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On Beam Splittings and Mathematical Construction of Quantum Logical Gate

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Abstracts

In usual computer, there exists an upper bound of computational speed because of irreversibility of logical gate. In order to avoid this demerit, Fredkin and Toffoli [3] proposed a conservative logical gate. Based on their work, Milburn [4] constructed a physical model of reversible quantum logical gate with beam splittings and a Kerr medium. This model is called FTM (Fredkin - Toffoli - Milburn gate) in this paper.This FTM gate was described by the quantum channel and the efficiency of information transmission of the FTM gate was discussed in [10]. FTM gate is using a photon number state as an input state for control gate. The photon number state might be difficult to realize physically. In this paper, we introduced a new device on symmetric Fock space in order to avoid this difficulty.

In Section 1, we briefly review quantum channels and beam splittings. In Section 2, we explain the quantum channel for FTM gate In Section 3, we intruduced a new device on symmetric Fock space and discuss the truth table for our gate.

1. Quantum channels

Let $(\mathbf{B}(\mathcal{H}_1), \mathfrak{S}(\mathcal{H}_1))$and $(\mathbf{B}(\mathcal{H}_2), \mathfrak{S}(\mathcal{H}_2))$ be input and output systems, respectively, where $\mathbf{B}(\mathcal{H}_k)$ is the set of all bounded linear operators on a separable Hilbert space $\mathcal{H}_k$ and $\mathfrak{S}(\mathcal{H}_k)$ is the set of all density operators on $\mathcal{H}_k$ $(k=1,2)$. Quantum channel Λ^* is a mapping from $\mathfrak{S}(\mathcal{H}_1)$ to $\mathfrak{S}(\mathcal{H}_2)$.

(1) Λ^* is linear if $\Lambda^*(\lambda\rho_1 + (1-\lambda)\rho_2) = \lambda\Lambda^*(\rho_1) + (1-\lambda)\Lambda^*(\rho_2)$ holds for any $\rho_1, \rho_2 \in \mathfrak{S}(\mathcal{H}_1)$ and any $\lambda \in [0,1]$.

(2) Λ^* is completely positive (C.P.) if Λ^* is linear and its dual $\Lambda : \mathbf{B}(\mathcal{H}_2) \to \mathbf{B}(\mathcal{H}_1)$ satisfies

$$\sum_{i,j=1}^{n} A_i^* \Lambda(\overline{A}_i^* \overline{A}_j) A_j \geq 0$$

for any $n \in \mathbf{N}$, any $\{\overline{A}_i\} \subset \mathbf{B}(\mathcal{H}_2)$ and any $\{A_i\} \subset \mathbf{B}(\mathcal{H}_1)$, where the dual map Λ of Λ^* is defined by

$$tr\Lambda^*(\rho)B = tr\rho\Lambda(B), \quad \forall\rho \in \mathfrak{S}(\mathcal{H}_1), \quad \forall B \in \mathbf{B}(\mathcal{H}_2). \tag{1.1}$$

Almost all physical transformation can be described by the CP channel [5], [7], [8]

Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two Hilbert spaces expressing noise and loss systems, respectively. Quantum communication process including the influence of noise and loss is denoted by the following scheme [6]: Let ρ be an input state in $\mathfrak{S}(\mathcal{H}_1)$, ξ be a noise state in $\mathfrak{S}(\mathcal{K}_1)$.

$$\begin{array}{ccc} & \xi \in \mathfrak{S}(\mathcal{K}_1) & \\ & \downarrow & \\ \mathfrak{S}(\mathcal{H}_1) \ni \rho & \xrightarrow{\hspace{6cm}} & \rho \in \Lambda^*\rho \in \mathfrak{S}(\mathcal{H}_2) \\ & \quad\quad\quad\quad\quad\quad \downarrow & \\ & \quad\quad\quad\quad\quad\quad \text{Loss} & \end{array}$$

$$\begin{array}{ccc} \mathfrak{S}(\mathcal{H}_1) & \xrightarrow{\Lambda^*} & \mathfrak{S}(\mathcal{H}_2) \\ \gamma^* \downarrow & & \uparrow a^* \\ \mathfrak{S}(\mathcal{H}_1 \otimes \mathcal{K}_1) & \xrightarrow[\Pi^*]{} & \mathfrak{S}(\mathcal{H}_2 \otimes \mathcal{K}_2) \end{array}$$

The above maps γ^*, a^* are given as

$$\gamma^*(\rho) = \rho \otimes \xi, \quad \rho \in \mathfrak{S}(\mathcal{H}_1), \tag{1.2}$$

$$a^*(\sigma) = tr_{\mathcal{K}_2}\sigma, \quad \sigma \in \mathfrak{S}(\mathcal{H}_2 \otimes \mathcal{K}_2), \tag{1.3}$$

The map Π^* is a channel from $\mathfrak{S}(\mathcal{H}_1 \otimes \mathcal{K}_1)$to $\mathfrak{S}(\mathcal{H}_2 \otimes \mathcal{K}_2)$ determined by physical properties of the device transmitting information. Hence the channel for the above process is given as

$$\Lambda^*(\rho) \equiv tr_{\mathcal{K}_2}\Pi^*(\rho \otimes \xi) = (a^* \circ \Pi^* \circ \gamma^*)(\rho) \tag{1.4}$$

for any $\rho \in \mathfrak{S}(\mathcal{H}_1)$. Based on this scheme, the attenuation channel and the noisy quantum channel are constructed as follows:

(1) Attenuation channel Λ_0^* was formulated such as

$$\begin{aligned}\Lambda_0^*(\rho) &\equiv tr_{\mathcal{K}_2}\Pi_0^*(\rho\otimes\xi_0)\\ &= tr_{\mathcal{K}_2}V_0(\rho\otimes|0\rangle\langle 0|)V_0^*, \end{aligned}\tag{1.5}$$

where $\xi_0 = |0\rangle\langle 0|$ is the vacuum state in $\mathfrak{S}(\mathcal{K}_1)$, V_0 is a mapping from $\mathcal{H}_1\otimes\mathcal{K}_1$ to $\mathcal{H}_2\otimes\mathcal{K}_2$ given by

$$V_0(|n_1\rangle\otimes|0\rangle) = \sum_j^{n_1} C_j^{n_1}|j\rangle\otimes|n_1-j\rangle, \tag{1.6}$$

$$C_j^{n_1} = \sqrt{\frac{n_1!}{j!(n_1-j)!}}\alpha^j\left(-\bar{\beta}\right)^{n_1-j}. \tag{1.7}$$

$|n_1\rangle$ is the n_1 photon number state vector in $\mathcal{H}_1$ and α and β are complex numbers satisfying $|\alpha|^2+|\beta|^2=1$. In particular, for the coherent input state $\rho = |\theta\rangle\langle\theta|\otimes|0\rangle\langle 0| \in \mathfrak{S}(\mathcal{H}_1\otimes\mathcal{K}_1)$, we obtain the output state of Π_0^* by

$$\Pi_0^*\left(|\theta\rangle\langle\theta|\otimes|0\rangle\langle 0|\right) = |\alpha\theta\rangle\langle\alpha\theta|\otimes\left|-\bar{\beta}\theta\right\rangle\left\langle-\bar{\beta}\theta\right|.$$

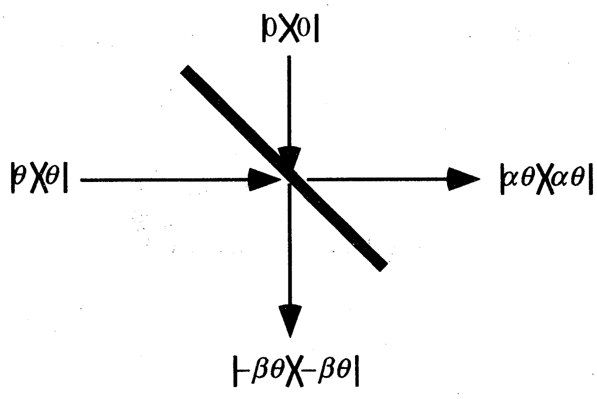

Fig 1.1 Beam Splitting π_0^*

Lifting $\mathcal{E}_0^*$ from $\mathfrak{S}(\mathcal{H})$ to $\mathfrak{S}(\mathcal{H}\otimes\mathcal{K})$ in the sense of Accardi and Ohya [1] is denoted by

$$\mathcal{E}_0^*\left(|\theta\rangle\langle\theta|\right) = |\alpha\theta\rangle\langle\alpha\theta|\otimes|\beta\theta\rangle\langle\beta\theta|.$$

$\mathcal{E}_0^*$ (or Π_0^*) isß called a beam splitting. Based on liftings, the beam splitting was studied by Accardi - Ohya and Fichtner - Freudenberg - Libsher [2].

(2) Noisy quantum channel Λ^* with a noise state ξ is defined by

$$\begin{aligned} \Lambda^*(\rho) &\equiv tr_{\mathcal{K}_2}\Pi^*(\rho\otimes\xi) \\ &= tr_{\mathcal{K}_2}V(\rho\otimes\xi)V^*, \end{aligned} \tag{1.8}$$

where $\xi = |m_1\rangle\langle m_1|$ is the m_1 photon number state in $\mathfrak{S}(\mathcal{K}_1)$ and V is a mapping from $\mathcal{H}_1\otimes\mathcal{K}_1$ to $\mathcal{H}_2\otimes\mathcal{K}_2$ denoted by

$$V(|n_1\rangle\otimes|m_1\rangle) = \sum_j^{n_1+m_1} C_j^{n_1,m_1}|j\rangle\otimes|n_1+m_1-j\rangle,$$

$$\begin{aligned} &C_j^{n_1,m_1} \\ &= \sum_{r=L}^{K}(-1)^{n_1+j-r}\frac{\sqrt{n_1!m_1!j!(n_1+m_1-j)!}}{r!(n_1-j)!(j-r)!(m_1-j+r)!} \\ &\quad\times\alpha^{m_1-j+2r}\left(-\bar{\beta}\right)^{n_1+j-2r} \end{aligned} \tag{1.9}$$

K and L are constants given by $K=\min\{n_1,j\}$, $L=\max\{m_1-j,0\}$. In particular for the coherent input state $\rho = |\theta\rangle\langle\theta|\otimes|\kappa\rangle\langle\kappa|\in\mathfrak{S}(\mathcal{H}_1\otimes\mathcal{K}_1)$, we obtain the output state of Π^* by

$$\Pi^*(|\theta\rangle\langle\theta|\otimes|\kappa\rangle\langle\kappa|) = |\alpha\theta+\beta\kappa\rangle\langle\alpha\theta+\beta\kappa|\otimes|-\bar{\beta}\theta+\alpha\kappa\rangle\langle-\bar{\beta}\theta+\alpha\kappa|.$$

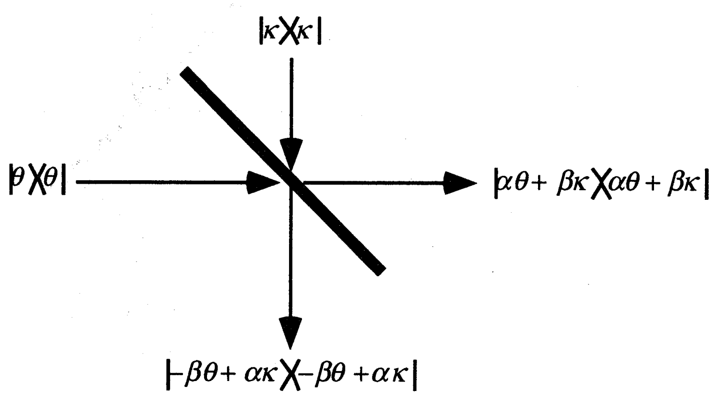

Fig 1.2 Generalized Beam Splitting π^*

Π^* was defined by Ohya - Watanabe [9], which is called a generalized beam splitting.

2. Quantum channel for Fredkin-Toffoli-Milburn gate

In usual computer, we could not determine two inputs for the logical gates AND and OR after we know the output for these gates. This property is called an irreversibility of logical gate. This property leads to the loss of information and the heat generation. Thus there exists an upper bound of computational speed.

Fredkin and Toffoli proposed a conservative gate, by which any logical gate is realized and it is shown to be a reversible gate in the sense that there is no loss of information. This gate was developed by Milburn as a quantum gate with quantum input and output. We call this gate Fredkin-Toffoli-Milburn (FTM) gate here. Recently, we reformulate a quantum channel for the FTM gate and we rigorously study the conservation of information for FTM gate [10].

The FTM gate is composed of two input gates I_1, I_2 and one control gate C. Two inputs come to the first beam splitter and one spliting input passes through the control gate made from an optical Kerr device, then two spliting inputs come in the second beam splitter and appear as two outputs (Fig.2.1). Two beam splitters and the optical Kerr medium are needed to describe the gate.

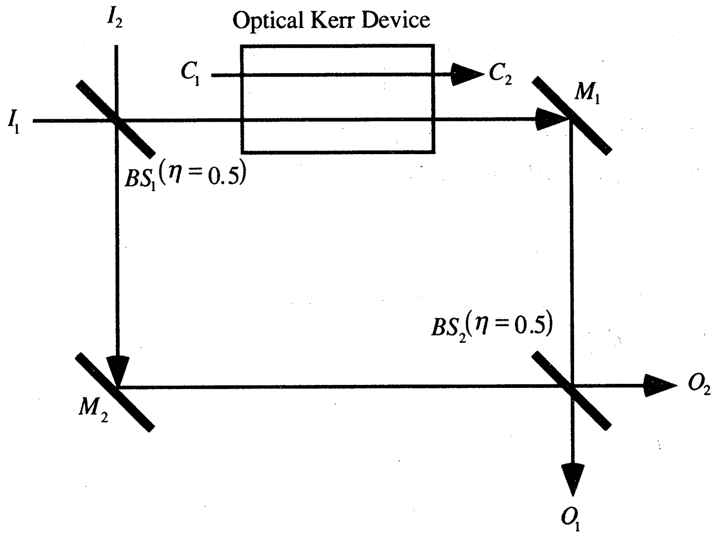

Fig 2.1 FTM gate

(1) Beam splitters: (a) Based on [9], let V_1 be a mapping from $\mathcal{H}_1 \otimes \mathcal{H}_2$ to $\mathcal{H}_1 \otimes \mathcal{H}_2$ with transmission rate η_1 given by

$$V_1(|n_1\rangle \otimes |n_2\rangle) \equiv \sum_{j=0}^{n_1+n_2} C_j^{n_1,n_2} |j\rangle \otimes |n_1+n_2-j\rangle \tag{2.1}$$

for any photon number state vectors $|n_1\rangle\otimes|n_2\rangle \in \mathcal{H}_1\otimes\mathcal{H}_2$. The quantum channel Π_{BS1}^* expressing the first beam splitter (beam splitter 1) is defined by

$$\Pi_{BS1}^*(\rho_1 \otimes \rho_2) \equiv V_1(\rho_1 \otimes \rho_2) V_1^* \tag{2.2}$$

for any states $\rho_1 \otimes \rho_2 \in \mathfrak{S}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. In particular, for an input state in two gates I_1 and I_2 given by the tensor product of two coherent states $\rho_1 \otimes \rho_2 = |\theta_1\rangle\langle\theta_1| \otimes |\theta_2\rangle\langle\theta_2|$, $\Pi_{BS1}^*(\rho_1 \otimes \rho_2)$ is written as

$$\begin{aligned}\Pi_{BS1}^*(\rho_1 \otimes \rho_2) &= \left|\sqrt{\eta_1}\theta_1 + \sqrt{1-\eta_1}\theta_2\right\rangle \left\langle\sqrt{\eta_1}\theta_1 + \sqrt{1-\eta_1}\theta_2\right| \\ &\otimes \left|-\sqrt{1-\eta_1}\theta_1 + \sqrt{\eta_1}\theta_2\right\rangle \left\langle-\sqrt{1-\eta_1}\theta_1 + \sqrt{\eta_1}\theta_2\right|. \end{aligned}\tag{2.3}$$

(b) Let V_2 be a mapping from $\mathcal{H}_1 \otimes \mathcal{H}_2$ to $\mathcal{H}_1 \otimes \mathcal{H}_2$ with transmission rate η_2 given by

$$V_2(|n_1\rangle \otimes |n_2\rangle) \equiv \sum_{j=0}^{n_1+n_2} C_j^{n_2,n_1} |n_1+n_2-j\rangle \otimes |j\rangle \tag{2.4}$$

for any photon number state vectors $|n_1\rangle\otimes|n_2\rangle \in \mathcal{H}_1\otimes\mathcal{H}_2$. The quantum channel Π_{BS2}^* expressing the second beam splitter (beam splitter 2) is defined by

$$\Pi_{BS2}^*(\rho_1 \otimes \rho_2) \equiv V_2(\rho_1 \otimes \rho_2) V_2^* \tag{2.5}$$

for any states $\rho_1 \otimes \rho_2 \in \mathfrak{S}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. In particular, for coherent input states $\rho_1 \otimes \rho_2 = |\theta_1\rangle\langle\theta_1| \otimes |\theta_2\rangle\langle\theta_2|$, $\Pi_{BS2}^*(\rho_1 \otimes \rho_2)$ is written as

$$\begin{aligned}\Pi_{BS2}^*(\rho_1 \otimes \rho_2) &= \left|\sqrt{\eta_2}\theta_1 - \sqrt{1-\eta_2}\theta_2\right\rangle \left\langle\sqrt{\eta_2}\theta_1 - \sqrt{1-\eta_2}\theta_2\right| \\ &\otimes \left|\sqrt{1-\eta_2}\theta_1 + \sqrt{\eta_2}\theta_2\right\rangle \left\langle\sqrt{1-\eta_2}\theta_1 + \sqrt{\eta_2}\theta_2\right|. \end{aligned}\tag{2.6}$$

(2) Optical Kerr medium: The interaction Hamiltonian in the optical Kerr medium is given by the number operators N_1 and N_c for the input system 1 and the Kerr medium, respectively, such as

$$H_{int} = \hbar\chi(N_1 \otimes I_2 \otimes N_c), \tag{2.7}$$

where $\hbar$ is the Plank constant divided by 2π, χ is a constant proportional to the susceptibility of the medium and I_2 is the identity operator on $\mathcal{H}_2$. Let T be the passing time of a beam through the Kerr medium and put $\sqrt{F} = \hbar\chi T$, a parameter exhibiting the power of the Kerr effect. Then the unitary operator U_K describing the evolution for time T in the Kerr medium is given by

$$U_K = \exp\left(-i\sqrt{F}\,(N_1 \otimes I_2 \otimes N_c)\right). \tag{2.8}$$

We assume that an initial (input) state of the control gate is the n photon number state $\xi = |n\rangle\langle n|$, a quantum channel Λ_K^* representing the optical Kerr effect is given by

$$\Lambda_K^*(\rho_1 \otimes \rho_2 \otimes \xi) \equiv U_K(\rho_1 \otimes \rho_2 \otimes \xi)U_K^* \tag{2.9}$$

for any state $\rho_1 \otimes \rho_2 \otimes \xi \in \mathfrak{S}(\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{K})$. In particular, for an initial state $\rho_1 \otimes \rho_2 \otimes \xi = |\theta_1\rangle\langle\theta_1| \otimes |\theta_2\rangle\langle\theta_2| \otimes |n\rangle\langle n|$, $\Lambda_K^*(\rho_1 \otimes \rho_2 \otimes \xi)$ is denoted by

$$\begin{aligned} &\Lambda_K^*(\rho_1 \otimes \rho_2 \otimes \xi) \\ = &\left| \exp\left(-i\sqrt{F}n\right)\theta_1 \right\rangle \left\langle \exp\left(-i\sqrt{F}n\right)\theta_1 \right| \\ &\otimes |\theta_2\rangle\langle\theta_2| \otimes |n\rangle\langle n|, \end{aligned} \tag{2.10}$$

Using the above channels, the quantum channel for the whole FTM gate is constructed as follows: Let both one input and output gates be described by $\mathcal{H}_1$, another input and output gates be described by $\mathcal{H}_2$ and the control gate be done by $\mathcal{K}$, all of which are Fock spaces. For a total state $\rho_1 \otimes \rho_2 \otimes \xi$ of two input states and a control state, the quantum channels $\Lambda_{BS1}^*, \Lambda_{BS2}^*$ from $\mathfrak{S}(\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{K})$ to $\mathfrak{S}(\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{K})$ are written by

$$\Lambda_{BSk}^*(\rho_1 \otimes \rho_2 \otimes \xi) = \Pi_{BSk}^*(\rho_1 \otimes \rho_2) \otimes \xi \quad (k = 1, 2) \tag{2.11}$$

Therefore, the whole quantum channel Λ_{FTM}^* of the FTM gate is defined by

$$\Lambda_{\mathrm{FTM}}^* \equiv \Lambda_{BS2}^* \circ \Lambda_K^* \circ \Lambda_{BS1}^*. \tag{2.12}$$

In particular, for an initial state $\rho_1 \otimes \rho_2 \otimes \xi = |\theta_1\rangle\langle\theta_1| \otimes |\theta_2\rangle\langle\theta_2| \otimes |n\rangle\langle n|$, $\Lambda_{\mathrm{FTM}}^*(\rho_1 \otimes \rho_2 \otimes \xi)$ is obtained by

$$\begin{aligned} &\Lambda_{\mathrm{FTM}}^*(\rho_1 \otimes \rho_2 \otimes \xi) \\ = &|\mu_n\theta_1 + \nu_n\theta_2\rangle\langle\mu_n\theta_1 + \nu_n\theta_2| \\ &\otimes |\nu_n\theta_1 + \mu_n\theta_2\rangle\langle\nu_n\theta_1 + \mu_n\theta_2| \otimes |n\rangle\langle n| \end{aligned} \tag{2.13}$$

where

$$\mu_k = \frac{1}{2}\left\{\exp\left(-i\sqrt{F}k\right)+1\right\}, \tag{2.14}$$

$$\nu_k = \frac{1}{2}\left\{\exp\left(-i\sqrt{F}k\right)-1\right\}, \quad (k=0,1,2,\cdots). \tag{2.15}$$

If $\sqrt{F}$ satisfies the conditions $\sqrt{F}=0$ or $\sqrt{F}=(2k+1)\,\pi$ $(k=0,1,2,\cdots)$, then one can obtain a complete truth table in FTM gate.

However, it might be difficult to realize the photon number state $|n\rangle\langle n|$ for the input of the Kerr medium physically. In stead of the Kerr medium, we introduce new device related to symmetric Fock space. We contruct a quantum logical gate mathematically with this new device in the next section.

3. Quantum logical gate on symmetric Fock space

In this section, we reformulate beam splittings on symmetric Fock space and we introduce a new operator instead of the Kerr medium on that space. We discuss the mathematical formulation of quantum logical gate by means of beam splittings and the new operator.

Let G be a complete separable metric space and $\mathcal{G}$ be a Borel σ-algebra of G. v is called a locally finite diffuse measure on the measurable space $(G,\mathcal{G})$ if v satisfies the conditions (1) $v(K)<\infty$ for bounded $K\in\mathcal{G}$ and (2) $v(\{x\})=0$ for any $x\in G$. We denote the set of all finite integer - valued measures φ on $(G,\mathcal{G})$ by M. For a set $K\in\mathcal{G}$ and a nutural number $n\in\mathbb{N}$, we put the set of φ satisfying $\varphi(K)=n$ as

$$M_{K,n}\equiv\{\varphi\in M;\varphi(K)=n\}.$$

Let $\mathfrak{M}$ be a σ-algebra generated by $M_{K,n}$. F is the σ-finite measure on $(G,\mathcal{G})$ defined by

$$F(Y)\equiv 1_Y(\varphi_0)+\sum_{n=1}\frac{1}{n!}\int_{G^n}1_Y\left(\sum_{j=1}^{n}\delta_{x_j}\right)v^n(dx_1\cdots dx_n),$$

where 1_Y is the characteristic function of a set Y, φ_0 is an empty configulation in M and δ_{x_j} is a Dirac measure in x_j. $\mathcal{M}\equiv L^2(M,\mathfrak{M},F)$ is called a (symmetric)

Fock space. We define an exponetal vector $\exp_g : M \to \mathbb{C}$ generated by a given function $g : G \to \mathbb{C}$ such that

$$\exp_g(\varphi) \equiv \begin{cases} 1 & (\varphi = \varphi_0), \\ \prod_{x \in \varphi} g(x) & (\varphi \neq \varphi_0), \end{cases} \quad (\varphi \in M).$$

3.1. Generalized beam splittings on Fock space

Let α, β be measurable mappings from G to $\mathbb{C}$ satisfying $\bar{\alpha}$

$$|\alpha(x)|^2 + |\beta(x)|^2 = 1, \quad x \in G.$$

We intoduce an unitary operator $V_{\alpha,\beta} : \mathcal{M} \otimes \mathcal{M} \to \mathcal{M} \otimes \mathcal{M}$ defined b

$$\begin{aligned} (V_{\alpha,\beta}\Phi)(\varphi_1, \varphi_2) \equiv & \sum_{\hat{\varphi}_1 \leq \varphi_1} \sum_{\hat{\varphi}_2 \leq \varphi_2} \exp_\alpha(\hat{\varphi}_1) \exp_\beta(\varphi_1 - \hat{\varphi}_1) \\ & \times \exp_{-\bar{\beta}}(\hat{\varphi}_2) \exp_{\bar{\alpha}}(\varphi_2 - \hat{\varphi}_2) \\ & \times \Phi(\hat{\varphi}_1 + \hat{\varphi}_2, \varphi_1 + \varphi_2 - \hat{\varphi}_1 - \hat{\varphi}_2) \end{aligned}$$

for $\Phi \in \mathcal{M} \otimes \mathcal{M}$ and $\varphi_1, \varphi_2 \in M$. Let $\mathcal{A} \equiv \mathbb{B}(\mathcal{H})$ be the set of all bounded operators on $\mathcal{M}$ and $\mathfrak{S}(\mathcal{A})$ be the set of all normal states on $\mathcal{A}$. $\mathcal{E}_{\alpha,\beta} : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ defined by

$$\mathcal{E}_{\alpha,\beta}(C) \equiv V_{\alpha,\beta}^* C V_{\alpha,\beta}, \quad \forall C \in \mathcal{A} \otimes \mathcal{A}$$

is the lifting in the sense of Accardi and Ohya [1] and the dual map $\mathcal{E}_{\alpha,\beta}^*$ of $\mathcal{E}_{\alpha,\beta}$ given by

$$\mathcal{E}_{\alpha,\beta}^*(\omega)(\bullet) \equiv \omega(\mathcal{E}_{\alpha,\beta}(\bullet)), \quad \forall \omega \in \mathfrak{S}(\mathcal{A} \otimes \mathcal{A})$$

is the CP channel from $\mathfrak{S}(\mathcal{A} \otimes \mathcal{A})$ to $\mathfrak{S}(\mathcal{A} \otimes \mathcal{A})$. Using the exponetial vectors, one can denote a coherent state θ^f ßby

$$\theta^f(A) \equiv \langle \exp_f, A \exp_f \rangle e^{-\|f\|^2}, \quad \forall f \in L^2(G, \nu), \ \forall A \in \mathcal{A}.$$

In particular, for the input coherent states $\eta_0 \otimes \omega_0 = \theta^f \otimes \theta^g$, two output states $\omega_1(\bullet) \equiv \eta_0 \otimes \omega_0 (\mathcal{E}_{\alpha,\beta}((\bullet) \otimes I))$ and $\eta_1(\bullet) \equiv \eta_0 \otimes \omega_0 (\mathcal{E}_{\alpha,\beta}(I \otimes (\bullet)))$ are obtained by

$$\begin{aligned} \omega_1 &= \theta^{\alpha f + \beta g}, \\ \eta_1 &= \theta^{-\bar{\beta} f + \bar{\alpha} g}. \end{aligned}$$

$\mathcal{E}^{*}_{\alpha,\beta}$ is called a generalized beam splitting on Fock space because it also hold the same properties satisfied by the generated beam splitting Π^{*}in Section 1.

Now we introduce a self-adjoint unitary operator $\tilde{U}$, which denotes a new device instead of the Kerr medium, defined by

$$\tilde{U}(\Phi)(\varphi_1,\varphi_2) \equiv (-1)^{|\varphi_1||\varphi_2|}\,\Phi(\varphi_1,\varphi_2)$$

for $\Phi \in \mathcal{M}\otimes\mathcal{M}$ and $\varphi_1,\varphi_2 \in G$, where $|\varphi_k| \equiv \varphi_k(G) \quad (k=1,2)$. For the input state $\omega_1 \otimes \kappa \equiv \theta^f \otimes \frac{1}{\|\psi\|^2}\langle\psi,\ \bullet\psi\rangle$, the output state ω_2 of new device is

$$\begin{aligned}\omega_2(A) &\equiv \omega_1\otimes\kappa\left(\tilde{U}(A\otimes I)\tilde{U}\right)\\ &= \frac{1}{\|\psi\|^2}\int_M F(d\varphi)\,|\psi(\varphi)|^2\,\theta^{(-1)^{|\varphi|^2}f}(A)\end{aligned}$$

for any $A \in \mathcal{A}$, $\psi \in \mathcal{M}$ $(\psi \neq 0)$ and $f \in L^2(G,\nu)$. If κ is given by the vacuum state θ^0, then the output state ω_2 is equals to ω_1 and if κ is given by one particle state, that is, $\kappa = \frac{1}{\|\psi\|^2}\langle\psi,\ \bullet\psi\rangle$ with $\psi\restriction_{M_1^c}$(where M_1 is the set of one-particle states), then ω_2 is obtained by θ^{-f}. Let M_o (resp. M_e) be the set of $\varphi \in M$ which satisfies that $|\varphi|$ is odd (resp. even) and M be the union of M_o and M_e. The output states ω_2 of the new device is written by

$$\omega_2(A) = \lambda_1\theta^{-f}(A) + \lambda_2\theta^{f}(A) \quad \forall A \in \mathcal{A},$$

where λ_1 and λ_2 are given by

$$\begin{cases}\lambda_1 = \frac{1}{\|\psi\|^2}\int_{M_o} F(d\varphi)\,|\psi(\varphi)|^2,\\ \lambda_2 = \frac{1}{\|\psi\|^2}\int_{M_e} F(d\varphi)\,|\psi(\varphi)|^2.\end{cases}$$

Two output states $\omega_3(\bullet) \equiv \omega_2\otimes\eta_2\left(\mathcal{E}_{\alpha_2,\beta_2}((\bullet)\otimes I)\right)$ and $\eta_3(\bullet) \equiv \omega_2\otimes\eta_2\left(\mathcal{E}_{\alpha_2,\beta_2}(I\otimes(\bullet))\right)$ of the total logical gate including two beam splittings $\mathcal{E}^{*}_{\alpha_k,\beta_k}$ with $\left(|\alpha_k|^2+|\beta_k|^2=1\right)$ $(k=1.2)$ and the new device instead of Kerr medium are obtained by

$$\begin{aligned}\omega_3 &= \lambda_1\theta^{\alpha_2(-(\alpha_1 f+\beta_1 g))+\beta_2\left(-\bar{\beta}_1 f+\bar{\alpha}_1 g\right)} + \lambda_2\theta^{\alpha_2(\alpha_1 f+\beta_1 g)+\beta_2\left(-\bar{\beta}_1 f+\bar{\alpha}_1 g\right)},\\ \eta_3 &= \lambda_1\theta^{-\bar{\beta}_2(-(\alpha_1 f+\beta_1 g))+\bar{\alpha}_2\left(-\bar{\beta}_1 f+\bar{\alpha}_1 g\right)} + \lambda_2\theta^{-\bar{\beta}_2(\alpha_1 f+\beta_1 g)+\bar{\alpha}_2\left(-\bar{\beta}_1 f+\bar{\alpha}_1 g\right)},\end{aligned}$$

where $\omega_2 = \lambda_1\theta^{-(\alpha_1 f+\beta_1 g)} + \lambda_2\theta^{\alpha_1 f+\beta_1 g}$ and $\eta_2 = \eta_1 = \theta^{-\bar{\beta}_1 f+\bar{\alpha}_1 g}$.

3.2. Complete truth table for the new logical gate

In this section, we show the complete truth table giving by the above logical gate on Fock space.

We put $\omega_1 = \theta^f$ and $\eta_1 = \theta^g$. If we assume the case (1) of $\lambda_1 = 0$ and $\lambda_2 = 1$, then one has

$$\begin{aligned} \omega_3 &= \theta^{\left(\alpha_1\alpha_2 - \bar{\beta}_1\beta_2\right)f + (\alpha_2\beta_1 + \bar{\alpha}_1\beta_2)g}, \\ \eta_3 &= \theta^{\left(-\alpha_1\bar{\beta}_2 - \bar{\alpha}_2\bar{\beta}_1\right)f + \left(-\beta_1\bar{\beta}_2 + \bar{\alpha}_1\bar{\alpha}_2\right)g}, \end{aligned}$$

and if we assume the case (2) of $\lambda_1 = 1$ and $\lambda_2 = 0$, then one has

$$\begin{aligned} \omega_3 &= \theta^{\left(-\alpha_1\alpha_2 - \bar{\beta}_1\beta_2\right)f + (\alpha_2\beta_1 + \bar{\alpha}_1\beta_2)g}, \\ \eta_3 &= \theta^{\left(\alpha_1\bar{\beta}_2 - \bar{\alpha}_2\bar{\beta}_1\right)f + \left(-\beta_1\bar{\beta}_2 + \bar{\alpha}_1\bar{\alpha}_2\right)g}. \end{aligned}$$

For example, we have the complete truth tables for the following two cases (I) and (II): (I) When $\alpha_1 = \alpha_2 = \beta_1 = \beta 2 = \frac{1}{\sqrt{2}}$ are satisfied, two output states of the new logical gate become $\omega_3 = \theta^g$ and $\eta_3 = \theta^{-f}$ under the case (1) and $\omega_3 = \theta^{-f}$ and $\eta_3 = \theta^g$ under the case (2). (II) When $\alpha_k = \frac{e^{i\gamma_k}}{\sqrt{2}}$ and $\beta_k = \frac{e^{i\delta_k}}{\sqrt{2}}$ with $\gamma_k, \delta_k \in [0, 2\pi]$ hold $\alpha_1\alpha_2 = \beta_1\bar{\beta}_2$, one has $\gamma_1 + \gamma_2 = \delta_2 - \delta_1$ and two output states of the new logical gate become $\omega_3 = \theta^g$ and $\eta_3 = \theta^{-f}$ under the case (1) and $\omega_3 = \theta^{-f}$ and $\eta_3 = \theta^g$ under the case (2).

The new logical gate treats three initial states ω_0, η_0 and κ corresponding to two input gates I_1, I_2 and the control gate C, respectively. The true T and false F of the input state ω (resp. η) are described by two different states ω^T and ω^F (resp. η^T and η^F), that is;

$$\begin{aligned} \text{True} &\Leftrightarrow \text{coherent state } \omega^T = \theta^f \text{ (resp. } \eta^T = \theta^g), \\ \text{False} &\Leftrightarrow \text{vacuum state.} \omega^F = \theta^0 \text{ (resp. } \eta^F = \theta^0). \end{aligned}$$

Moreover, the truth state κ^T and the false state κ^F are denoted by the control states of the case (1) and (2), respectively. When the initial control state κ is κ^F under the above case (I) or (II), the final states of the new logical gate corresponding to two input gates O_1, O_2 are obtained as the following truth table:

I_1	I_2	C	O_1	O_2
T	T	F	T	T
F	T	F	F	T
T	F	F	T	F
F	F	F	F	F

(3.1)

When the initial control state κ is κ^T under the above case (I) or (II), the final states of the new logical gate corresponding to two input gates O_1, O_2 are obtained as the following truth table:

I_1	I_2	C	O_1	O_2
T	T	T	T	T
F	T	T	T	F
T	F	T	F	T
F	F	T	F	F

(3.2)

It means that the new logical gate performs the complete truth table. Further results will be appear in our joint paper [11].

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