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Forcing NS_{ω_1} Completely Bounded via Semiproper Iterations

Tadatoshi MIYAMOTO

南山大学、経営学部、宮元 忠敏

Abstract

We consider the combinatorial principle CB. We discuss its consequences, consistency and negation.

§0. Introducing CB

We begin by defining the combinatorial principle of our concern. This originates from [B-M], [Y] and [W].

0.0 Definition. We say NS_{ω_1} is *completely bounded*, if for any $g:\omega_1\longrightarrow\omega_1$, there is a sequence $\langle X_i \mid i<\omega_1\rangle$ and an ordinal γ s.t.

- For any $i<\omega_1$, X_i is a countable subset of γ with $g(i)<\text{o.t.}(X_i)$. (the order type of X_i is larger than $g(i)$.)
- For any $i<j<\omega_1$, $X_i\subseteq X_j$. (increasing)
- For any limit ordinal $i<\omega_1$, $X_i=\bigcup\{X_l \mid l<i\}$. (continuous)
- $\gamma=\bigcup\{X_i \mid i<\omega_1\}$.

We say *CB* for short to express NS_{ω_1} is completely bounded. We also say any sequence $\langle X_i \mid i<\omega_1\rangle$ is a *CB-sequence for g at γ* for short to express the above 4 conditions on the sequence. Notice that once we have a CB-sequence for g at γ, then we may raise the value of γ upward anywhere below ω_2. So CB iff for any $g\in {}^{\omega_1}\omega_1$, there is a CB-sequence for g at some $\omega_1<\gamma<\omega_2$. Hence we may restrict our attention to those γ's with $\omega_1<\gamma<\omega_2$.

§1. Consequences of CB

1.0 Theorem. *CB implies that there are no $(\omega_1,1)$-morasses.*

Proof. By contradiction. Suppose $\mathcal{A}$ is an $(\omega_1,1)$-morass. We may define $g:\omega_1\longrightarrow\omega_1$ from $\mathcal{A}$ as follows: Given any $i<\omega_1$, take any $A\in\mathcal{A}$ s.t. the rank of A in $\mathcal{A}$ is i. We then set $g(i)=\text{o.t.}(A)$ (the order type of A). Since $\mathcal{A}$ is an $(\omega_1,1)$-morass, this is well-defined. Now let $\langle X_i \mid i<\omega_1\rangle$ be any possible CB-sequence for g at any γ with $\omega_1<\gamma<\omega_2$. We find i s.t. $g(i)>\text{o.t.}(X_i)$ so that these X_i's never satisfy CB for g. To this end, we take a sequence $\langle A_i \mid i<\omega_1\rangle$ s.t. $A_i\in\mathcal{A}$, $\gamma\in A_i$ and the rank of A_i in $\mathcal{A}$ is i. Since $\mathcal{A}$ is a morass, we know that $\langle A_i\cap\gamma \mid i<\omega_1\rangle$ is continuously increasing to γ. Since we have two continuously increasing sequences, we certainly have $i < \omega_1$ s.t. $A_i \cap \gamma = X_i$. Since $\gamma \in A_i$, we have $g(i) > \text{o.t.}(A_i \cap \gamma) = \text{o.t.}(X_i)$.

□

1.1 Theorem. *If CB is ever consistent, then we may construct a universe of set theory where the following hold simultaneously.*

- $\Box_{\omega_1}$ *holds.*
- *A Kurepa tree exists.*
- *No* $(\omega_1, 1)$*-morasses exist.*

1.2 Note. It is known that the existence of an $(\omega_1, 1)$-morass implies both $\Box_{\omega_1}$ and the existence of a Kurepa tree.

Proof. We may start with the ground model where CB holds. We first force $\Box_{\omega_1}$ via a σ-closed and ω_2-Baire p.o.set ([J]). It is clear that CB remains. We then force a Kurepa tree via a c.c.c. forcing. This is possible due to $\Box_{\omega_1}$ ([B]). Both CB ([B-M] and [Y]) and $\Box_{\omega_1}$ remain in the final model.

□

§2. The Partially Ordered Set $Q(g, \gamma)$

2.0 Definition. Let g be any function with $g : \omega_1 \longrightarrow \omega_1$ and γ be any ordinal with $\omega_1 < \gamma$. We want to force a CB-sequence $\langle X_i \mid i < \omega_1 \rangle$ for g at γ. To do so, we may define a p.o.set $Q(g, \gamma)$ as follows: $p = \langle X_i^p \mid i \leq i^p \rangle \in Q(g, \gamma)$, if

- $i^p < \omega_1$. (p is a sequence of countable length with the last entry.)
- For any $i \leq i^p$, X_i^p is a countable subset of γ with $g(i) < \text{o.t.}(X_i^p)$ (= the order type of X_i^p).
- For any $i < j \leq i^p$, we demand $X_i^p \subseteq X_j^p$. (X_i^p's are increasing.)
- For any limit ordinal i with $i \leq i^p$, $X_i^p = \bigcup \{X_l^p \mid l < i\}$. ($X_i^p$'s are continuously increasing.)

For p, $q \in Q(g, \gamma)$, we set $q \leq p$, if $q \supseteq p$. So $p \in Q(g, \gamma)$ iff p is a continuously increasing sequence of countable subsets of γ with right order types and p is of countable length with the last listing. We consider the obvious order on them. Notice that $Q(g, \gamma)$ does not have the greatest element as defined. But there is no need to worry.

2.1 Lemma. *Let $g : \omega_1 \longrightarrow \omega_1$ be any and γ be any ordinal with $\omega_1 < \gamma$. The following are equivalent.*

(1) There is a σ-Baire and semiproper p.o.set Q s.t. Q forces a CB-sequence for g at γ.

(2) For all sufficiently large regular cardinals θ and all countable elementary substructures N of H_θ with g, $\gamma \in N$, there is a countable elementary substructure M of H_θ s.t. $N \subseteq M$, $N \cap \omega_1 = M \cap \omega_1$ and $g(M \cap \omega_1) < o.t.(M \cap \gamma)$.

(3) $Q(g, \gamma)$ is semiproper.

The situation here is very much similar to semiproper seal forcing in the context of ω_2-saturation of NS_{ω_1}. But the relevant large cardinal strength need here appears to be much lower as we see later.

We then consider the class of p.o.sets which preserve the stationary subsets of ω_1. We remind you that the semiproper p.o.sets are included in this class. They may coincide with depending on the universes.

2.2 Lemma. *Let $g : \omega_1 \longrightarrow \omega_1$ be any and γ be any ordinal with $\omega_1 < \gamma$. The following are equivalent.*

(1) There is a σ-Baire p.o.set Q s.t. Q preserves every stationary subset of ω_1 (with Boolean value 1) and that Q forces a CB-sequence for g at γ.

(2) For any stationary subset S of ω_1, $A(S) = \{X \in [\gamma]^\omega \mid X \cap \omega_1 \in S$ and $\forall\, i \leq X \cap \omega_1$ $g(i) < o.t.(X)\}$ is stationary in $[\gamma]^\omega$.

(3) $Q(g, \gamma)$ preseves every stationary subset of ω_1 (with Boolean value 1).

We lastly consider the situation with properness. It is hard to come by with a proper p.o.set, unless g is very simple.

2.3 Lemma. *Let $g : \omega_1 \longrightarrow \omega_1$ be any and γ be any ordinal with $\omega_1 < \gamma$. The following are equivalent.*

(1) There is a (σ-Baire, may omit this condition) proper p.o.set Q s.t. Q forces a CB-sequence for g at γ.

(2) For all sufficiently large regular cardinals θ and all countable elementary substructures N of H_θ with g, $\gamma \in N$, we have $g(N \cap \omega_1) < o.t.(N \cap \gamma)$.

(3) $Q(g, \gamma)$ is proper.

We eventually consider an iterated forcing to get CB. But we provide an observation due to [T] that proper p.o.sets do not work for establishing CB in the latter section. We also know ([S]) that stationary preserving p.o.sets may collapse ω_1, if they are iterated ω-times regardless of the limit. Hence what left is the class of semiproper amongest these three. Since we are interested in semiproper p.o.sets, we provide a proof for the first lemma alone. Others are more or less the same and left to the interested readers.

Proof of 2.1 Lemma. For (1) implies (2): Suppose Q is a σ-Baire and semiproper p.o. set s.t. Q forces a CB-sequence $\langle \dot{X}_i \mid i < \omega_1 \rangle$ for g at γ. Let us write $H = H_{\gamma^+}$ for short. We first show the following:

Claim 1. *$\mathcal{A} = \{A \in [H]^\omega \mid \exists X$ s.t. X is countable, $A \subseteq X$, $A \cap \omega_1 = X \cap \omega_1$, $X \prec H$ and $g(X \cap \omega_1) <$ o.t.$(X \cap \gamma)\}$ contains a club C in $[H]^\omega$.*

Proof. Let S be any stationary set in $[H]^\omega$. It suffices to show $S \cap \mathcal{A} \neq \emptyset$. Since every stationary set in $[H]^\omega$ is semistationary and Q is semiproper, S remains semistationary in V^Q. Namely, we have in V^Q: $S^* = \{X \in [H]^\omega \mid \exists\, A \in S$ s.t. $A \subseteq X$, $A \cap \omega_1 = X \cap \omega_1$ and $X \prec H\}$ is stationary in $[H]^\omega$.

Let χ be a sufficiently large regular cardinal. We may take a countable $\dot{M} \prec \dot{H}_\chi$ (both calculated in V^Q) s.t. $\dot{M} \cap H$ is in the stationary set S^* and $\langle \dot{X}_i \mid i < \omega_1 \rangle \in \dot{M}$. Let $X = \dot{M} \cap H$ and take $A \in S$ which witnesses that $X \in S^*$. Since Q is σ-Baire, we have $X \in V$. It is easy to check $A \in S \cap \mathcal{A}$ due to X.

□

Now θ be any regular cardinal s.t. $H \in H_\theta$. Suppose g, $\gamma \in N \prec H_\theta$. Since $\mathcal{A}$ is definable in H_θ from g and γ, we may assume $C \in N$. Hence $N \cap H \in C$. So there is a countable X s.t. $N \cap H \subseteq X$, $N \cap \omega_1 = X \cap \omega_1$, $X \prec H$ and $g(X \cap \omega_1) <$ o.t.$(X \cap \gamma)$. Let $M = \{f(\vec{x}) \mid \vec{x} \in X \cap \gamma,\ f \in N\}$. Then this M works. Namely, we have

Claim 2. *(1) $N \subseteq M \prec H_\theta$, $N \cap \omega_1 = M \cap \omega_1$ and $X \cap \gamma \subseteq M$.*

And so,

(2) $g(M \cap \omega_1) <$ o.t.$(M \cap \gamma)$.

Proof. To show $N \subseteq M$, take any $n \in N$. Then, say, let $f = \{(\xi, n) \mid \xi < \gamma\} : \gamma \longrightarrow \{n\}$. We have $f \in N$ and $n = f(0) \in M$. To show $X \cap \gamma \subseteq M$, take any $x \in X \cap \gamma$. Then let $f = \{(\xi, \xi) \mid \xi \in \gamma\}$. We have $x = f(x) \in M$. To show $N \cap \omega_1 = M \cap \omega_1$, take any $j \in M \cap \omega_1$. So $j = f(\vec{x})$ for some $\vec{x} \in X \cap \gamma$ and $f \in N$. Since $f \in N \prec H_\theta$, we may assume $f : {}^{<\omega}\gamma \longrightarrow \omega_1$. Since $f \in N \cap ({}^{{}^{<\omega}\gamma}\omega_1) \subseteq N \cap H \subseteq X$. So $f(\vec{x}) \in X \cap \omega_1$. Notice that we in fact had $X \cap \gamma = M \cap \gamma$ above. To show $M \prec H_\theta$, we may use the Tarski's criterion. The following is not precise but typical.

Claim 3. *For any formula $\varphi(y, z)$, if $m = f(x) \in M$ s.t. $f \in N$, $x \in X \cap \gamma$ and $H_\theta \models$ "$\exists\, y\ \varphi(y, m)$", then there is such y in M.*

Proof. Take $h \in N$ s.t. $H_\theta \models$ " for any $\xi \in \gamma$, if $\exists\, y\ \varphi(y, f(\xi))$, then $\varphi(h(\xi), f(\xi))$". This is possible as $f, \gamma \in N \prec H_\theta$. Let $y = h(x) \in M$. This y works.

□

(2) implies (3): We first show density (without assuming (2)).

Claim 4. *For any $p \in Q(g, \gamma)$, any α with $i^p < \alpha < \omega_1$ and any $\xi < \gamma$, there is $q \leq p$ s.t. $i^q = \alpha$ and $\xi \in X^q_\alpha$.*

Proof. By induction on α for all p, ξ.

Case 1. For $\alpha = 0$: It is vacuously true.

Case 2. For $\alpha+1$: Take $p_1 \leq p$ with $\delta^{p_1} = \alpha$. We may already have $\xi \in X_\alpha^{p_1}$ by induction. Then for any X s.t. $X_\alpha^{p_1} \cup \{\xi\} \subseteq X \in [\gamma]^\omega$ and $g(\alpha+1) <$ o.t.(X), let $q = p_1 \cup \{(\alpha+1, X)\}$. This q works.

Case 3. For limit α: Take a strictly increasing sequence of ordinals $\langle \alpha_n \mid n < \omega\rangle$ s.t. $\alpha_0 = i^p$ and $\sup\{\alpha_n \mid n < \omega\} = \alpha$. Then take a sufficiently large regular cardinal θ and any countable $N \prec H_\theta$ with $\{\xi, \gamma, g, p, \langle \alpha_n \mid n < \omega\rangle, Q(g,\gamma)\} \subset N$. This just meant that N contains every relevant parameters. Since $\alpha \in N$ and $\omega_1 < \gamma \in N$, we have $g(\alpha) \in N \cap \omega_1 <$ o.t.$(N \cap \gamma)$. So we may place $N \cap \gamma$ at the α-th, as long as we make sure the continuity. To this end, we enumerate $N \cap \gamma$ by $\langle \xi_n \mid n < \omega\rangle$. We construct a discending sequence of conditions $\langle p_n \mid n < \omega\rangle$ s.t.

- $p_0 = p$, $i^{p_0} = \alpha_0$.
- $p_n \leq p$, $p_n \in N$ and $i^{p_n} = \alpha_n$.
- ξ_n gets captured by p_{n+1}. Namely, $\xi_n \in X_{\alpha_{n+1}}^{p_{n+1}}$.

Now by construction, we have $\bigcup\{X_{i^{p_n}}^{p_n} \mid n < \omega\} = N \cap \gamma$. Hence $q = \bigcup\{p_n \mid n < \omega\} \cup \{(\alpha, N \cap \gamma)\} \in Q(g,\gamma)$. This q works.

□

We now assume (2) and proceed to show $Q(g,\gamma)$ is σ-Baire and semiproper. Fix a sufficiently large regular cardinal θ as in (2) and take any countable $N \prec H_\theta$ with $\{g,\gamma\} \subset N$. And so $Q(g,\gamma) \in N$. Fix any $p \in Q(g,\gamma) \cap N$. Then we may take M as in (2). Construct any $(Q(g,\gamma), M)$-generic sequence $\langle q_n \mid n < \omega\rangle$ with $q_0 = p$. It suffices to find a lower bound q of these conditions. This is because q is $(Q(g,\gamma), M)$-generic so $q \Vdash_{Q(g,\gamma)}$ "$N \cap \omega_1 = M \cap \omega_1 = M[\dot{G}] \cap \omega_1 \supseteq N[\dot{G}] \cap \omega_1$" and so q is $(Q(g,\gamma), N)$-semi-generic. Let $q = \bigcup\{q_n \mid n < \omega\} \cup \{(M \cap \omega_1, M \cap \gamma)\}$. By Claim 4, $q \in Q(g,\gamma)$ and this q works.

□

(3) implies (1): We first note that $Q(g,\gamma)$ is σ-Baire iff $Q(g,\gamma)$ preserves ω_1. To see this, suppose $Q(g,\gamma)$ preserved ω_1. Then $\bigcup \dot{G}$ must be of length ω_1, where $\dot{G}$ is any generic filter. This is because, given any $p \in Q(g,\gamma)$ and any $\xi \in \gamma$, it takes nothing to get $q \leq p$ with $\xi \in X_{i^q}^q$. Similarly, given any countably many open dense subsets D_n's of $Q(g,\gamma)$, there are $\dot{p}_n$'s in the $\dot{G} \cap D_n$'s. But dom$(\bigcup\{\dot{p}_n \mid n < \omega\}) < \omega_1$. Otherwise they would collapse ω_1. Hence there must be a condition $q \in \dot{G}$ which extends every $\dot{p}_n \in \dot{G} \cap D_n$. So q must be in the intersection of the D_n's.

Now suppose $Q(g,\gamma)$ is semiproper. In particular, $Q(g,\gamma)$ preserves ω_1. So $Q(g,\gamma)$ is σ-Baire. We want a CB-sequence for g at γ. But as we see above $\bigcup \dot{G}$ is of length ω_1 and so it is a CB-sequence for g at γ.

□

§3. Using a Measurable Cardinal and Products

3.0 Lemma. *Let κ be a measurable cardinal with a normal measure D. For any regular cardinal $\theta \geq (2^\kappa)^+$, any N s.t. $D \in N \prec H_\theta$ and $|N| < \kappa$, and any ξ with $sup(N \cap \kappa) \leq \xi < \kappa$, we have $M \prec H_\theta$ s.t.*

(1) $N \subset M$ and $|M| = |N|$.

(2) $(M \setminus N) \cap \kappa \neq \emptyset$ and if s is the $<$-least element of $(M \setminus N) \cap \kappa$ then $\xi < s$.

(3) For any $\eta \in N \cap \kappa$, we have $N \cap V_\eta = M \cap V_\eta$.

Proof. Take $s \in \bigcap(N \cap D)$ with $s > \xi$. Let $M = \{f(s) \mid f \in N\}$. Then this M works. We provide some details. We first show that $M \prec H_\theta$ via the Tarski's criterion. Namely,

Claim 1. *For any $f_1(s), \cdots, f_n(s) \in M$, if $H_\theta \models$ "$\exists y \varphi(y, f_1(s), \cdots, f_n(s))$", then there is $f(s) \in M$ s.t. $H_\theta \models$ "$\varphi(f(s), f_1(s), \cdots, f_n(s))$".*

Proof. Note that $H_\theta \models$ "$\exists f : \kappa \longrightarrow \mathrm{ran}(f)\ \forall \alpha < \kappa$, if $\exists y \varphi(y, f_1(\alpha), \cdots, f_n(\alpha))$, then $\varphi(f(\alpha), f_1(\alpha), \cdots, f_n(\alpha))$". This may be expressed as $H_\theta \models$ "$\exists f \Phi(f, f_1, \cdots, f_n)$" for some formula Φ. But $f_1, \cdots, f_n \in N \prec H_\theta$, so we may fix such f in N. Hence if $H_\theta \models$ "$\exists y \varphi(y, f_1(s), \cdots, f_n(s))$" holds, then $H_\theta \models$ "$\varphi(f(s), f_1(s), \cdots, f_n(s))$" holds.

□

For (1): Take any $n \in N$ and let $f = \{(\alpha, n) \mid \alpha < \kappa\}$. Since $D \in N$, we may take $A \in D \cap N$. We have $\kappa = \bigcup A \in N$. So $f \in N \prec H_\theta$ and $n = f(s) \in M$. Hence $N \subset M$. It is clear that N and M are of same size.

For (2): Let $f = \{(\alpha, \alpha) \mid \alpha \in \kappa\}$. Then $f \in N$ and $s = f(s) \in M$. By the choice of ξ and s, we have $s \in (M \setminus N) \cap \kappa$. So it suffices to show that if $g(s) < s$ with $g \in N$, then $g(s) \in N$. We may assume $g : \kappa \longrightarrow \kappa$ is a regressive function. Since D is a normal measure, we have $A \in D$ and $v < \kappa$ s.t. $g''A = \{v\}$. Since relevant parameters are all in N, we may assume that both A and v are in N. So $g(s) = v \in N$.

For (3): It is clear that for any $\tau \in N \cap \kappa$, $N \cap \tau = M \cap \tau$ holds by (2). Since $\langle V_\eta \mid \eta \leq \kappa \rangle \in H_\theta$ is definable from κ in H_θ, we have $\langle V_\eta \mid \eta \leq \kappa \rangle \in N$. Now take any $\eta \in N \cap \kappa$. So $V_\eta \in N$. Let $\tau = |V_\eta| < \kappa$ and fix an onto map $e : \tau \longrightarrow V_\eta$. We may assume both τ and e are in N. To observe $M \cap V_\eta \subseteq N$, take any $m \in M \cap V_\eta$. Since e is onto, there is $i < \tau$ s.t. $m = e(i)$. Since m, τ, e are all in $M \prec H_\theta$, we may assume $i \in M \cap \tau = N \cap \tau$. So $m = e(i) \in N$.

□

3.1 Corollary. *Let κ be a measurable cardinal. Then for any $g : \omega_1 \longrightarrow \omega_1$, the p.o.set $Q(g, \kappa)$ is σ-Baire, semiproper and forces a CB-sequence for g at κ. For any α with $\omega_1 \leq \alpha \leq \kappa$, we have $|\alpha| = \omega_1$ in the generic extensions.*

Proof. By repeatedly applying 3.0 Lemma, we may make sure the second condition (2) in 2.1 Lemma. So $Q(g, \kappa)$ is σ-Baire and semiproper. By the proof of (3) implies (1) in 2.1 Lemma, $Q(g, \kappa)$ forces a CB-sequence for g at κ.

□

In order to take care of all the g's in the ground model at a time (rather than using a book-keeping method in iterated forcing), we may consider the countable support product of the $Q(g,\kappa)$'s for all g. Namely,

3.2 Definition. Let κ be a measurable cardinal. Let $p \in Q(\kappa)$, if p is a countable function s.t. $\mathrm{dom}(p) \subset {}^{\omega_1}\omega_1$ and for all $g \in \mathrm{dom}(p)$, $p(g) \in Q(g,\kappa)$.

For $p, q \in Q(\kappa)$, let $q \leq p$, if $\mathrm{dom}(q) \supseteq \mathrm{dom}(p)$ and for all $g \in \mathrm{dom}(p)$, $q(g) \leq p(g)$ hold in $Q(g,\kappa)$.

$Q(\kappa)$ is a p.o.set with the greatest element $\emptyset$ and satisfies the following:

3.3 Lemma. *(1) For any $g \in {}^{\omega_1}\omega_1$, any $(\delta, p) \in \omega_1 \times Q(\kappa)$ s.t. $\forall f \in dom(p)\ i^{p(f)} < \delta$, and any $\xi < \kappa$, there is (X, q) s.t. $X \in [\kappa]^\omega$, $q \leq p$, $g \in dom(q)$, and for all $f \in dom(q)$, we uniformly have $i^{q(f)} = \delta$ and $\xi \in X = X_\delta^{q(f)}$.*

(2) $Q(\kappa)$ is σ-Baire and semiproper.

(3) In the generic extensions, every $g \in V \cap {}^{\omega_1}\omega_1$ has a CB-sequence at κ.

Proof. It is identical to 2.1 Lemma. We provide some details.

For (1): We proceed by induction on δ for all g, p, ξ.

Case 1. For $\delta = 0$: Vacuously true.

Case 2. For $\delta + 1$: By applying induction hypothesis to $p\lceil\{f \in \mathrm{dom}(p) \mid i^{p(f)} < \delta\}$, we may assume for all $f \in \mathrm{dom}(p)$, $i^{p(f)} = \delta$. Now take any $Y \in [\kappa]^\omega$ s.t. for all $f \in \mathrm{dom}(p)$, $X_\delta^{p(f)} \subseteq Y$ and $f(\delta+1) < \mathrm{o.t.}(Y)$. It is easy to construct q via this Y.

Case 3. For limit δ: Fix a countable $N \prec H_{(2^\kappa)^+}$ s.t. relevant parameters are all in N. We may assume $g \in \mathrm{dom}(p)$. Since $\mathrm{dom}(p)$ is countable, we may assume $\mathrm{dom}(p) \subset N$ and for any $f \in \mathrm{dom}(p)$, we may assume $Q(f,\kappa) \in N$ and $p(f) \in Q(f,\kappa) \cap N$. Hence we may construct $q(f) \leq p(f)$ s.t. $i^{q(f)} = \delta$ and $X_\delta^{q(f)} = N \cap \kappa$ as $f(\delta) \in N \cap \omega_1 < \mathrm{o.t.}(N \cap \kappa)$. This q works.

For (2): Take any countable $N \prec H_{(2^\kappa)^+}$ with $Q(\kappa) \in N$ and any $p \in Q(\kappa) \cap N$. We may assume for all $f \in N \cap {}^{\omega_1}\omega_1$, $f(N \cap \omega_1) < \mathrm{o.t.}(N \cap \kappa)$, while $N \cap \omega_1$ and $N \cap {}^{\omega_1}\omega_1$ remain unchanged. Let $\langle p_n \mid n < \omega \rangle$ be any $(Q(\kappa), N)$-generic sequence with $p_0 = p$. Then we have the following by (1):

- $\bigcup\{\mathrm{dom}(p_n) \mid n < \omega\} = N \cap {}^{\omega_1}\omega_1$.
- $\forall f \in N \cap {}^{\omega_1}\omega_1\ \ \bigcup\{X_{i^{p_n(f)}}^{p_n(f)} \mid f \in \mathrm{dom}(p_n),\ n < \omega\} = N \cap \kappa$.
- $\forall f \in N \cap {}^{\omega_1}\omega_1\ \ \bigcup\{i^{p_n(f)} \mid f \in \mathrm{dom}(p_n),\ n < \omega\} = N \cap \omega_1$.

So we may define $q \in Q(\kappa)$ s.t.

- $\mathrm{dom}(q) = N \cap {}^{\omega_1}\omega_1$.
- $\forall f \in N \cap {}^{\omega_1}\omega_1\ \ q(f) = \bigcup\{p_n(f) \mid f \in \mathrm{dom}(p_n),\ n < \omega\} \cup \{(N \cap \omega_1, N \cap \kappa)\}$.

Then q is a lower bound of the p_n's. In particular, q is $(Q(\kappa), N)$-generic for this N and so q is $(Q(\kappa), N)$-semi-generic in general.

For (3): Let $\dot{G}$ be a $Q(\kappa)$-generic filter over the ground model. For any $g \in V \cap {}^{\omega_1}\omega_1$, let $\bigcup\{p(g) \mid p \in \dot{G}, g \in \mathrm{dom}(p)\} = \langle \dot{X}_i^g \mid i < \omega_1 \rangle$. This sequence works.

□

§4. Consistency of CB

We recap [M] in order to define our iterated forcing. We construct iterated forcing $\langle P_\alpha \mid \alpha \leq \rho \rangle$ together with $\langle \dot{Q}_\alpha \mid \alpha < \rho \rangle$ by recursion on α. The construction is carried out as usual by specifing what Q_α is in V^{P_α} at each successor stage. But we take the following limit.

4.0 Definition. Let ν be a limit ordinal and an iterated forcing $I = \langle P_\alpha \mid \alpha < \nu \rangle$ (together with $\langle \dot{Q}_\alpha \mid \alpha < \nu \rangle$) has been specified. Then the *simple limit* P of I is a suborder of the inverse limit I^* of I s.t. $p \in P$, if there is a sequence of I^*-names $\langle \dot{\alpha}_n \mid n < \omega \rangle$ s.t.

- $\Vdash_{I^*}$ "$\dot{\alpha}_n \leq \dot{\alpha}_{n+1} \leq \nu$".
- If $x \Vdash_{I^*}$ "$\dot{\alpha}_n = \xi$", then $x\lceil\xi^\frown 1 \Vdash_{I^*}$ "$\dot{\alpha}_n = \xi$".
- $p \Vdash_{I^*}$ "$\dot{\alpha}_n < \nu$".
- $\Vdash_{I^*}$ "If $\dot{\alpha} = \sup\{\dot{\alpha}_n \mid n < \omega\}$ and $p\lceil\dot{\alpha} \in \dot{G}\lceil\dot{\alpha}$, then $p \in \dot{G}$", where $\dot{G}$ denotes the canonical I^*-name of the I^*-generic filters.

So each condition in this limit has its own countable (Boolean valued) stages $\dot{\alpha}_n$'s. The stages are required to have some simple dependencies on the generic filters. The $\dot{\alpha}_n$'s are I^*-names but they natually give rise to corresponding P-named stages. When $\mathrm{cf}(\alpha) = \omega$, we have $P = I^*$. So nothing new happens. But when $\mathrm{cf}(\alpha) \geq \omega_1$, there is a chance that the limit is somewhat larger than the direct limit of I.

4.1 Definition. If we take the simple limit at every limit stage, then the iteration is called a *simple iteration.*

We quote the following technical lemma on the simple iterations from [M].

4.2 Lemma. *Let $\langle P_\alpha \mid \alpha \leq \rho \rangle$ be any simple iteration s.t. $\forall \alpha < \rho \Vdash_{P_\alpha}$ "$\dot{Q}_\alpha$ is semiproper". Then*

(1) For any α, β with $\alpha \leq \beta \leq \rho$, we have $\Vdash_{P_\alpha}$ "$P_{\alpha\beta}$ is semiproper".
(2) If $cf(\beta) = \omega_1$, then the direct limit of $\langle P_\alpha \mid \alpha < \beta \rangle$ is dense in P_β.
(3) If ρ is a regular uncountable cardinal and $\forall \alpha < \rho \mid P_\alpha \mid < \rho$, then the direct limt of $\langle P_\alpha \mid \alpha < \rho \rangle$ is dense in P_ρ. (This takes no semiproperness.)
And so,
(4) If ρ is a regular cardinal with $\rho \geq \omega_2$ and $\forall \alpha < \rho \mid P_\alpha \mid < \rho$, then P_ρ has the ρ-c.c.

Now we may state our main observation.

4.3 Theorem. *Let ρ be the $<$-least strongly inaccessible cardinal s.t. $\{\kappa < \rho \mid \kappa$ is measurable$\}$ is cofinal below ρ. Then we have a simple iteration $\langle P_\alpha \mid \alpha \leq \rho \rangle$ s.t.*

(1) P_ρ is semiproper and so preserves ω_1 and the stationary subsets of ω_1.

(2) P_ρ has the ρ-c.c.

(3) In V^{P_ρ}, CB holds and $2^{\omega_1} = \omega_2 = \rho$.

Proof. Let $\langle \kappa_\alpha \mid \alpha < \rho \rangle$ enumerate $\{\kappa < \rho \mid \kappa$ is measurable$\}$ in increasing order. Notice that for any limit β, we have $\sup\{\kappa_\alpha \mid \alpha < \beta\} < \kappa_\beta$. Construct $\langle P_\alpha \mid \alpha \leq \rho \rangle$ together with $\langle \dot{Q}_\alpha \mid \alpha < \rho \rangle$ by recursion so that

(4) $P_0 = \{\emptyset\}$

(5) $P_\alpha \in H_{\kappa_\alpha}$ and $\Vdash_{P_\alpha}$"$\dot{Q}_\alpha$ is the countable support product of the $Q(g, \kappa_\alpha)$ for $g \in {}^{\omega_1}\omega_1 \cap V[\dot{G}_\alpha]$" and so,

- $\Vdash_{P_\alpha}$"$\dot{Q}_\alpha$ is σ-Baire and semiproper".
- $\Vdash_{P_\alpha}$"$\dot{Q}_\alpha \subset H_{\kappa_\alpha}^{V[\dot{G}_\alpha]}$".

So we may assume

- $P_{\alpha+1} \subset H_{\kappa_\alpha} \in H_{\kappa_{\alpha+1}}$.

(6) For limit β, P_β is the simple limit of $\langle P_\alpha \mid \alpha < \beta \rangle$ and so

- $| P_\beta | \leq \Pi_{\alpha<\beta} | P_\alpha | \leq 2^{\Sigma_{\alpha<\beta}|P_\alpha|} < \kappa_\beta$.

This completes the construction. By 4.2 Lemma, we know that (1) and (2) hold.

For (3): Suppose $g : \omega_1 \longrightarrow \omega_1$ in $V[G_\rho]$, where G_ρ is any P_ρ-generic filter over the ground model V. Since P_ρ has the ρ-c.c, there is a stage $\alpha < \rho$ s.t. $g \in V[G_\alpha]$. Then in $V[G_{\alpha+1}]$, there is a CB-sequence for g at κ_α. This is upward absolute. So remains in $V[G_\rho]$. Notice that $| \kappa_\alpha | = \omega_1$ in $V[G_{\alpha+1}]$. But ρ remains a cardinal. Hence $\rho = \omega_2^{V[G_\rho]}$. Since the direct limit of $\langle P_\alpha \mid \alpha < \rho \rangle$ is dense in P_ρ, we may conclude that the value of 2^{ω_1} in $V[G_\rho]$ is exactly ρ by counting the number of the nice names for the subsets of ω_1 in V.

□

4.4 Question. (1) CB implies the existence of some large cardinal ([D-L]). So we need some large cardinal to get CB. Can we get the equiconsistency here. It would be very interesting because this situation sits below the picture: A Woodin cardinal (+ a measurable cardinal above it) vs. the saturation of NS_{ω_1} ([W]).

(2) It is easy to arrange $2^\omega = 2^{\omega_1} = \omega_2$. But can you arrange so that $2^\omega = \omega_1, 2^{\omega_1} = \omega_2$? In particular, we do not know the value of 2^ω in this model. The approach in [Chaper XI, say, p. 546 in S] does not seem to work in this case. So the positive solution to this problem would lead to a new technique in iterated forcing. The negative solution would shed light on the nature of the universes of set theory.

§5. Negation of CB

This section is based on [T]. We first rephrase CB using stationary sets.

5.0 Proposition. *The following are equivalent.*

(1) CB fails.

(2) $\exists g : \omega_1 \longrightarrow \omega_1 \forall \gamma \in (\omega_1, \omega_2)$ $\{X \in [\gamma]^\omega \mid g(X \cap \omega_1) \geq o.t.(X)\}$ *is stationary in* $[\gamma]^\omega$.

Proof. Any club in $[\gamma]^\omega$, with $\omega_1 < \gamma < \omega_2$, contains a continously increasing sequence of length ω_1 s.t. the union of those countable subsets of γ listed in the sequence is exactly γ.

□

We get a strong failure of CB.

5.1 Lemma. *If we force with the set of countable initial segments* ${}^{<\omega_1}\omega_1$*, then in the generic extensions, we have*

- $\exists g : \omega_1 \longrightarrow \omega_1 \forall \gamma > \omega_1 \{X \in [\gamma]^\omega \mid g(X \cap \omega_1) \geq o.t.(X)\}$ *is stationary in* $[\gamma]^\omega$.

Proof. Let $P = {}^{<\omega_1}\omega_1$ and define $g = \bigcup G$, where G is a P-generic filter. We observe this g works. Suppose $p \Vdash_P$ "$\dot{f} : {}^{<\omega}\gamma \longrightarrow \gamma$". We want to find $X \in [\gamma]^\omega$ and $q \leq p$ s.t. $q \Vdash_P$ "X is closed under $\dot{f}$ and $\dot{g}(X \cap \omega_1) \geq$ o.t.(X)". To this end, let θ be a sufficiently large regular cardinal and take a countable $N \prec H_\theta$ s.t. $P, p, \dot{f} \in N$. Define $X = N \cap \gamma$. Fix a (P, N)-generic sequence $\langle p_n \mid n < \omega \rangle$ with $p_0 = p$. Let $q = \bigcup \{p_n \mid n < \omega\} \cup \{(N \cap \omega_1, v)\}$, where $v \in [\text{o.t}(N \cap \gamma), \omega_1)$. Then $q \leq p$ is (P, N)-generic and $q \Vdash_P$ "$\dot{g}(N \cap \omega_1) = v \geq$o.t.$(X)$". In particular, $q \Vdash_P$ "$X = N \cap \gamma = N[\dot{G}] \cap \gamma$ is closed under $\dot{f} \in N[\dot{G}]$". We are done.

□

So the strong failure of CB is preserved by any notion of forcing which is proper. Accordingly, we have

5.2 Theorem. *It is consistent that no proper forcing construction produce a model of CB even if large cardinals are available.*

Proof. Consider the universe V^P, where $P = {}^{<\omega_1}\omega_1$. We have the strong failure of CB. Since proper forcing preserves every stationary set, no proper forcing over this model would ever produce CB.

□

5.3 Corollary. ([T]) *The following are all consistent provided that a supercompact cardinal exists.*

- *PFA*$^+ + \neg$ *CB.*
- *PFA*$^+ + \neg$*(*NS_{ω_1} *is saturated).*

- *PFA$^+$ + ¬SRP(Strong Reflection Principle).*
- *PFA$^+$ + ¬MM (Martin's Maximum).*

Proof. We simply note the following well-known implications (see [B]). MM $\Rightarrow$ SRP $\Rightarrow$ saturation $\Rightarrow$ CB.

□

The last implication is due to [B-M] and likely to [W].

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Mathematics
Nanzan University
18, Yamazato-cho,
Showa-ku, Nagoya
466-8673, Japan
e-mail: miyamoto@iq.nanzan-u.ac.jp