}\rangle + 1)}\exp\left(-\frac{|\alpha|^2}{\langle n_{\rm ch}\rangle + 1}\right), \tag{9}$$

where $\langle n_{\rm ch}\rangle$ is the mean number of photons. The mean photon number of thermal (blackbody) radiation at thermal equilibrium at temperature T, is given by $\langle n_{\rm ch}\rangle = \{\exp(\hbar\omega/k_B T) - 1\}^{-1}$, where k_B is the Boltzmann constant. The entropic density of the

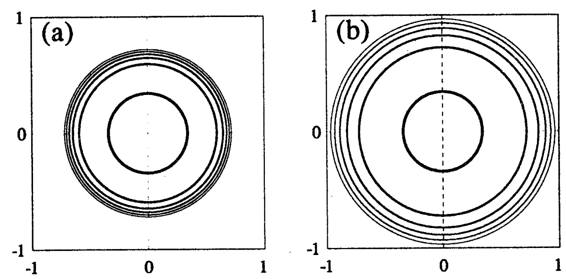

FIG. 1. Entropic density for (a) Fock states and (b) chaotic fields with the mean photon numbers $n = \langle n_{\rm ch}\rangle$ equal to: 0 (inner circle), 10, 20, 30, 40, 50 (outer circle).

chaotic light is then given by

$$S_\theta = \frac{1}{2\pi} S_{\rm W} = \frac{1}{2\pi}\left[1 + \ln\pi + \ln(\langle n_{\rm ch}\rangle + 1)\right]. \tag{10}$$

The Wehrl entropy $S_{\rm W}$, given by Eq. (10), was discussed in Refs. [7,20,18]. Perfectly circular representations of the phase-independent entropic densities for chaotic states are plotted in Fig. 1(b). For comparison, the entropic densities for Fock states in Fig. 1(a) and for chaotic states in Fig. 1(b) are shown for the same mean-photon numbers $\langle n_{\rm ch}\rangle = \langle n_{\rm Fock}\rangle \equiv n$.

The Pegg-Barnett phase distribution [24] and the marginal quasiprobability phase distributions [23], including the Husimi phase function P_θ, are equal to $1/(2\pi)$ for any state with random phase. The entropic density (4), although independent of phase, remains dependent on the mean number of photons in the field with random phase. Hence, the density S_θ contains more information than the phase distributions, but still fulfills the requirement for a good measure of phase properties or decoherence.

IV. ENTROPIC DESCRIPTION OF PARTIAL DECOHERENCE AND PHASE LOCKING

We will show, by referring to the examples of pure or "noisy" coherent fields, that the entropic density describes properly the influence of noise on coherent field (decoherence effect) as well as phase locking effect observed with the increasing mean number of photons of the partial phase states (amplification of coherence).

A. Signal with noise

A superposition of coherent signal with complex amplitude α_0 and chaotic noise with the mean photon number $\langle n_{\rm ch}\rangle$ can be described by the following Husimi Q-function [20]:

$$Q(\alpha) = \frac{1}{\pi(\langle n_{\rm ch}\rangle + 1)} \exp\left(-\frac{|\alpha - \alpha_0|^2}{\langle n_{\rm ch}\rangle + 1}\right). \tag{11}$$

After integration, according to Def. (4), we find the closed-form expression of the entropic density for the superposition of coherent signal and noise as follows:

$$S_\theta \equiv S_\theta(\alpha_0, n_{\rm ch}) = \frac{1}{2\pi}\exp[-(X_0^2 - X^2)]\Big\{\exp(-X^2)f_2 + \sqrt{\pi}X\,[1+\mathrm{erf}(X)]f_1\Big\}, \qquad (12)$$

where

$$f_j = X_0^2 - X^2 + \ln[\langle n_{\rm ch}\rangle + 1] + \ln\pi + j/2, \quad \text{and} \quad X = \frac{|\alpha_0|}{\sqrt{\langle n_{\rm ch}\rangle + 1}}\cos(\theta - \theta_0), \qquad (13)$$

which, for the special choice of θ, is denoted by $X_0 = X(\theta = \theta_0) = |\alpha_0|/\sqrt{\langle n_{\rm ch}\rangle + 1}$, where θ_0 is the phase of α_0. Analogously, for the field described by Eq. (11), we derive the following Husimi phase distribution

$$P_\theta \equiv P_\theta(\alpha_0, n_{\rm ch}) = \frac{1}{2\pi}\exp[-(X_0^2 - X^2)]\Big\{\exp(-X^2) + \sqrt{\pi}X\,[1+\mathrm{erf}(X)]\Big\}, \qquad (14)$$

where X is defined by Eq. (13). It is seen, by comparing Eqs. (12) and (14), that the entropic density S_θ differs from the Husimi phase distribution $P_{\rm H}(\theta)$ by the factors f_j. Eq. (12) goes over into Eq. (14) by putting $f_1 = f_2 = 1$. The Wehrl entropy for the field (11) reads as [7,20]:

$$S_{\rm W} = 1 + \ln\pi + \ln(\langle n_{\rm ch}\rangle + 1), \qquad (15)$$

which can be obtained by direct integration of the entropic density (12). The Wehrl entropy (15) is independent of the complex amplitude of the coherent signal. Eq. (12) reduces to the phase-independent density (10) for chaotic field without coherent signal ($\alpha_0 = 0$). The entropic density for the Glauber coherent state is another special case of Eq. (13).

B. Coherent states

The Husimi Q-function for the Glauber coherent state $|\alpha_0\rangle$ reads as

$$Q(\alpha) = \frac{1}{\pi} \exp\left\{-|\alpha - \alpha_0|^2\right\}. \tag{16}$$

Coherent state (16) is the most common example of the partial phase state [24]. Eq. (16) is the special case of Eq. (11) for the signal field without noise ($\langle n_{\rm ch}\rangle = 0$). Thus, the entropic density for the coherent state is given by Eq. (12), while the Husimi phase distribution by Eq. (14), where $f_j = X_0^2 - X^2 + \ln\pi + j/2$ and $X = |\alpha_0|\cos(\theta - \theta_0)$. The s-parametrized phase distribution for the coherent state was obtained by Tanaś et al. [26]. Their solution in the special case for $s = -1$ reduces to the Husimi phase distribution $P_\theta(\alpha_0, n_{\rm ch} = 0)$, given by Eq. (14).

The first term in braces in Eq. (12) plays an essential role for small $|\alpha_0|$. By expanding Eq. (12) in power series of $|\alpha_0|$, and by putting $\langle n_{\rm ch}\rangle = 0$, we find

$$S_\theta = \frac{1+\ln\pi}{2\pi} + \frac{1+2\ln\pi}{4\sqrt{\pi}}|\alpha_0|\cos(\theta-\theta_0) + \mathcal{O}(|\alpha_0|^2), \tag{17}$$

which is the approximation of the entropic density (12) for coherent state with small amplitude. The corresponding Husimi phase distribution

$$P_\theta = \frac{1}{2\pi} + \frac{1}{2\sqrt{\pi}}|\alpha_0|\cos(\theta-\theta_0) + \mathcal{O}(|\alpha_0|^2) \tag{18}$$

is obtained from the exact Eq. (11). For $|\alpha_0| = 0$, both S_θ and P_θ are phase independent. For large $|\alpha_0|$, the second term of Eq. (12) predominates resulting in the following asymptotic formula

$$S_\theta \approx \frac{f_1}{\sqrt{\pi}} X \exp[-(X_0^2 - X^2)]. \tag{19}$$

The error function $\mathrm{erf}(X)$ in Eq. (12) was replaced by unity in the derivation of Eq. (19). Similarly, the asymptotic Husimi phase distribution is

$$P_\theta \approx \frac{1}{\sqrt{\pi}} X \exp[-(X_0^2 - X^2)]. \tag{20}$$

The asymptotic formulas (19) and (20) are valid for $-\pi/2 \le (\theta-\theta_0) \le \pi/2$ only. However,

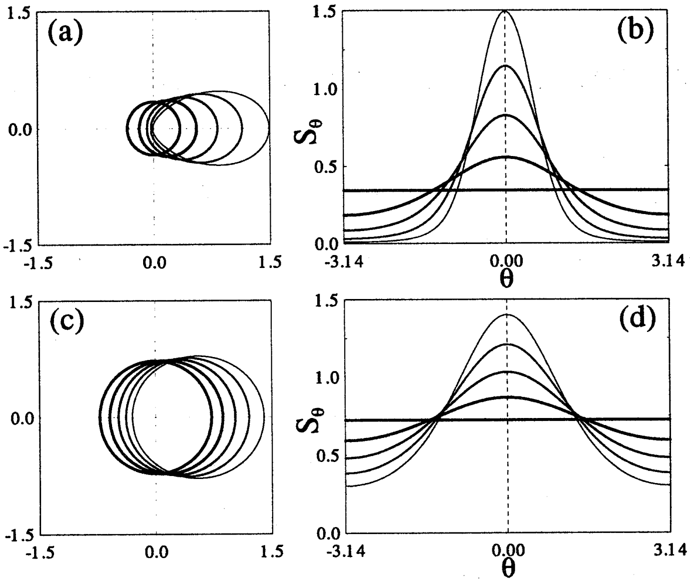

FIG. 2. Entropic density for coherent states [figs. (a) and (b)] and for superpositions of coherent signal and noise with $\langle n_{\rm ch}\rangle = 10$ [figs. (c) and (d)] for the coherent amplitude α_0 equal to: 0 (thickest solid line), 0.4, 0.8, 1.2, 1.6 (thinnest line).

Eq. (12), which is given in terms of the error function, holds for arbitrary phase θ. From Eq. (15) follows that the Wehrl entropy for a coherent state is constant, i.e. $S_{\rm W} = 1 + \ln\pi$, although its density is dependent on the amplitude α_0.

Entropic densities are depicted for the coherent signals in Figs. 2(a,b) and for the superpositions of the coherent signal and noise in Figs. 2(c,d) for various values of the mean photon numbers. It is seen that, for the fixed mean number of chaotic photons, the entropic density functions become sharper with the increasing mean number of coherent photons. It is a signature of the so-called phase locking. This phenomenon can be described within the optical phase formalisms [24,23]. Here, we have presented alternative entropic description of the phase locking. It is also evident, by comparing Figs. 2(b) and 2(d) for fixed number of photons $|\alpha_0|^2$ in the coherent signal, that (i) the area covered by the entropic density (which is the Wehrl entropy) increases, while (ii) the entropic density itself becomes less phase dependent with increasing noise. Thus, the entropic density serves as a measure of both (i) noise (phase-space uncertainty) and (ii) decoherence.

V. ENTROPIC DESCRIPTION OF PHASE BIFURCATION

The bifurcation phenomenon arising in the phase probability distribution of highly squeezed states was discovered by Schleich et al. [28]. We explain this effect by using the entropic density concept.

A. Squeezed states

The ideal squeezed states (two-photon coherent states) are defined to be [27]

$$|\alpha_0,\zeta\rangle = \widehat{D}(\alpha_0)\widehat{S}(\zeta)|0\rangle, \tag{21}$$

where $\widehat{D}(\alpha_0) = \exp(\alpha_0\widehat{a}^\dagger - \alpha_0^*\widehat{a})$ is the displacement operator with the complex displacement parameter α_0, and $\widehat{S}(\zeta) = \exp\left(\frac{1}{2}\zeta^*\widehat{a}^2 - \frac{1}{2}\zeta\widehat{a}^{\dagger 2}\right)$ is the unitary squeeze operator with the squeeze parameter ζ. For simplicity, we assume that ζ is real. The Husimi Q-function for the state (21) is

$$Q(\alpha) = \frac{1}{\pi\sigma_1\sigma_2}\exp\left\{-\frac{\mathrm{Im}^2(\alpha-\alpha_0)}{\sigma_1^2} - \frac{\mathrm{Re}^2(\alpha-\alpha_0)}{\sigma_2^2}\right\}, \tag{22}$$

where $\sigma_{1,2} = \sqrt{\frac{1}{2}(\mathrm{e}^{\pm 2\zeta}+1)}$. We find, by assuming that α_0 is real, the following entropic density for the squeezed state:

$$S_\theta = \frac{1}{2\pi}\frac{\sigma_1\sigma_2}{\sigma^2}\exp[-(X_0^2 - X^2)]\left\{\mathrm{e}^{-X^2}f_2 + \sqrt{\pi}X\,[1+\mathrm{erf}(X)]f_1\right\}, \tag{23}$$

where

$$f_j = \sigma_2^2\left(\frac{X_0^2}{\sigma^2} - \frac{X^2}{\sigma_1^2}\right) + \ln(\pi\sigma_1\sigma_2) + \frac{j}{2}, \qquad \text{and} \quad X = X(\theta) = \frac{\sigma_1}{\sigma_2\sigma}\alpha_0\cos\theta. \tag{24}$$

Moreover, $X_0 = X(\theta=0) = \alpha_0/\sigma_2$ and $\sigma^2 = \sigma_1^2\cos^2\theta + \sigma_2^2\sin^2\theta$. Eq. (23) for the squeezed state has the same structure as Eq. (12) for the coherent signal field with noise, apart from the extra coefficient $\sigma_1\sigma_2\sigma^{-2}$, and the modified definitions of the functions f_j and X. For $\zeta = 0$, which results in $\sigma = \sigma_1 = \sigma_2 = 1$, Eq. (23) reduces to a special case of Eq. (12) for real coherent state ($\theta_0 = \langle n_{\rm ch}\rangle = 0$). If we put $f(i) \to 1$, Eq. (23) will describe the Husimi phase distribution P_θ for the squeezed state obtained by Tanaś et al. [26,23]. By integrating the Husimi function (23), one readily obtains the Wehrl entropy

$$S_{\rm W} = 1 + \ln(\pi \cosh\zeta) \tag{25}$$

in agreement with the well-known results for the squeezed states obtained by Lee [17] and Orłowski [18]. Eq. (23) is valid for arbitrary value of the displacement parameter α_0. The entropic density S_θ for a highly displaced ($\alpha_0^2 \gg 1$) squeezed state can be approximated by

$$S_\theta \approx \frac{f_1}{\sqrt{\pi}} \frac{\sigma_1\sigma_2}{\sigma^2} X \exp[-(X_0^2 - X^2)]. \tag{26}$$

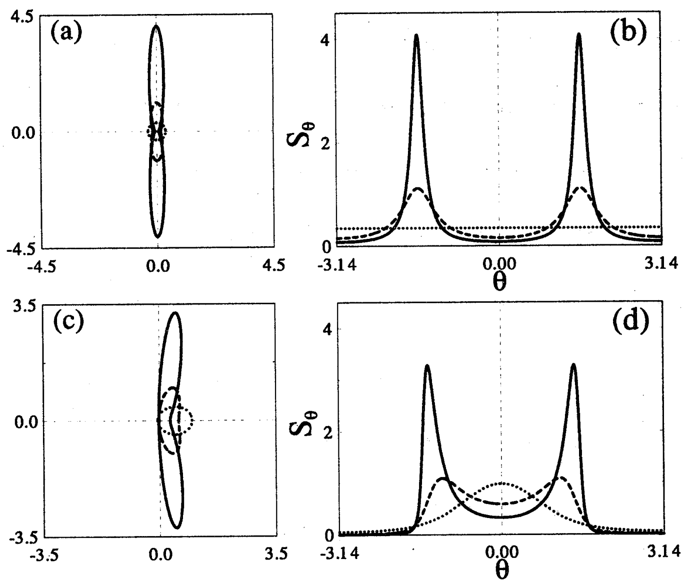

FIG. 3. Bifurcation of entropic density for squeezed states with: $\alpha_0 = 0$ (squeezed vacuum) [figs. (a) and (b)] and $\alpha_0 = 1$ [figs. (c) and (d)] for different values of the squeeze parameter: $\zeta = 0$ (dotted line), $\zeta = 1$ (dashed line), and $\zeta = 2$ (solid line).

For $\alpha_0^2 \ll 1$, Eq. (23) can be expanded in power series in α_0. We find the following small-amplitude entropic density

$$S_\theta = \frac{1}{2\pi}\frac{\sigma_1\sigma_2}{\sigma^2}\Big\{1 + \ln(\pi\sigma_1\sigma_2) + \sqrt{\pi}X\left[1/2 + \ln(\pi\sigma_1\sigma_2)\right]\Big\} + \mathcal{O}(\alpha_0^2). \tag{27}$$

For the squeezed vacuum, i.e., in the limit $\alpha_0 \to 0$, Eq. (27) reduces to $S_\theta = 1/(2\pi)\sigma_1\sigma_2\sigma^{-2}[1 + \ln(\pi\sigma_1\sigma_2)]$. The entropic densities are plotted in Figs. 3(a) and 3(b) for squeezed vacua and in Figs. 3(c) and 3(d) for squeezed states with the displacement parameter $\alpha_0 = 1$ and different values of the squeeze parameter ζ. The entropic density for a squeezed vacuum has a two-peak structure, which goes into a flat distribution in the limit of no squeezing ($\zeta = 0$). However, the entropic density (23) for the squeezed state with nonzero displacement parameter undergoes a transition from a single- to double-peak distribution either by increasing the squeeze parameter ζ or by decreasing the displacement parameter α_0. This behavior of the entropic density (23) arises from the competition between the coherent component exhibiting a single-peak structure and the squeezed-vacuum component having a double-peak structure. The above transition of the entropic density indicates the phase bifurcation phenomenon discovered by Schleich et al. [28] by analyzing the Pegg-Barnett phase distribution with increasing value of the product of the squeeze and displacement parameters for the state (21). Alternatively, the phase bifurcation can be analyzed in terms of the marginal quasiprobability distributions [26,23]. The squeezed-state phase bifurcation has also a simple physical interpretation according to the principle of the area of overlap in the phase space [28].

In the limit of strong squeezing $\zeta \gg \alpha_0^2$, the peaks of the entropic density (23) are centered at $\theta = \pm\pi/2$ and can approximately be expressed by

$$S_\theta \approx \frac{\exp(\zeta - 2\alpha_0^2)}{2\pi}[\zeta + 2\alpha_0^2 + \ln(\pi/2) + 1], \tag{28}$$

which shows that the peak height of S_θ is proportional to ζe^ζ if we neglect the parameter α_0 small in comparison with ζ. In the same squeezing limit but for $\theta \neq \pm\pi/2$, Eq. (23) simplifies to

$$S_\theta \approx \frac{\mathrm{e}^{-\zeta}}{2\pi}\sec^2\theta\Big\{\mathrm{e}^{-2\alpha_0^2}f_2 + \sqrt{2\pi}\alpha_0\left[\frac{\cos\theta}{|\cos\theta|} + \mathrm{erf}(\sqrt{2}\alpha_0)\right]f_1\Big\}, \qquad (29)$$

where $f_j \approx \ln(\pi/2) + s + j/2$ is obtained as the approximation of Eq. (24). For squeezed vacuum, Eq. (29) reduces to $S_\theta \approx 1/(2\pi)f_2\mathrm{e}^{-\zeta}\sec^2\theta$. Eq. (29) is proportional to $\exp(-\zeta)$, so the entropic density is negligible for θ not close to $\pm\pi/2$. The exact entropic density (23) is presented graphically both in polar [Figs. 3(a,c)] and in Cartesian coordinates [Figs. 3(b,d)] in agreement with the approximate Eqs. (28)–(29) for the case fulfilling the condition $\zeta \gg \alpha_0^2$. Finally, we conclude that the entropic density (23), having a structure with two sharp peaks in the limit of strong squeezing, properly describes the phase bifurcation effect.

VI. ENTROPIC DESCRIPTION OF SUPERPOSITION PRINCIPLE

Schrödinger cat or cat-like states (kittens) are the striking manifestations of the superposition principle in mesoscopic systems. Although, the Schrödinger cats have been experimentally generated only recently [30,31], they have attracted lots of interest in quantum and atom optics, quantum cryptography or quantum computing by allowing controlled studies of quantum measurement or quantum entanglement and decoherence. We will analyze simple prototypes of the Schrödinger cats and kittens states to show that the entropic density is a good indicator of their formation and properties.

A. Schrödinger cat states

Superposition of two coherent states $|\alpha_0\rangle$ and $|-\alpha_0\rangle$ in the form

$$|\alpha_0,\gamma\rangle = c(\gamma)\left\{|\alpha_0\rangle + \exp(\mathrm{i}\gamma)|-\alpha_0\rangle\right\}, \qquad (30)$$

with the normalization $c(\gamma) = \left\{2\left[1+\cos\gamma\exp(-2|\alpha_0|^2)\right]\right\}^{-1/2}$, is an example of the Schrödinger cat (see, e.g., [32]). For the special choices of the superposition phase γ, the state (30) reduces to the well-known Schrödinger cats, including the Yurke-Stoler coherent state for $\gamma = \pi/2$ [33]:

$$|\alpha_0\rangle_{\rm YS} = |\alpha_0, \pi/2\rangle = \frac{1}{\sqrt{2}}\left(|\alpha_0\rangle + {\rm i}| - \alpha_0\rangle\right), \tag{31}$$

and the even ($\gamma = 0$) and odd ($\gamma = \pi$) coherent states [20]:

$$|\alpha_0, 0\rangle = c(0)\left(|\alpha_0\rangle + | - \alpha_0\rangle\right) = \frac{1}{\sqrt{\cosh|\alpha_0|^2}}\sum_{n=0}^{\infty}\frac{\alpha_0^{2n}}{\sqrt{(2n)!}}|2n\rangle, \tag{32}$$

$$|\alpha_0, \pi\rangle = c(\pi)\left(|\alpha_0\rangle - | - \alpha_0\rangle\right) = \frac{1}{\sqrt{\sinh(|\alpha_0|^2)}}\sum_{n=0}^{\infty}\frac{\alpha_0^{2n+1}}{\sqrt{(2n+1)!}}|2n+1\rangle, \tag{33}$$

respectively. The Husimi function for the cat (30) can be given as the sum

$$Q(\alpha) = c(\gamma)^2[Q_1(\alpha) + 2Q_{12}(\alpha) + Q_2(\alpha)] \tag{34}$$

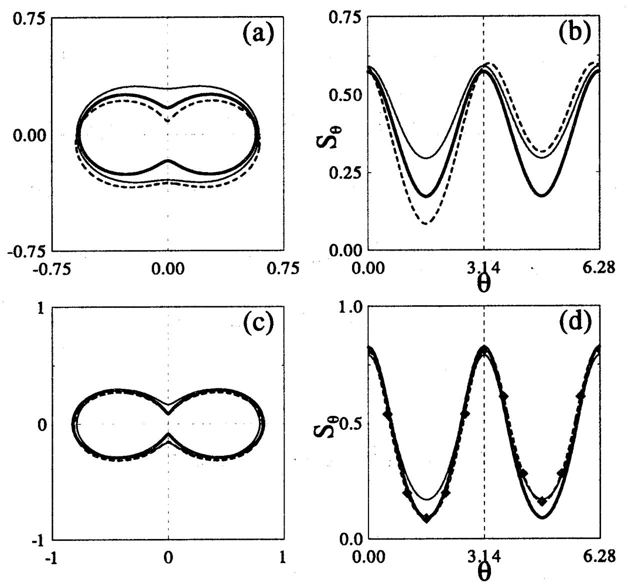

FIG. 4. Entropic density for the three types of Schrödinger cats: even coherent states (thick solid line), odd coherent states (thin solid line), and Yurke-Stoler coherent states (dashed line, additionally with diamonds in figure (d)) for the two values of the coherent amplitude: $\alpha_0 = 0.8$ [figs. (a) and (b)] and $\alpha_0 = 1.2$ [figs. (c) and (d)].

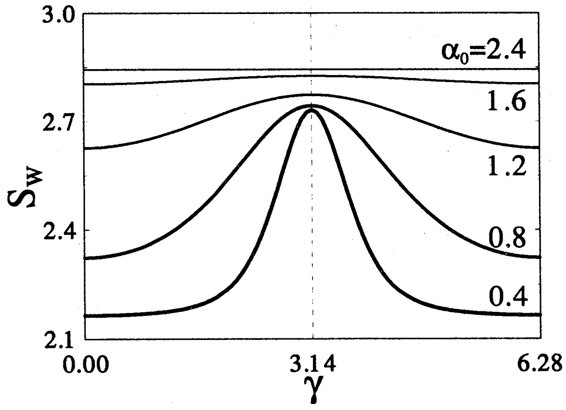

FIG. 5. Wehrl entropy for Schrödinger cat $|\alpha_0, \gamma\rangle$ in its dependence on the superposition phase γ for different values of the coherent amplitude α_0.

of coherent terms ($k = 1, 2$)

$$Q_k(\alpha) = \frac{1}{\pi} \exp\left\{ - \left|\alpha + (-1)^k \alpha_0\right|^2 \right\} \tag{35}$$

and the interference term

$$Q_{12}(\alpha) = \frac{1}{\pi} \exp(-|\alpha|^2 - |\alpha_0|^2) \cos[\gamma + 2|\alpha|\,|\alpha_0| \sin(\theta - \theta_0)]. \tag{36}$$

There is no compact-form exact expression for the entropic densities for the cat (30). The states analyzed in the former sections are among a few examples, for which entropic densities can be expressed analytically in compact form. Nevertheless, the entropic density for the well separated ($|\alpha_0| \gg 1$) coherent states $|\alpha_0\rangle$ and $|-\alpha_0\rangle$ in the superposition (30) can be approximated

$$S_\theta \approx \frac{1}{2}\Big\{ S_\theta(\alpha_0, 0) + S_\theta(-\alpha_0, 0) + \left[P_\theta(\alpha_0, 0) + P_\theta(-\alpha_0, 0)\right] \ln 2 \Big\}, \tag{37}$$

where the entropic densities for $S_\theta(\pm\alpha_0, 0)$ are given by Eq. (12) and the Husimi phase distributions $P_\theta(\pm\alpha_0, 0)$ are defined by Eq. (14) in the special case for $\langle n_{\rm ch}\rangle = 0$. In Fig. 4, we have presented the entropic densities for the Yurke-Stoler coherent state, and the even and odd coherent states. Figs. 4(a) and 4(c) are plotted in polar coordinates to show more explicitly the two-peak structures of the cats, whereas Figs. 4(b) and 4(d) are depicted in Cartesian coordinates for better comparison of the values of the entropic densities. It is seen that the maximum values of the entropic densities are dependent on the amplitude. By analyzing Figs. 4(a,b) with Figs. 4(c,d), one can conclude that differences between the cats (31)–(33) diminish with the increasing amplitude α_0.

The Wehrl entropy for the cat (30) has been studied numerically by Bužek et al. [10] for arbitrary superposition phase γ. Whereas, Jex and Orłowski [19] and Vaccaro and Orłowski [13] studied the Wehrl entropy for the Yurke-Stoler coherent state generated in a Kerr-like medium. In Fig. 5, we show the Wehrl entropies $S_{\rm W}$ for the cat (30) in their dependence on the superposition phase γ for various values of the separation amplitude α_0. The curve for $\alpha_0 = 0.8$ corresponds to the case analyzed by Bužek et al. [10,21]. The differences between Wehrl entropies for the cats (31), (32) and (33) vanish with increasing $|\alpha_0|$. The Wehrl entropy in the amplitude limit, $|\alpha_0| \to \infty$, tends to $S_{\rm W} = 1 + \ln(2\pi)$. This value can be obtained by integrating the approximate entropic density (37). On the scale of Fig. 5, the curve representing the Wehrl entropy for $\alpha_0 = 2.4$ is practically indistinguishable from $1 + \ln(2\pi)$, i.e., the entropy in the infinite-amplitude limit.

B. Schrödinger cat-like states

The Schrödinger cat-like state (kitten state [29]) is a generalization of the Schrödinger cat for macroscopic quantum-superposition state with more than two components. In particular, the normalized superposition of N coherent states:

$$|\psi\rangle = \sum_{k=1}^{N} c_k |\exp(\mathrm{i}\phi_k)\alpha_0\rangle \tag{38}$$

is the standard example the Schrödinger cat for $N = 2$ and the Schrödinger cat-like state for $N > 2$. The cat (30) consists of two coherent states with the same amplitude, but opposite phases ($\phi_2 - \phi_1 = \pi$). Eq. (38) generalizes Eq. (30) for arbitrary number, amplitude and phase of the states in the superposition. The Husimi Q-function for the superposition state (38) reads as [34,32]:

$$Q(\alpha) = Q_0(\alpha) + Q_{\rm int}(\alpha) = \sum_{k=1}^{N} |c_k|^2\, Q_k(\alpha) + 2\sum_{k>l} |c_k||c_l|\, Q_{kl}(\alpha), \tag{39}$$

where the free part $Q_0(\alpha)$ is the sum of the coherent terms

$$Q_k(\alpha) = \frac{1}{\pi}\exp\left\{-\left|\alpha - {\rm e}^{{\rm i}\phi_k}\alpha_0\right|^2\right\}, \tag{40}$$

and the interference part $Q_{\rm int}(\alpha)$ is given in terms of

$$Q_{kl}(\alpha) = \sqrt{Q_k Q_l}\cos\left[\gamma_k - \gamma_l + 2|\alpha|\;|\alpha_0|\cos(\phi_{kl}^{(+)} + \theta_0 - \theta)\sin(\phi_{kl}^{(-)})\right]. \tag{41}$$

The phases in Eq. (41) are defined as $\gamma_k = {\rm Arg}\, c_k$, $\theta = {\rm Arg}\,\alpha$, $\theta_0 = {\rm Arg}\,\alpha_0$, and $\phi_{kl}^{(\pm)} = \frac{1}{2}(\phi_k \pm \phi_l)$, where ϕ_k occurs in Eq. (38). The Husimi function (39) is a generalization of Eq. (34). In particular, Eq. (41) reduces to Eq. (36) for $N = 2$. There exists exact analytical compact-form expression neither for the entropic density nor the Wehrl entropy. However, for well separated states, the entropic density for the Schrödinger cat-like state (38) is approximately equal to

$$S_\theta \approx \frac{1}{N}\sum_{k=1}^{N}\left\{S_\theta({\rm e}^{{\rm i}\phi_k}\alpha_0, 0) + P_\theta({\rm e}^{{\rm i}\phi_k}\alpha_0, 0)\ln N\right\}, \tag{42}$$

where the coherent-field entropic densities $S_\theta(\exp\{{\rm i}\phi_k\}\alpha_0, 0)$ are given by Eq. (12), while the coherent-field Husimi phase distributions $P_\theta(\exp\{{\rm i}\phi_k\}\alpha_0, 0)$ read as Eq. (14). In Eq. (42), we have assumed for simplicity that superposition coefficients are the same for all components of the cat-like state, i.e., $c_k = {\rm const} = 1/\sqrt{N}$. The approximate entropic density (42) leads, after integration according to Def. (4), to the Wehrl entropy $S_{\rm W} \approx 1 + \ln(N\pi)$ in agreement with the Jex and Orlowski result [19,21].

There have been proposed several methods to produce Schrödinger's cat or cat-like states (see Refs. [30,31] and [32] for review). In particular, it has been predicted that the coherent light propagating in a Kerr-like medium, described by the Hamiltonian

$$\widehat{H} = \hbar\omega\widehat{a}^\dagger\widehat{a} + \frac{1}{2}\hbar\kappa\widehat{a}^{\dagger 2}\widehat{a}^2, \tag{43}$$

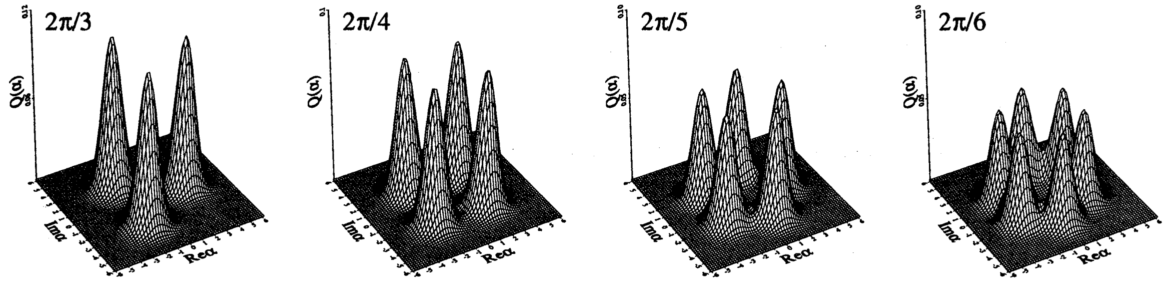

FIG. 6. Husimi Q-function for Schrödinger cat-like states generated in Kerr medium at different evolution times $\kappa t = 2\pi/N$ for the field initially coherent with the mean photon number $|\alpha_0|^2 = 9$.

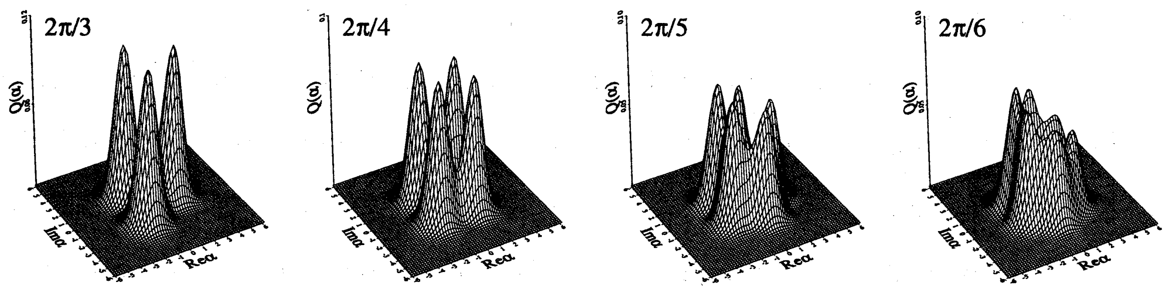

FIG. 7. Same as in Fig. 6, but for the initial condition $|\alpha_0|^2 = 4$.

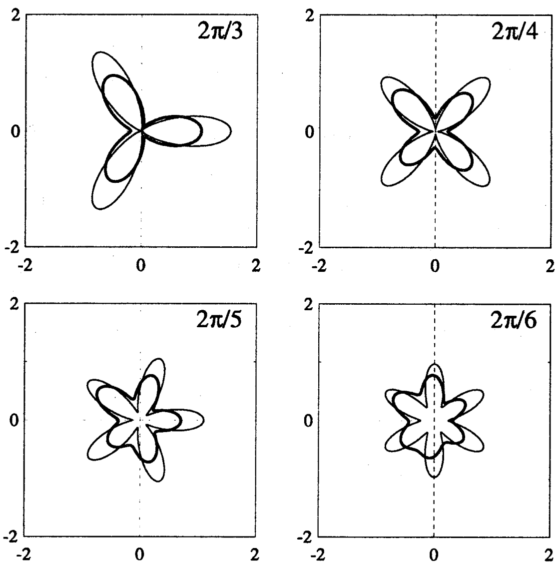

FIG. 8. Entropic density for Schrödinger cat-like states for the same cases as in Fig. 6 (depicted by thin lines) and in Fig. 7 (thick lines).

can be transformed into the cat (Yurke-Stoler coherent state) [33] but also into the Schrödinger kittens [34]. Jex and Orłowski [19] and Vaccaro and Orłowski [13] have shown, by analyzing the model (43), that the Wehrl entropy gives a signature of the formation of the Schrödinger cat-like states. The entropic density, in comparison to the Wehrl entropy, offers more detailed description of cats and kittens, showing explicitly the phase configuration and amplitudes of the components. The Husimi Q-function and the entropic densities for the Schrödinger cat-like states generated in the model (43) are presented in Figs. 6–8 for different evolution times κt, where κ is the coupling constant in the Hamiltonian (43). The number of peaks in S_θ clearly corresponds to the number of components in the superposition state. The entropic densities also show how the amplitude of the incident coherent beam determines the maximum number of well-distinguishable states. For example, both the Husimi Q-function in Fig. 6 and the entropic density depicted by thin line in Fig. 8 have regular and well-distinguishable structures even of six-component superposition for the initial amplitude $|\alpha_0| = 3$. However, as plotted in Figs. 7 and 8 (thick-line plots), the six-component superposition for the initial condition $|\alpha_0| = 2$ is highly deformed. Thus, the entropic density describes the influence of the interference terms (41) on formation of the Schrödinger cat-like states.

VII. CONCLUSIONS

We have proposed a definition of the phase-density of the Wehrl entropy as a measure of the various quantum-mechanical and information-theoretic properties of optical fields. We have shown that the entropic density is more sensitive measure of, e.g., phase properties than the Wehrl entropy. In particular, our measure clearly describes phase decoherence (phase randomization), enhancement of coherence (including phase locking) and phase bifurcation of quantum systems. The conventional Wehrl entropy offers a crude estimation of decoherence only [11,15,16], and it describes neither the high-intensity coherent-state phase locking nor the squeezed-state phase bifurcation. Moreover, the entropic density of Wehrl entropy uniquely distinguishes the number and phase-space configuration of the two- (Schrödinger cats) or multicomponent (Schrödinger kittens) quantum superpositions of macroscopically distinguishable states. In contrast, the conventional Wehrl entropy contains no information about the phase-space configuration and does not always distinguish macroscopic quantum superpositions with different number or amplitude of their components.

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