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A new method for iterating σ–centered forcing

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The purpose of this note is to describe a general iteration technique for σ–centered forcing notions. Note that σ–centered forcing can be iterated using either finite or countable supports. The first method has the disadvantage of adding generics for large Cohen algebras, while the second allows of a continuum of size at most $\aleph_2$ only. Our method, which can be thought of as being somewhere in between finite and countable support, avoids the second drawback and, depending on which order(s) we iterate, may avoid the first one too. It has been motivated by similar constructions, due mainly to Shelah, in particular by [FShS], [Sh3], and [DžSh]. Unfortunately, we do not know yet of any interesting application of our method, for the one we originally had in mind did not work (see the discussion at the end of the note). I thank Lajos Soukup and Tadatoshi Miyamoto for comments and discussions, as well as Shizuo Kamo for organizing the stimulating meeting during which the material in here was presented and discussed.

Let $\mathbb{P}$ be a p.o. Recall that $P \subseteq \mathbb{P}$ is said to be *centered* if any finite subset of P has a lower bound in $\mathbb{P}$. In case the lower bound can always be taken from P, P is *directed* (and so generates a filter). $\mathbb{P}$ is *σ–centered* if it can written as a union of countably many centered subsets. It's *σ–filtered* (or *σ–directed*) if it's a union of countably many directed subsets. Any σ–centered p.o. is σ–filtered, but the converse does not hold (see [Mi1, Theorem 9]). However, for Boolean algebras, the two notions are equivalent so that we may confine ourselves to σ–filtered forcing notions in what follows.

Definition Define by recursion $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha;\ \alpha < \delta \rangle$ to be an *iteration of σ–centered forcing with mixed support* if all $\mathbb{P}_\alpha$'s are p.o.'s whose elements are pairs of functions (f, p) with domain a countable subset of α, all $\dot{\mathbb{Q}}_\alpha$'s are $\mathbb{P}_\alpha$–names for σ–filtered forcing notions, say, $\Vdash_\alpha \dot{\mathbb{Q}}_\alpha = \bigcup_n \dot{Q}_{\alpha,n}$, and the following conditions are satisfied.

i. Basic step. $\mathbb{P}_0 = \{\emptyset\}$.

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ii. Successor step. $\mathbb{P}_{\alpha+1} = \{(f,p);\ (f\restriction\alpha, p\restriction\alpha) \in \mathbb{P}_\alpha$ and either $\alpha \notin \mathrm{dom}(f,p)$ or $f(\alpha) \in \omega$ and $(f\restriction\alpha, p\restriction\alpha) \Vdash_\alpha p(\alpha) \in \dot{Q}_{\alpha,f(\alpha)}\}$, and we put $(g,q) \le (f,p)$ iff $(g\restriction\alpha, q\restriction\alpha) \le (f\restriction\alpha, p\restriction\alpha)$ and either $\alpha \notin \mathrm{dom}(f,p)$ or $(g\restriction\alpha, q\restriction\alpha) \Vdash_\alpha q(\alpha) \le p(\alpha)$ as usual.

iii. Limit step. $\mathbb{P}_\beta = \{(f,p);\ (f\restriction\alpha, p\restriction\alpha) \in \mathbb{P}_\alpha$ for all $\alpha < \beta$ and $|\mathrm{dom}(f,p)| \le \aleph_0\}$, and we put $(g,q) \le (f,p)$ iff $(g\restriction\alpha, q\restriction\alpha) \le (f\restriction\alpha, p\restriction\alpha)$ for all $\alpha < \beta$ and $\{\alpha < \beta;\ \alpha \in \mathrm{dom}(f,p)$ and $f(\alpha) \neq g(\alpha)\}$ is finite.

We emphasize two main features of the iteration. First, only the second coordinate of a condition is a name; this will give us the $\aleph_2$–cc almost for free. Second, the order allows changes only on finitely many first coordinates of the domain; this is used to prove properness. As a first step however, we show that our construction is an iteration in the usual sense.

Lemma 1 $\mathbb{P}_{\alpha+1} \cong \mathbb{P}_\alpha \star \dot{\mathbb{Q}}_\alpha$.

Proof. Given $(f,p) \in \mathbb{P}_{\alpha+1}$, let $e(f,p) = ((f\restriction\alpha, p\restriction\alpha), p(\alpha))$. Since $(f\restriction\alpha, p\restriction\alpha) \in \mathbb{P}_\alpha$ and $\Vdash_\alpha p(\alpha) \in \dot{\mathbb{Q}}_\alpha$, we have $e(f,p) \in \mathbb{P}_\alpha \star \dot{\mathbb{Q}}_\alpha$. Similarly $(g,q) \le (f,p)$ iff $e(g,q) \le e(f,p)$.

So it suffices to check e is dense. To this end, let $((f,p),\dot{q}) \in \mathbb{P}_\alpha \star \dot{\mathbb{Q}}_\alpha$. Then $(f,p) \Vdash_\alpha \dot{q} \in \dot{\mathbb{Q}}_\alpha = \bigcup_n \dot{Q}_{\alpha,n}$. Hence there are $(f',p') \le (f,p)$ and n such that $(f',p') \Vdash_\alpha \dot{q} \in \dot{Q}_{\alpha,n}$. Put $f'' = f' \cup \{\langle\alpha,n\rangle\}$ and $p'' = p' \cup \{\langle\alpha,\dot{q}\rangle\}$. Then clearly $(f'',p'') \in \mathbb{P}_{\alpha+1}$ and $e(f'',p'') \le ((f,p),\dot{q})$. $\square$

The following easy lemma exhibits a crucial property of the iteration, necessary for the proofs of both Lemmata 4 and 5. That we work with σ–filtered forcing notions is paramount for the proof.

Lemma 2 *Any (f,p) and $(g,q) \in \mathbb{P}_\delta$ such that f and g agree on their common domain are compatible.*

Proof. By recursion on α produce a common extension (h,r) with $h = f \cup g$ and domain $\mathrm{dom}(f,p) \cup \mathrm{dom}(g,q)$ as follows. Assume $(h\restriction\alpha, r\restriction\alpha)$ has been defined. If α belongs only to the domain of one of the conditions, say, (f,p), let $h(\alpha) = f(\alpha)$ and $r(\alpha) = p(\alpha)$. If it belongs to the domain of both, let $h(\alpha) = f(\alpha) = g(\alpha)$, recall that

$$(h\restriction\alpha, r\restriction\alpha) \Vdash_\alpha \text{“}p(\alpha), q(\alpha) \in \dot{Q}_{\alpha,h(\alpha)} \text{ and } \dot{Q}_{\alpha,h(\alpha)} \text{ is directed”}$$

so that there is a $\mathbb{P}_\alpha$–name $\dot{r}$ such that

$$(h\restriction\alpha, r\restriction\alpha) \Vdash_\alpha \text{“}\dot{r} \le p(\alpha), q(\alpha) \text{ and } \dot{r} \in \dot{Q}_{\alpha,h(\alpha)}.\text{”}$$

This means $r(\alpha) = \dot{r}$ is as required. $\square$

Lemma 3 *Let $(f,p) \in \mathbb{P}_\delta$. Then the set $\{(g,q) \le (f,p);\ \mathrm{dom}(g,q) = \mathrm{dom}(f,p)\}$ is σ–filtered.*

Proof. Given any finite $F \subseteq \mathrm{dom}(p)$ and any $s \in \omega^F$, we form the set $\mathbb{P}_s^{f,p}$ of all conditions $(g,q) \leq (f,p)$ with domain $\mathrm{dom}(f,p)$ such that

$$g(\alpha) = \begin{cases} s(\alpha) & \text{if } \alpha \in F \\ f(\alpha) & \text{otherwise} \end{cases}$$

and note that $\mathbb{P}_s^{f,p}$ is directed (possibly empty) by the previous lemma. □

Lemma 4 *(CH) Any $\mathbb{P}_\delta$ has the $\aleph_2$–cc.*

Proof. Let $\{(f_\zeta, p_\zeta);\ \zeta < \omega_2\}$ be a collection of $\aleph_2$ many conditions of $\mathbb{P}_\delta$. By CH and the Δ–system lemma, we may assume that $\{\mathrm{dom}(f_\zeta, p_\zeta);\ \zeta < \omega_2\}$ forms a Δ–system, say with root R. Again by CH, we may assume that there is $g \in \omega^R$ such that $f_\zeta{\restriction}R = g$ for all ζ. By Lemma 2, this means, however, that all (f_ζ, p_ζ) are pairwise compatible. □

We say $(g,q) \leq_0 (f,p)$ iff $\mathrm{dom}(g,q) \supseteq \mathrm{dom}(f,p)$, and $g(\alpha) = f(\alpha)$ and $q(\alpha) = p(\alpha)$ for $\alpha \in \mathrm{dom}(f,p)$. This easily entails $(g,q) \leq (f,p)$. Note that the ordering $\leq_0$ is σ–closed. The proof of the following lemma by a pressing–down argument is a variation on a theme originally invented by Shelah (see e.g. [FShS] or [Sh3]).

Crucial Lemma 5 *Let $(f,p) \in \mathbb{P}_\delta$ and let $\dot\gamma$ be a $\mathbb{P}_\delta$–name for an ordinal. Then there is $(g,q) \leq_0 (f,p)$ such that $\{(h,r) \leq (g,q);\ \mathrm{dom}(h,r) = \mathrm{dom}(g,q)$ and (h,r) decides $\dot\gamma\}$ is predense below (g,q).*

Proof. Assume not, and construct recursively sequences $\langle (f_\zeta, p_\zeta);\ \zeta < \omega_1\rangle$, $\langle (g_\zeta, q_\zeta);\ \zeta < \omega_1\rangle$ of conditions such that

i. $(f_0, p_0) \leq_0 (f,p)$

ii. $(f_\xi, p_\xi) \leq_0 (f_\zeta, p_\zeta)$ for $\xi \geq \zeta$

iii. $(g_\zeta, q_\zeta) \leq (f_\zeta, p_\zeta)$ and $\mathrm{dom}(g_\zeta, q_\zeta) = \mathrm{dom}(f_\zeta, p_\zeta)$ and $g_\zeta(\alpha) = f_\zeta(\alpha)$ for all $\alpha \notin \bigcup_{\xi<\zeta} \mathrm{dom}(f_\xi, p_\xi)$

iv. (g_ζ, q_ζ) decides $\dot\gamma$

v. all (g_ζ, q_ζ) are pairwise incompatible.

To carry out step ζ of the recursion, do the following. Put $(f'_\zeta, p'_\zeta) = \bigcup_{\xi<\zeta}(f_\xi, p_\xi)$ ($(f'_0, p'_0) = (f,p)$ at the basic step, resp.). Note that $(f'_\zeta, p'_\zeta) \leq_0 (f,p)$. Hence (f'_ζ, p'_ζ) is not as required in the lemma and there is $(g_\zeta, q_\zeta) \leq (f'_\zeta, p'_\zeta)$ deciding $\dot\gamma$ and incompatible with all $(h,r) \leq (f'_\zeta, p'_\zeta)$ with $\mathrm{dom}(h,r) = \mathrm{dom}(f'_\zeta, p'_\zeta)$ which decide $\dot\gamma$.

Define (f_ζ, p_ζ) with domain $\mathrm{dom}(g_\zeta, q_\zeta)$ by recursion on α such that

- $(g_\zeta{\restriction}\alpha, q_\zeta{\restriction}\alpha) \leq (f_\zeta{\restriction}\alpha, p_\zeta{\restriction}\alpha) \leq_0 (f'_\zeta{\restriction}\alpha, p'_\zeta{\restriction}\alpha)$ and
- $f_\zeta(\alpha) = g_\zeta(\alpha)$ for all $\alpha \in \mathrm{dom}(g_\zeta) \setminus \mathrm{dom}(f'_\zeta)$.

Assume $(f_\zeta{\restriction}\alpha, p_\zeta{\restriction}\alpha)$ has been defined as required. If $\alpha \notin \mathrm{dom}(g_\zeta, q_\zeta)$ or $\alpha \in \mathrm{dom}(f'_\zeta, p'_\zeta)$ there is nothing to do. So suppose $\alpha \in \mathrm{dom}(g_\zeta, q_\zeta) \setminus \mathrm{dom}(f'_\zeta, p'_\zeta)$. Then $(g_\zeta{\restriction}\alpha, q_\zeta{\restriction}\alpha) \Vdash_\alpha q_\zeta(\alpha) \in \dot{Q}_{\alpha, g_\zeta(\alpha)}$. So there is a name $\dot r$ such that

- $(f_\zeta \restriction \alpha, p_\zeta \restriction \alpha) \Vdash_\alpha \dot{r} \in \dot{Q}_{\alpha, g_\zeta(\alpha)}$ and
- $(g_\zeta \restriction \alpha, q_\zeta \restriction \alpha) \Vdash_\alpha \dot{r} = q_\zeta(\alpha)$.

Let $p_\zeta(\alpha) = \dot{r}$. Then $(f_\zeta \restriction \alpha+1, p_\zeta \restriction \alpha+1)$ is as required. Clearly (g_ζ, q_ζ) and (f_ζ, p_ζ) satisfy (i) through (v) above.

Now define $F : \omega_1 \to \omega_1$ such that $F(\zeta) = \min\{\xi \leq \zeta;\ f_\zeta(\alpha) = g_\zeta(\alpha)$ for all $\alpha \notin \mathrm{dom}(f_\xi, p_\xi)\}$. By (iii) and by definition of the p.o. in the iteration, F is a regressive function. Hence, by Fodor's lemma, there are $S \subseteq \omega_1$ stationary and $\xi < \omega_1$ such that $F(\zeta) = \xi$ for $\zeta \in S$. There are $\xi \leq \zeta_0 < \zeta_1$, both in S, such that $g_{\zeta_0} \restriction \mathrm{dom}(f_\xi, p_\xi) = g_{\zeta_1} \restriction \mathrm{dom}(f_\xi, p_\xi)$. By construction and by Lemma 2, this means $(g_{\zeta_0}, q_{\zeta_0})$ and $(g_{\zeta_1}, q_{\zeta_1})$ are compatible, a contradiction. □

Corollary 6 *Any $\mathbb{P}_\delta$ is proper.*

Proof. This is immediate from Lemmata 3 and 5. □

Corollary 7 *(CH) Any $\mathbb{P}_\delta$ preserves cardinals and cofinalities.*

We proceed to look at examples for our technique. First consider the iteration $\mathbb{P}_\delta$ of Cohen forcing $\mathbb{C}$ with mixed support. Recall that $\mathbb{C} = 2^{<\omega}$, ordered by $s \leq t$ iff $s \supseteq t$. Putting $C_s = \{s\}$ we get a decomposition $\mathbb{C} = \bigcup_{s \in 2^{<\omega}} C_s$ of $\mathbb{C}$ into countably many trivially directed sets.

Let us now review the *pseudo–product of Cohen forcing*, as introduced by Fuchino, Shelah and Soukup [FShS]. Given a cardinal κ, $\mathbb{Q}_\kappa$ consists of partial functions $f : \kappa \times \omega \to 2$ with countable domain and such that $\mathrm{dom}(f(\alpha, \cdot)) = \{n;\ f(\alpha, n)$ is defined$\}$ is finite for all α. Order $\mathbb{Q}_\kappa$ by $f \leq g$ iff $f \supseteq g$ and $\{\alpha \in \mathrm{proj}_\kappa \mathrm{dom}(g);\ \mathrm{dom}(f(\alpha, \cdot)) \neq \mathrm{dom}(g(\alpha, \cdot))\}$ is finite where $\mathrm{proj}_\kappa \mathrm{dom}(f) = \{\alpha;\ f(\alpha, n)$ is defined for some $n\}$. It is well–known [FShS] that $\mathbb{Q}_\kappa$ is proper (it even satisfies Axiom A) and that it has the $(2^{\aleph_0})^+$–cc so that it preserves cardinals and cofinalities under CH. Since $\mathbb{P}_\kappa$ consists of functions (f, p) with countable domain such that, for $\alpha \in \mathrm{dom}(f, p)$, we have $f(\alpha) \in 2^{<\omega}$ and $(f \restriction \alpha, p \restriction \alpha) \Vdash_\alpha p(\alpha) \in C_{f(\alpha)}$ which means $p(\alpha) = f(\alpha)$, and since $(f, p) \leq (g, q)$ iff $p(\alpha) \leq q(\alpha)$ for all α and $\{\alpha < \kappa;\ \alpha \in \mathrm{dom}(g, q)$ and $f(\alpha) \neq g(\alpha)\}$ is finite, we see immediately

Proposition 8 $\mathbb{P}_\kappa \cong \mathbb{Q}_\kappa$.

So we get nothing new, yet this also means our construction generalizes theirs. Note this is similar to the connection between the usual (finite support) product of Cohen forcing and finite support iteration in general, for the latter boils down to the former in case of Cohen forcing. (This is far from being true for countable supports: the countable support product of Cohen forcing collapses $2^{\aleph_0}$ to $\aleph_0$ while the countable support iteration of Cohen forcing is proper.)

The original motivation for considering $\mathbb{Q}_\kappa$ was to construct models with large continuum in which the combinatorial principle ♣ and some fragment of Martin's axiom MA hold simultaneously. Recall that ♣ says there is a sequence $\langle A_\alpha \subseteq \alpha;\ \alpha < \omega_1$ is a limit ordinal and A_α is cofinal in $\alpha\rangle$ such that for all uncountable $A \subseteq \omega_1$ there is α with $A_\alpha \subseteq A$.

Theorem 9 (Fuchino, Shelah, Soukup [FShS]) $\Vdash_{\mathbb{Q}_\kappa} \clubsuit$. *So, if CH holds and κ is of uncountable cofinality,* $\Vdash_{\mathbb{Q}_\kappa}$ "$2^{\aleph_0} = \kappa + MA(\text{countable}) + \clubsuit$".

It would be interesting to know whether a similar result is true for $\mathbb{P}_\kappa$ when the iterands are more complicated than mere Cohen forcing, for this might give us the consistency of $\clubsuit$ with stronger forms of MA and large continuum. In fact, the author originally conjectured that $\Vdash_{\mathbb{P}_\kappa} \clubsuit$ holds in case $\Vdash_\alpha$ "$\dot{\mathbb{Q}}_\alpha$ is Hechler forcing $\dot{\mathbb{D}}$", but this turned out to be false. Recall that $\mathbb{D}$ is the set of pairs (s, f) such that $s \in \omega^{<\omega}$, $f \in \omega^\omega$ and $s \subseteq f$, ordered by $(s, f) \leq (t, g)$ iff $s \supseteq t$ and $f(n) \geq g(n)$ for all n. Putting $D_s = \{(s, f);\ s \subseteq f \in \omega^\omega\}$ we get a decomposition $\mathbb{D} = \bigcup_{s \in \omega^{<\omega}} D_s$ of $\mathbb{D}$ into countably many directed sets.

Proposition 10 $\mathbb{P}_{\omega_1+1}$ *adds an uncountable subset of ω_1 which contains no countable subset of the ground model.*

Proof. It is easy to see that $\mathbb{P}_{\omega_1+1}$ is equivalent to the collection of all $p = \langle (s^p_\gamma, \dot{f}^p_\gamma);\ \gamma \in \text{dom}(p)\rangle$ where $\text{dom}(p) \subseteq \omega_1 + 1$ is countable, $s^p_\gamma \in \omega^{<\omega}$, $\dot{f}^p_\gamma$ is a $\mathbb{P}_\gamma$–name for a member of ω^ω, and $\Vdash_\gamma s^p_\gamma \subseteq \dot{f}^p_\gamma \in \omega^\omega$, ordered by $p \leq q$ iff $\text{dom}(p) \supseteq \text{dom}(q)$, $s^p_\gamma \supseteq s^q_\gamma$ and $p{\restriction}\gamma \Vdash_\gamma$ "$\dot{f}^p_\gamma(n) \geq \dot{f}^q_\gamma(n)$ for all n" for all $\gamma \in \text{dom}(q)$, as well as $s^p_\gamma = s^q_\gamma$ for all but finitely many $\gamma \in \text{dom}(q)$. Let $\dot{d}_\gamma$ be the $\mathbb{P}_{\gamma+1}$–name for the γ–th Hechler generic.

Say "$p \in \mathbb{P}_{\omega_1+1}$ forces $\gamma \in \dot{A}$" iff

- $\{0\} \cup [\gamma, \gamma + \omega) \cup \{\omega_1\} \subseteq \text{dom}(p)$
- $p{\restriction}(\gamma+n) \Vdash_{\gamma+n} \dot{f}^p_{\gamma+n}(i) > \dot{d}_0(n)$ for all n and all i (this means in particular $s^p_{\gamma+n}(i) > s^p_0(n)$ whenever $i \in \text{dom}(s^p_{\gamma+n})$, and also $n \in \text{dom}(s^p_0)$ whenever $\text{dom}(s^p_{\gamma+n}) \neq \emptyset$)
- $p{\restriction}\omega_1 \Vdash_{\omega_1} \dot{f}^p_{\omega_1}(i) > \max_{n \leq i} \dot{d}_{\gamma+n}(i)$ for all $i \geq \text{dom}(s^p_{\omega_1})$.

Since any strengthening of a condition forcing $\gamma \in \dot{A}$ also forces $\gamma \in \dot{A}$, $\dot{A}$ is indeed a name for a subset of ω_1.

Claim 10.1 $\Vdash_{\omega_1+1}$ "*$\dot{A}$ is uncountable*".

Proof. Fix $p \in \mathbb{P}_{\omega_1+1}$, and let $\gamma < \omega_1$ be such that $\text{dom}(p) \cap \omega_1 \subseteq \gamma$. Without loss $0, \omega_1 \in \text{dom}(p)$. Define a condition q such that $\text{dom}(q) = \text{dom}(p) \cup [\gamma, \gamma + \omega)$ and

- $s^q_\alpha = s^p_\alpha$, $\dot{f}^q_\alpha = \dot{f}^p_\alpha$ for $\alpha \in \text{dom}(p) \cap \omega_1$
- $s^q_{\omega_1} = s^p_{\omega_1}$
- $q{\restriction}\omega_1 \Vdash_{\omega_1} \dot{f}^q_{\omega_1}(i) > \max\{\max_{n \leq i} \dot{d}_{\gamma+n}(i), \dot{f}^p_{\omega_1}(i)\}$ for all $i \geq \text{dom}(s^p_{\omega_1})$
- $s^q_{\gamma+n} = \langle\rangle$
- $q{\restriction}(\gamma + n) \Vdash_{\gamma+n} \dot{f}^q_{\gamma+n}(i) > \dot{d}_0(n)$ for all n and all i.

Since $[\gamma, \gamma + \omega) \cap \text{dom}(p) = \emptyset$, the choice in the last two clauses can indeed be made, and we see easily that $q \leq p$ and q forces $\gamma \in \dot{A}$. $\square$

Claim 10.2 *Given $B \subseteq \omega_1$ countable and $p \in \mathbb{P}$, there is $q \leq p$ such that $q \Vdash_{\omega_1+1} B \not\subseteq \dot{A}$.*

Proof. Assume $p \Vdash_{\omega_1+1} B \subseteq \dot{A}$, i.e. $\{0\} \cup \bigcup_{\gamma\in B}[\gamma, \gamma+\omega) \cup \{\omega_1\} \subseteq \mathrm{dom}(p)$ and

- $p\lceil(\gamma+n) \Vdash_{\gamma+n} \dot{f}^p_{\gamma+n}(i) > \dot{d}_0(n)$ for all n, all i and all $\gamma \in B$, and
- $p\lceil\omega_1 \Vdash_{\omega_1} \dot{f}^p_{\omega_1}(i) > \max_{n\leq i} \dot{d}_{\gamma+n}(i)$ for all $i \geq \mathrm{dom}(s^p_{\omega_1})$ and all $\gamma \in B$.

Hence $p\lceil\omega_1 \Vdash_{\omega_1}$ "$\max_{n\leq i, \gamma\in B} \dot{d}_{\gamma+n}(i)$ exists for all $i \geq \mathrm{dom}(s^p_{\omega_1})$". $(\star)$

Let $k = \max\{\mathrm{dom}(s^p_0), \mathrm{dom}(s^p_{\omega_1})\}$. Then $s^p_{\gamma+k} = \langle\rangle$ for all $\gamma \in B$ by a previous remark. Therefore it is easy to find $q \leq p$ such that $q\lceil(\gamma_j + k) \Vdash_{\gamma_j+k} \dot{f}^q_{\gamma_j+k}(k) \geq j$ where $B = \{\gamma_j;\ j \in \omega\}$ is an enumeration. Hence $q\lceil(\gamma_j + k + 1) \Vdash_{\gamma_j+k+1} \dot{d}_{\gamma_j+k}(k) \geq j$ for all j which contradicts $(\star)$. □

This completes the proof of the proposition. □

It is well-known (and easy to see) that after adding one Hechler real every uncountable subset of ω_1 still contains a countable set of the ground model. The same is true for $\mathbb{D}_\alpha$, the finite support iteration of $\mathbb{D}$ of length α, where $\alpha < \omega_1$ [Sh1]. An elaboration of the above argument shows for regular uncountable κ:

Proposition 11 *(CH) $\mathbb{P}_\kappa$ forces the size of the least family $\mathcal{F}$ of countable subsets of ω_1 such that each uncountable subset of ω_1 contains a member of $\mathcal{F}$ is equal to κ. In particular, if $\kappa \geq \omega_2$, $\clubsuit$ fails.*

A natural variation of the iteration considered here strenghtens the definition of the order relation in part (iii) to: $(g, q) \leq (f, p)$ iff $(g\lceil\alpha, q\lceil\alpha) \leq (f\lceil\alpha, p\lceil\alpha)$ for all $\alpha < \beta$ and $\{\alpha < \beta;\ \alpha \in \mathrm{dom}(f, p)$ and $(f(\alpha) \neq g(\alpha)$ or $p(\alpha) \neq q(\alpha))\}$ is finite. An iteration technique of this kind has been used several times, e.g. in [FShS] and [DžSh]. Its disadvantage is the $\aleph_2$–cc will in general only hold for iterations of orderings of size $\leq \aleph_1$ and of length $\leq \aleph_2$. Our approach avoids this problem and is in this respect similar to the much more complicated historic iteration of [Sh3] which also sets up a novel iteration technique which goes beyond $\aleph_2$.

Iterations with "mixed" support have been considered early on, e.g. in the work of Groszek and Jech [GJ] which makes the continuum "fat" but no "taller" than $\aleph_2$, and thus also restricting the value of, say, $\mathfrak{b}$ to at most $\aleph_2$. It is unclear whether our method can be used to make $\mathfrak{b}$ larger than $\aleph_2$ over an arbitrary model satisfying CH, without adding Cohen reals, a problem originally considered by Judah [Mi2, Problem 16.1] and still unsolved. The main obstacle seems to be one may not be able to guarantee the existence of a σ–centered forcing not adding Cohen reals along the iteration. In fact it is known there may not be such a forcing (see [BłSh] and [Sh2]).

References

[BłSh] A. Błaszczyk and S. Shelah, *Regular subalgebras of complete Boolean algebras*, preprint.

[DžSh] M. Džamonja and S. Shelah, *Similar but not the same: various versions of ♣ do not coincide*, J. Symbolic Logic 64 (1999), 180-198.

[FShS] S. Fuchino, S. Shelah and L. Soukup, *Sticks and clubs*, Ann. Pure Appl. Logic 90 (1997), 57-77.

[GJ] M. Groszek and T. Jech, *Generalized iteration of forcing*, Trans. Amer. Math. Soc. 324 (1991), 1-26.

[Mi1] A. Miller, *Special sets of reals*, in: Set Theory of the Reals (H. Judah, ed.), Israel Math. Conf. Proc. 6 (1993), 415-431.

[Mi2] A. Miller, *Arnie Miller's problem list*, in: Set Theory of the Reals (H. Judah, ed.), Israel Math. Conf. Proc. 6 (1993), 645-654.

[Sh1] S. Shelah, *Can you take Solovay's inaccessible away?*, Israel J. Math. 48 (1984), 1-47.

[Sh2] S. Shelah, *There may be no nowhere dense ultrafilter*, in: Logic Colloquium '95 (J.A. Makowsky and E.V. Ravve, eds.), Lecture Notes in Logic 11, Springer, Berlin (1998), 305-324.

[Sh3] S. Shelah, *Historic iteration with $\aleph_\epsilon$–support*, preprint.