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Quantum chaos induced by measurements[1]

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Abstract

We study the dynamics of a "kicked" quantum system undergoing repeated measurements of momentum. A diffusive behavior is obtained for a large class of Hamiltonians, even when the dynamics of the classical counterpart is not chaotic. These results can be interpreted in classical terms by making use of a "randomized" classical map. We compute the transition probability for the action variable and consider the semiclassical limit.

1 Introduction

The classical and quantum dynamics of bound Hamiltonian systems under the action of periodic "kicks" are in general very different. Classical systems can follow very complicated trajectories in phase space, while the evolution of the wave function in the quantum case is more regular. In the classical case, in those regions of the phase space that are stochastic, the evolution of the system can be well described in terms of the action variable alone and one of the most distinctive features of an underlying chaotic behavior is just the diffusion of the action variable in phase space. On the other hand, in the quantum case, such a diffusion is always suppressed after a sufficiently long time [1, 2]. This phenomenon, known as the quantum mechanical suppression of classical chaos, can be framed in a proper context in terms of the semiclassical approximation $\hbar \to 0$ [3, 4].

The "kicked" rotator is a pendulum that evolves under the action of a gravitational field that is "switched on" at periodic time intervals. It is a very useful system, able to elucidate many different features between the classical and the quantum case. By studying this model, Kaulakis and Gontis [5] showed that a diffusive behavior of the action variable takes place even in the quantum case, if a quantum measurement is performed after every kick. This interesting observation was investigated in some detail in a recent paper [6], where it was proven that quantum mechanical measurements of the action variable provoke diffusion in a very large class of "kicked" systems, even when the corresponding classical dynamics is regular. In this paper we shall first briefly review some of our general results and then corroborate our findings by concentrating our attention on the particular case of the kicked rotator.

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2 Kicks interspersed with quantum measurements

We consider the Hamiltonian

$$H = H_0(p) + \lambda V(x)\delta_T(t), \tag{2.1}$$

where p and $x \in [-\pi, \pi]$ are the action and angle variable, respectively, and

$$\delta_T(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT), \tag{2.2}$$

T being the period of the perturbation. We impose periodic boundary conditions on the interaction $V(x)$. This Hamiltonian gives rise to the so-called radial twisting map, that describes the local behavior of a perturbed integrable map near resonance [7]. The free Hamiltonian H_0 has a discrete spectrum and a countable complete set of eigenstates $\{|m\rangle\}$:

$$\langle x|m\rangle = \frac{1}{\sqrt{2\pi}} \exp(imx), \qquad m = 0, \pm 1, \pm 2, \dots . \tag{2.3}$$

We shall consider the evolution engendered by the Hamiltonian (2.1) interspersed with quantum measurements, in the following sense: the system evolves under the action of the free Hamiltonian for $(N-1)T + \tau < t < NT$ $(0 < \tau < T)$, undergoes a "kick" at $t = NT$, evolves again freely and then undergoes a "measurement" of p at $t = NT + \tau$. The evolution of the system is best described in terms of the density matrix: between successive measurements one has

$$\begin{aligned} \rho_{NT+\tau} &= U_{\rm free}(\tau) U_{\rm kick} U_{\rm free}(T-\tau) \rho_{(N-1)T+\tau} U^\dagger_{\rm free}(T-\tau) U^\dagger_{\rm kick} U^\dagger_{\rm free}(\tau), & (2.4)\\ & U_{\rm kick} = \exp(-i\lambda V/\hbar), \quad U_{\rm free}(t) = \exp(-iH_0 t/\hbar). & (2.5) \end{aligned}$$

At each measurement, the wave function is "projected" onto the nth eigenstate of p with probability $P_n(NT+\tau) = \mathrm{Tr}(|n\rangle\langle n|\rho_{NT+\tau})$ and the off-diagonal terms of the density matrix disappear. The occupation probabilities $P_n(t)$ change discontinuously at times NT and their evolution is governed by the master equation

$$P_n(N) = \sum_m W_{nm} P_m(N-1), \tag{2.6}$$

where

$$W_{nm} \equiv |\langle n|U_{\rm free}(\tau) U_{\rm kick} U_{\rm free}(T-\tau)|m\rangle|^2 = |\langle n|U_{\rm kick}|m\rangle|^2 \tag{2.7}$$

are the transition probabilities and we defined, with a little abuse of notation,

$$P_n(N) \equiv P_n(NT+\tau). \tag{2.8}$$

The map (2.6) depends on λ, V, H_0 in a complicated way. However, interestingly, very general conclusions can be drawn about the average value of a generic regular function of momentum $g(p)$ [6]. Let

$$\langle g(p)\rangle_t \equiv \mathrm{Tr}(g(p)\rho(t)) = \sum_n g(p_n) P_n(t), \tag{2.9}$$

where $p|n\rangle = p_n|n\rangle$ ($p_n = n\hbar$), and consider the average value of g after N kicks

$$\langle g(p)\rangle_N \equiv \langle g(p)\rangle_{NT+\tau} = \sum_n g(p_n)P_n(N) = \sum_{n,m} g(p_n)W_{nm}P_m(N-1). \tag{2.10}$$

Substituting W_{nm} from (2.7) one obtains

$$\begin{aligned}\langle g(p)\rangle_N &= \sum_{n,m} g(p_n)\langle m|U_{\text{kick}}^\dagger|n\rangle\langle n|U_{\text{kick}}|m\rangle P_m(N-1) \\ &= \sum_m \langle m|U_{\text{kick}}^\dagger g(p)U_{\text{kick}}|m\rangle P_m(N-1),\end{aligned} \tag{2.11}$$

where we used $g(p)|n\rangle = g(p_n)|n\rangle$. We are mostly interested in the evolution of the quantities p and p^2 (momentum and kinetic energy). By the Baker-Hausdorff lemma

$$U_{\text{kick}}^\dagger g(p)U_{\text{kick}} = g(p) + i\frac{\lambda}{\hbar}[V, g(p)] + \frac{1}{2!}\left(\frac{i\lambda}{\hbar}\right)^2 [V,[V,g(p)]] + ..., \tag{2.12}$$

we obtain the exact expressions

$$U_{\text{kick}}^\dagger pU_{\text{kick}} = p + i\frac{\lambda}{\hbar}[V,p], \tag{2.13}$$

$$U_{\text{kick}}^\dagger p^2 U_{\text{kick}} = p^2 + i\frac{\lambda}{\hbar}[V,p^2] + \lambda^2 (V')^2, \tag{2.14}$$

where prime denotes derivative. We observe, incidentally, that in general, for polynomial $g(p)$, the highest order of λ appearing in (2.12) is the degree of the polynomium.

Substituting (2.13) and (2.14) in (2.11) and then iterating on the number of kicks we obtain

$$\langle p\rangle_N = \langle p\rangle_{N-1} = \langle p\rangle_0, \tag{2.15}$$

$$\langle p^2\rangle_N = \langle p^2\rangle_{N-1} + \lambda^2\langle f^2\rangle = \langle p^2\rangle_0 + \lambda^2\langle f^2\rangle N, \tag{2.16}$$

where $f = -V'(x)$ is the force and

$$\langle f^2\rangle = \text{Tr}\left(f^2\rho_{NT+\tau}\right) = \sum_n \langle n|f^2|n\rangle P_n(N) = \frac{1}{2\pi}\int_{-\pi}^{\pi} dx\ f^2(x) \tag{2.17}$$

is a constant that does not depend on N: Indeed $\langle n|f^2|n\rangle$ is independent of the state $|n\rangle$ [see (2.3)] and $\sum P_n = 1$. In particular, the kynetic energy $K = p^2/2m$ grows at a constant rate: $\langle K\rangle_N = \langle K\rangle_0 + \lambda^2\langle f^2\rangle N/2m$. By using (2.15)-(2.16) we obtain the friction (F) and the diffusion (D) coefficients

$$F = \frac{\langle p\rangle_N - \langle p\rangle_0}{NT} = 0, \tag{2.18}$$

$$D = \frac{\langle \Delta p^2\rangle_N - \langle \Delta p^2\rangle_0}{NT} = \frac{\lambda^2\langle f^2\rangle}{T}, \tag{2.19}$$

where $\langle \Delta p^2\rangle_N = \langle p^2\rangle_N - \langle p\rangle_N^2$. We stress that the above results are *exact*: their derivation involves no approximation. This shows that Hamiltonian systems of the type (2.1) (radial twisting maps), in the quantum case, if "measured" after every kick, have a constant diffusion rate in momentum with no friction, for *any* perturbation $V = V(x)$. In particular, the seminal kicked-rotator model $H_0 = p^2/2I, V = \cos x$ has the diffusion coefficient

$$D = \frac{\lambda^2}{2T}, \tag{2.20}$$

which is nothing but the result obtained in the classical case [1, 5].

The above results are somewhat puzzling, essentially because one finds that in the quantum case, when repeated measurements of momentum (action variable) are performed on the system, a chaotic behavior is obtained for *every* value of λ and for *any* potential $V(x)$. On the other hand, in the classical case, diffusion occurs only for some $V(x)$, when λ exceeds some critical value λ_{crit}. (For instance, the kicked rotator displays diffusion for $\lambda \geq \lambda_{\text{crit}} \simeq 0.972$ [1, 7].) It appears, therefore, that quantum measurements not only yield a chaotic behavior in a quantum context, they even produce chaos when the classical motion is regular. In order to bring to light the causes of this peculiar situation, one has to look at the classical case. The classical map for the Hamiltonian (2.1) reads

$$\begin{aligned} x_N &= x_{N-1} + H_0'(p_{N-1})T, \\ p_N &= p_{N-1} - \lambda V'(x_N). \end{aligned} \tag{2.21}$$

A quantum measurement of p yields an exact determination of momentum p and, as a consequence, makes position x completely undetermined (uncertainty principle). This situation has no classical analog: it is inherently quantal. However, the classical "map" that best mymics this physical picture is obtained by assuming that position x_N at time τ after each kick (i.e. when the quantum counterpart undergoes a measurement) behaves like a random variable ξ_N uniformly distributed over $[-\pi, \pi]$:

$$\begin{aligned} x_N &= \xi_N, \\ p_N &= p_{N-1} - \lambda V'(x_N). \end{aligned} \tag{2.22}$$

Introducing the ensemble average $\langle\langle \cdots \rangle\rangle$ over the stochastic process (i.e. over the set of independent random variables $\{\xi_k\}_{k\leq N}$), we obtain

$$\langle\langle p_N \rangle\rangle = \langle\langle p_{N-1} \rangle\rangle - \lambda \langle V'(\xi_N) \rangle, \tag{2.23}$$

where

$$\langle g(\xi) \rangle \equiv \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\xi) d\xi \tag{2.24}$$

is the average over the single random variable ξ [this coincides with the quantum average: see for instance the last term of (2.17)]. The average of $V'(\xi_N)$ in (2.23) vanishes due to the periodic boundary conditions on V, so that

$$\langle\langle p_N \rangle\rangle = \langle\langle p_{N-1} \rangle\rangle, \tag{2.25}$$

which is the same as Eq. (2.15). Moreover, using (2.22) and (2.25) we get

$$\langle\langle \Delta p_N^2 \rangle\rangle = \langle\langle p_N^2 \rangle\rangle - \langle\langle p_N \rangle\rangle^2 = \langle\langle \Delta p_{N-1}^2 \rangle\rangle + \lambda^2 \langle V'(\xi_N)^2 \rangle - 2\lambda \langle\langle p_{N-1} \rangle\rangle \langle V'(\xi_N) \rangle. \tag{2.26}$$

In writing (2.26), the average of $V'(\xi_N)p_{N-1}$ has been factorized because p_{N-1} depends only on $\{\xi_k\}_{k\leq N-1}$, as can be evinced from (2.22). Using again the periodic boundary condition on V, one finally gets

$$\langle\langle \Delta p_N^2\rangle\rangle = \langle\langle \Delta p_{N-1}^2\rangle\rangle + \lambda^2\langle f^2\rangle \tag{2.27}$$

and the momentum diffuses at the rate (2.19), as in the quantum case with measurements. We obtain in this case a diffusion taking place in the whole phase space, without effects due to the presence of adiabatic islands.

It is interesting to compare the different cases analyzed: (A) a classical system, under the action of a suitable kicked perturbation, displays a diffusive behavior if the coupling constant exceeds a certain threshold (KAM theorem); (B) on the other hand, in its quantum counterpart, this diffusion is always suppressed. (C) The introduction of measurements between kicks encompasses this limitation, yielding diffusion in the quantum case. More so, diffusion takes place for any potential and all values of the coupling constant (namely, even when the classical motion is regular). (D) The same behavior is displayed by a "randomized classical map," in the sense explained above. These conclusions are sketched in Table 1.

Table 1: Classical vs quantum diffusion

A	classical	diffusion for $\lambda > \lambda_{\mathrm{crit}}$
B	quantum	no diffusion
C	quantum + measurements	diffusion $\forall\lambda$
D	classical + random	diffusion $\forall\lambda$

3 Semiclassical limit

As we have seen, the effect of quantum measurements is basically equivalent to a complete randomization of the classical angle variable x, at least if one's attention is limited to the calculation of the diffusion coefficient in the chaotic regime. One might therefore naively think that the randomized classical map (2.22) and the quantum map with measurements (2.6), (2.15)-(2.19) are identical. This expectation would be wrong: there are in fact corrections in $\hbar$. It is indeed straightforward, using Eqs. (2.11)-(2.12), to obtain in the quantum case

$$\begin{aligned}
\langle p^3\rangle_N &= \langle p^3\rangle_{N-1} + 3\lambda^2\langle f^2\rangle\langle p\rangle_{N-1} + \lambda^3\langle f^3\rangle, \\
\langle p^4\rangle_N &= \langle p^4\rangle_{N-1} + 6\lambda^2\langle f^2\rangle\langle p^2\rangle_{N-1} + 4\lambda^3\langle f^3\rangle\langle p\rangle_{N-1} + \lambda^4\langle f^4\rangle + \lambda^2\hbar^2\langle (f')^2\rangle.
\end{aligned} \tag{3.1}$$

On the other hand, using (2.22) and the periodic boundary conditions, one gets for the randomized classical map

$$\begin{aligned}
\langle\langle p_N^3\rangle\rangle &= \langle\langle p_{N-1}^3\rangle\rangle + 3\lambda^2\langle f^2\rangle\langle\langle p_{N-1}\rangle\rangle + \lambda^3\langle f^3\rangle, \\
\langle\langle p_N^4\rangle\rangle &= \langle\langle p_{N-1}^4\rangle\rangle + 6\lambda^2\langle f^2\rangle\langle\langle p_{N-1}^2\rangle\rangle + 4\lambda^3\langle f^3\rangle\langle\langle p_{N-1}\rangle\rangle + \lambda^4\langle f^4\rangle.
\end{aligned} \tag{3.2}$$

Hence the two maps have equal moments up to third order, while the fourth moment displays a difference of order $O(\hbar^2)$:

$$\langle p^4\rangle_N - \langle p^4\rangle_{N-1} = \langle\langle p_N^4\rangle\rangle - \langle\langle p_{N-1}^4\rangle\rangle + \lambda^2\hbar^2\langle (f')^2\rangle. \tag{3.3}$$

In order to understand better the similarities and differences between the two maps, as well as the quantum mechanical corrections, we focus our attention on the particular case of the kicked rotator $H_0 = p^2/2$, $V(x) = \cos x$, which gives rise to the so-called standard map

$$\begin{aligned} x_N &= x_{N-1} + p_{N-1}T, \\ p_N &= p_{N-1} + \lambda \sin x_N. \end{aligned} \tag{3.4}$$

The conditional probability density W_{cl} that an initial state (p', x') evolves after one step into the final state (p, x) is, from (3.4),

$$\begin{aligned} W_{\text{cl}}(p, x|p', x') &= \delta(p - p' - \lambda \sin x)\, \delta(x - x' - p'T) \\ &= \delta(p - p' - \lambda \sin[x' + p'T])\, \delta(x - x' - p'T). \end{aligned} \tag{3.5}$$

This is a completely deterministic evolution. On the other hand, if one randomizes the standard map, as in (2.22),

$$\begin{aligned} x_N &= \xi_N, \\ p_N &= p_{N-1} + \lambda \sin x_N, \end{aligned} \tag{3.6}$$

the conditional probability density becomes

$$W_{\text{cl}}(p, x|p', x') = W_{\text{cl}}(p, x|p') = P(x)\, \delta(p - p' - \lambda \sin x) = \frac{1}{2\pi}\delta(p - p' - \lambda \sin x) \tag{3.7}$$

and is independent of the initial position x'. It is therefore possible to describe the dynamics by considering only the momentum distribution

$$\begin{aligned} W_{\text{cl}}(p|p') &= \frac{1}{2\pi}\int_{-\pi}^{\pi} dx\, \delta(p - p' - \lambda \sin x) = \frac{1}{\lambda\pi}\int_{-1}^{+1} \frac{dy}{\sqrt{1-y^2}}\, \delta\left(y - \frac{p-p'}{\lambda}\right) \\ &= \frac{1}{\pi}\frac{1}{\sqrt{\lambda^2 - (p-p')^2}}\, \theta(\lambda - |p - p'|). \end{aligned} \tag{3.8}$$

Notice that $W_{\text{cl}}(p|p')$ is a function of the momentum transfer $|\Delta p| = |p - p'|$ and vanishes for $|\Delta p| > \lambda$.

Consider now the kicked quantum rotator with measurements. From Eq. (2.7), the transition probability reads

$$W_{\text{q}}(p = \hbar n|p' = \hbar n') = \frac{1}{\hbar}W_{nn'} = \frac{1}{\hbar}\left|\langle n|e^{-i\lambda\cos x/\hbar}|n'\rangle\right|^2 \tag{3.9}$$

and by using the definition (2.3) one obtains

$$\begin{aligned}\langle n|e^{-i\lambda\cos x/\hbar}|n'\rangle &= \int_{-\pi}^{\pi} dx \langle n|x\rangle e^{-i\lambda\cos x/\hbar}\langle x|n'\rangle \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} dx\, e^{-i(n-n')x} e^{-i\lambda\cos x/\hbar} = i^{n-n'} J_{n-n'}\left(\frac{\lambda}{\hbar}\right), \end{aligned} \tag{3.10}$$

where $J_m(z)$ is the Bessel function of order m. Therefore, in the quantum case, from (3.9) and (3.10), we can write

$$W_{\mathrm{q}}(p=\hbar n|p'=\hbar n') = \frac{1}{\hbar}J_\nu\left(\frac{\lambda}{\hbar}\right)^2 \qquad (\Delta p = p-p' = \hbar\nu;\ \ \nu\equiv n-n'). \tag{3.11}$$

There are two important differences between the classical case (3.8) and its quantum counterpart (3.11): i) the quantum mechanical transition probability W_{q} admits only quantized values of momentum $\hbar n$, while the classical one W_{cl} is defined on the real line; ii) momentum can change by any value in the quantum case (notice however that this occurs with very small probability for $|\Delta p| = \hbar|\nu| \gg \lambda$ [1]), while in the classical case this change is strictly constrained by $|\Delta p| \leq \lambda$. These features have an interesting physical meaning: see Figure 1. The transition probability of classical momentum appears as an "average" of its quantum counterpart, which explains the strong analogy discussed in Section 2. At the same time, the quantum mechanical transition probability has a small nonvanishing tail for $|\Delta p| = \hbar|\nu| > \lambda$: this is at the origin of the difference (3.3).

Finally, let us show how one recovers the transition probability W_{cl} starting from W_{q}, in the semiclassical limit. We look at the limit $\hbar\to 0$, while keeping $\Delta p = \hbar\nu$ finite:

$$\hbar\to 0, \qquad \nu\to\infty \quad \text{with} \quad \Delta p = \hbar\nu = \text{const.} \tag{3.12}$$

In this limit, the argument and the order of the Bessel function in (3.11) are infinities of the same order. For $|\Delta p|\leq\lambda$, setting $\Delta p/\lambda \equiv \cos\beta$, one gets

$$\frac{\lambda}{\hbar} = \frac{\lambda}{\Delta p}\frac{\Delta p}{\hbar} = \nu\sec\beta. \tag{3.13}$$

Hence, by using the asymptotic limit of the Bessel function [8]

$$J_\nu(\nu\sec\beta) \overset{\nu\ \text{large}}{\sim} \sqrt{\frac{2}{\nu\pi\tan\beta}}\left[\cos\left(\nu\tan\beta - \nu\beta - \frac{\pi}{4}\right) + \mathrm{O}(\nu^{-1})\right], \tag{3.14}$$

Eq. (3.11) becomes, in the limit (3.12),

$$\begin{aligned} W_{\mathrm{q}}(p|p') &= \frac{1}{\hbar}J_{\frac{\Delta p}{\hbar}}\left(\frac{\lambda}{\hbar}\right)^2 = \frac{1}{\hbar}J_\nu\left(\nu\sec\beta\right)^2 \\ &\sim \frac{1}{\hbar}\frac{2}{\frac{\Delta p}{\hbar}\pi\sqrt{\frac{\lambda^2}{\Delta p^2}-1}}\left[\cos^2\left(\frac{\Delta p}{\hbar}\sqrt{\frac{\lambda^2}{\Delta p^2}-1} - \frac{\Delta p}{\hbar}\arccos\frac{\Delta p}{\lambda} - \frac{\pi}{4}\right) + \mathrm{O}\left(\frac{\hbar}{\Delta p}\right)\right] \\ &\sim W_{\mathrm{cl}}(p|p')\left[1+\sin\left(\frac{2\sqrt{\lambda^2-\Delta p^2}}{\hbar} - \frac{2\Delta p}{\hbar}\arccos\frac{\Delta p}{\lambda}\right) + \mathrm{O}\left(\frac{\hbar}{\Delta p}\right)\right], \\ &\qquad\qquad (|\Delta p|\leq\lambda) \end{aligned} \tag{3.15}$$

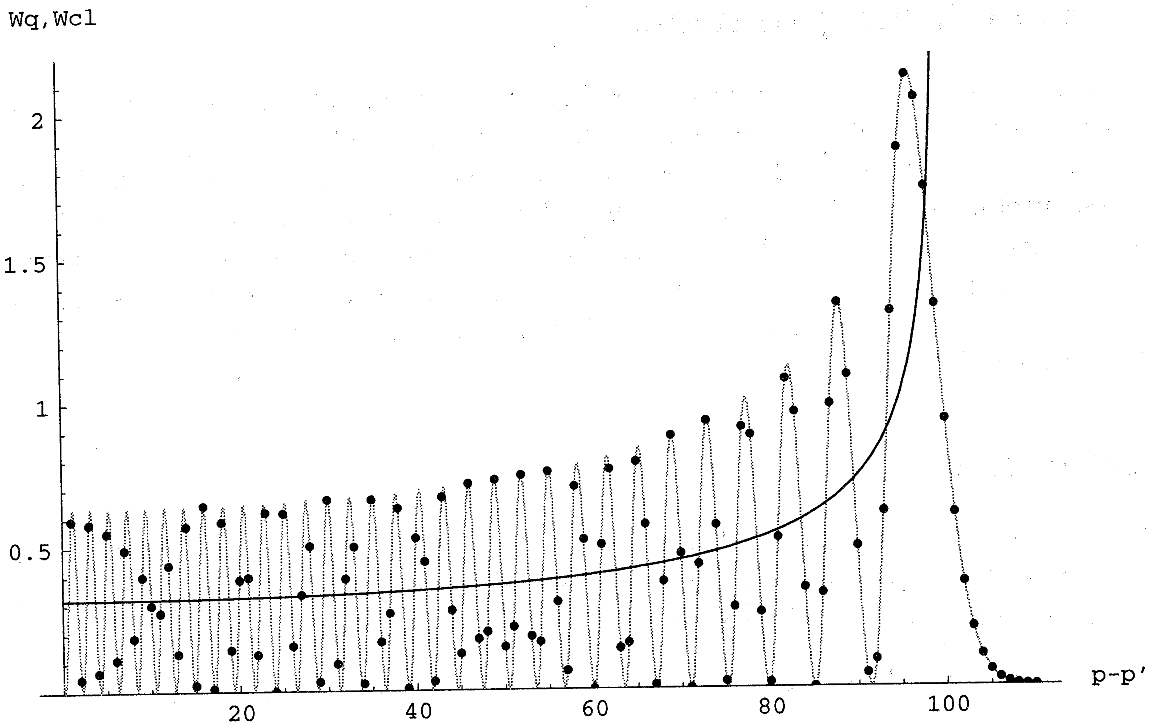

Figure 1: Momentum transition probabilities for the kicked rotator ($\lambda = 100\hbar$ and the momentum transfer $p - p'$ is expressed in units $\hbar$). The thick line is the classical expression (3.8): it diverges for $p - p' = \lambda$ and vanishes for $p - p' > \lambda$. The quantum mechanical transition probability (3.11) is defined only for integer values of $p - p'$ (dots). The interpolating line (obtained by treating the order of the Bessel function as a continuos variable) oscillates around its classical counterpart and is nonvanishing (although very small) outside the classical range, i.e. for $p - p' > \lambda$.

that, due to Riemann-Lebesgue lemma, tends to $W_{\rm cl}$ in the sense of distributions.

On the other hand, for $|\Delta p| > \lambda$, setting $\Delta p/\lambda \equiv \cosh\alpha$ and using the asymptotic formula [8]

$$J_\nu\left(\frac{\nu}{\cosh\alpha}\right) \overset{\nu\ \text{large}}{\sim} \frac{\exp(\nu\tanh\alpha - \nu\alpha)}{\sqrt{2\nu\pi\tan\beta}}\left[1 + \mathrm{O}(\nu^{-1})\right], \tag{3.16}$$

we get

$$W_{\rm q}(p|p') \sim \frac{1}{2\pi\sqrt{\Delta p^2 - \lambda^2}} \exp\left\{-\frac{2\Delta p}{\hbar}\left[\operatorname{arccos}\frac{\Delta p}{\lambda} - \sqrt{1 - \left(\frac{\lambda}{\Delta p}\right)^2}\right]\right\}\left[1 + \mathrm{O}\left(\frac{\hbar}{\Delta p}\right)\right], \quad (|\Delta p| > \lambda) \tag{3.17}$$

which vanishes exponentially (remember that $\tanh\alpha < \alpha$). Equations (3.15) and (3.17) corroborate the results of Section 2 and enable us to conclude that the "randomized" classical kicked rotator is just the semiclassical limit of the "measured" quantum kicked rotator.

4 Concluding remarks

The conclusion drawn in the previous section for the kicked rotator can be generalized to an arbitrary radial twisting map. The calculation and the techniques utilized are more involved and will be presented elsewhere. There are also a number of related problems that deserve attention and a careful investigation. Among these, we just mention the case of imperfect quantum measurements, yielding a partial loss of quantum mechanical coherence, the relation to disordered systems, Anderson localization [9] and quantum Zeno effect [10] and finally the extension to a different class of Hamiltonians [11].

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