数理解析研究所講究録
1145 巻 2000 年 1-9

Numerical Analysis of Nedelec's Edge Elements

Fumio Kikuchi (菊地 文雄) *
Graduate School of Mathematical Sciences
University of Tokyo, Tokyo 153–8914 Japan

1 Introduction

To solve electromagnetic problems effectively by FEM, the Nedelec edge elements [12],[13] is in wide use, and such use is now considered to be essential [4],[8],[14]. This may be mainly attributed to the facts that such elements are usually free from the spectral pollution by spurious modes and they are robust to geometric singularities such as caused by reentrant corners. Moreover, they are easy to calculate the rotations of vector fields and to deal with the electromagnetic boundary conditions. However, it has not been easy to show their mathematical validity since the formulations using edge elements are usually based on some mixed variational principles and hence we must prove various conditions such as the inf-sup one, the discrete compactness, etc. for respective schemes.

Thus we have performed theoretical analysis of the edge elements, and, in particular, we showed the discrete compactness properties for the simplest Nedelec simplex elements [8],[9]. Especially, such properties play essential roles in showing that the associated finite element schemes for electromagnetic spectral problems are free from the spectral pollution. However, the corresponding properties for more general edge elements have been quite difficult to prove. Recently, Prof. Boffi has obtained remarkable results [2],[3] on the discrete compactness, and this work is devoted to giving some related results such as an alternative proof supplementing his original one.

2 Physical formulations of a model problem

To explain our finite element method, we will use a model problem. That is, let us consider the cavity resonator eigenvalue problem, which is essentially to determine non-trivial time-harmonic electromagnetic fields $\vec{E}$ and $\vec{H}$ satisfying the Maxwell equations in a vacuum cavity (Ω) surrounded by a perfectly conducting wall ($\partial\Omega$).

*email: kikuchi@ms.u-tokyo.ac.jp

More specifically, the Maxwell equations for a vacuum region with the above boundary conditions are givend by

$$\operatorname{rot}\vec{E} + \frac{\partial \vec{B}}{\partial t} = \vec{0}\,,\ \operatorname{div}\vec{D} = \rho\,(=0)\,,\ \operatorname{rot}\vec{H} - \frac{\partial \vec{D}}{\partial t} = \vec{j}\,(=\vec{0})\,,\ \operatorname{div}\vec{B} = 0\ \ in\ \Omega\,, \tag{1}$$

$$\vec{E}\times\vec{n} = \vec{0}\,,\quad \vec{B}\cdot\vec{n} = 0\ \ on\ \partial\Omega\,, \tag{2}$$

where $\vec{E}$ = electric field, $\vec{H}$ = magnetic field, $\vec{D}$ = electric flux density = $\varepsilon_0\vec{E}$, $\vec{B}$ = magnetic flux density = $\mu_0\vec{H}$, ε_0 = dielectric constant of vacuum > 0, μ_0 = magnetic permeability of vacuum > 0, ρ = electric charge density, $\vec{j}$ = electric current density, $\vec{n}$ = outward unit normal on $\partial\Omega$, and t = time variable.

Introducing the time-harmonic assumption, i. e., the unknown fields vary like $e^{i\omega t}$ in time, we have

$$\operatorname{rot}\vec{E} = -i\omega\mu_0\vec{H}\,,\quad \operatorname{div}\vec{E} = 0\,,\quad \operatorname{rot}\vec{H} = i\omega\varepsilon_0\vec{E}\,,\quad \operatorname{div}\vec{H} = 0\ \ in\ \Omega\,, \tag{3}$$

$$\vec{E}\times\vec{n} = \vec{0}\,,\quad \vec{H}\cdot\vec{n} = 0 \qquad on\ \partial\Omega\,, \tag{4}$$

where i = imaginary unit, ω = angular frequency, and the vector fields are now functions of space variables only.

It is now possible to give formulations in terms of $\vec{E}$ or $\vec{H}$ only.

$\vec{E}$-formulation : *Find non-trivial $\vec{E}$ and $\lambda = \varepsilon_0\mu_0\omega^2 (\in \mathbf{R})$ such that*

$$\operatorname{rot}\,\operatorname{rot}\vec{E} = \lambda\vec{E}\,,\quad \operatorname{div}\vec{E} = 0\ \ in\ \Omega\,;\quad \vec{E}\times\vec{n} = \vec{0}\ \ on\ \partial\Omega\,. \tag{5}$$

$\vec{H}$-formulation : *Find non-trivial $\vec{H}$ and $\lambda = \varepsilon_0\mu_0\omega^2 (\in \mathbf{R})$ such that*

$$\operatorname{rot}\,\operatorname{rot}\vec{H} = \lambda\vec{H}\,,\quad \operatorname{div}\vec{H} = 0\ \ in\ \Omega\,;\quad \vec{H}\cdot\vec{n} = 0\,,\quad (\operatorname{rot}\vec{H})\times\vec{n} = \vec{0}\ \ on\ \partial\Omega\,. \tag{6}$$

The equivalence of these two formulations is well known for $\lambda \neq 0$ at least physically, and can be also shown mathematically under appropriate setting of functions spaces for vector functions.

3 Mathematical preliminaries

Let $\Omega \subset \mathbf{R}^3$ be a bounded domain with Lipschitz boundary $\partial\Omega$. Furthermore, we also assume that Ω is simply-connected and $\partial\Omega$ is connected. Then we can assure the existence of both the scalar and vector potentials under appropriate boundary conditions.

Besides the usual Sobolev spaces such as $H^1(\Omega)$, $H_0^1(\Omega)$ and $L_p(\Omega)$ $(1 \le p < +\infty)$, we will also use some Sobolev-like spaces for vector fields:

$$H(\operatorname{rot};\Omega) = \{\vec{u}\in L_2(\Omega)^3;\ \operatorname{rot}\vec{u}\in L_2(\Omega)^3\}, \tag{7}$$

$$H_0(\operatorname{rot};\Omega) = \{\vec{u}\in H(\operatorname{rot};\Omega);\ \vec{u}\times\vec{n} = \vec{0}\ on\ \partial\Omega\}, \tag{8}$$

$$
\begin{aligned}
H(\mathrm{rot}^0;\Omega) &= \{\vec{u} \in H(\mathrm{rot};\Omega);\ \mathrm{rot}\ \vec{u} = \vec{0}\}, & (9)\\
H_0(\mathrm{rot}^0;\Omega) &= H_0(\mathrm{rot};\Omega) \cap H(\mathrm{rot}^0;\Omega), & (10)\\
H(\mathrm{div};\Omega) &= \{\vec{u} \in L_2(\Omega)^3;\ \mathrm{div}\ \vec{u} \in L_2(\Omega)\}, & (11)\\
H_0(\mathrm{div};\Omega) &= \{\vec{u} \in H(\mathrm{div};\Omega);\ \vec{u}\cdot\vec{n} = 0\ \mathit{on}\ \partial\Omega\}, & (12)\\
H(\mathrm{div}^0;\Omega) &= \{\vec{u} \in H(\mathrm{div};\Omega);\ \mathrm{div}\ \vec{u} = 0\}, & (13)\\
H_0(\mathrm{div}^0;\Omega) &= H_0(\mathrm{div};\Omega) \cap H(\mathrm{div}^0;\Omega), & (14)
\end{aligned}
$$

where the subscript "0" means that the tangential or normal components of the vector functions vanish on $\partial\Omega$, and the superscript "0" does that the vector fields are divergence- or rotation-free. These function spaces become Hilbert spaces when equipped with appropriate inner products. Moreover, we will use $(\cdot,\cdot)$ and $\|\cdot\|$ respectively as the notations of the inner product and the norm of $\{L_2(\Omega)\}^3$ as well as those of $L_2(\Omega)$. For details of the above spaces, especially the definitions of boundary conditions, cf. [5],[7]. It is also to be noted that, for the present Ω, the existence of the scalar potentials is assured in the sense

$$H(\mathrm{rot}^0;\Omega) = \mathrm{grad}\, H^1(\Omega)\,, \quad H_0(\mathrm{rot}^0;\Omega) = \mathrm{grad}\, H_0^1(\Omega)\,, \tag{15}$$

where $\mathrm{grad}\, H^1(\Omega)$ for example implies $\{\mathrm{grad}\,\varphi;\ \varphi \in H^1(\Omega)\}$.

To analyze the Maxwell operator, the following compactness properties stated in [1] are essential:

$H_0(\mathrm{rot};\Omega)\cap H(\mathrm{div}^0;\Omega)$ *and* $H(\mathrm{rot};\Omega)\cap H_0(\mathrm{div}^0;\Omega)$ *are compactly imbedded to* $\{L_2(\Omega)\}^3$.

Here the divergence-free conditions are essential and may be expressed weakly as follows: *A vector field* $\vec{u} \in H_0(\mathrm{rot};\Omega)$ ($H(\mathrm{rot};\Omega)$, *resp.*) *satisfies*

$$(\vec{u}, \mathrm{grad}\ \varphi) = 0\,;\quad \forall\varphi \in H_0^1(\Omega)\ \ (H^1(\Omega),\ \mathit{resp.})\,. \tag{16}$$

As above and henceforth, we will use $\vec{u}$ instead of $\vec{E}$ and $\vec{H}$.

We can now give fundamental variational formulations for (5) and (6) as follows.

[E] *Find* $\{\lambda, \vec{u}\} \in \mathbf{R}\times\{H_0(\mathrm{rot};\Omega) \cap H(\mathrm{div}^0;\Omega)\}$ *such that* $\vec{u} \neq \vec{0}$ *and*

$$(\mathrm{rot}\ \vec{u}, \mathrm{rot}\ \vec{v}) = \lambda(\vec{u}, \vec{v})\ ;\ \forall\vec{v} \in H_0(\mathrm{rot};\Omega)\ . \tag{17}$$

[H] *Find* $\{\lambda, \vec{u}\} \in \mathbf{R}\times\{H(\mathrm{rot};\Omega) \cap H_0(\mathrm{div}^0;\Omega)\}$ *such that* $\vec{u} \neq \vec{0}$ *and*

$$(\mathrm{rot}\ \vec{u}, \mathrm{rot}\ \vec{v}) = \lambda(\vec{u}, \vec{v})\ ;\ \forall\vec{v} \in H(\mathrm{rot};\Omega)\ . \tag{18}$$

These formulations are not symmetric with respect to $\vec{u}$ and $\vec{v}$, i. e, the divergence-free conditions are a priori imposed on $\vec{u}$ but not on $\vec{v}$. However, for $\lambda \neq 0$, $\vec{u}$ automatically satisfies them in the sense of (16) even when they are not imposed, since $\mathrm{grad}\,\varphi \in H_0(\mathrm{rot}^0;\Omega)$ ($H(\mathrm{rot}^0;\Omega)$, resp.) for $\varphi \in H_0^1(\Omega)$ ($H^1(\Omega)$, resp.), cf. [8]. Thus we will use formulations with the divergence-free conditions omitted for numerical analysis. Of course, such conditions play essential roles in theoretical analysis of spectral problems: under appropriate settings, our problems become spectral problems of symmetric positive compact operators, i. e., very standard and typical problems in functional analysis. In fact, we have nice properties such as:

(a) the set of eigenvalues is countable; they can be numbered as $\{\lambda_i\}_{i=1}^{\infty}$,

(b) $\lambda_i > 0$ for $\forall i \in \mathbf{N}$, and hence $\{\lambda_i\}_{i=1}^{\infty}$ may be numbered in the increasing order with $\lim_{i\to\infty} \lambda_i = +\infty$,

(c) finite multiplicity of each λ_i,

(d) completeness of the eigenfunctions in some associated spaces, etc.

Thus an essential point of numerical analysis is how to approximate the divergence-free conditions appropriately, hence arises the concept of discrete compactness.

It is also possible to use the Lagrange multiplier to deal with the divergence-free conditions. However, we can see that the multiplier is essentially zero for the present problems, and hence we can avoid its use at least formally: see [8] for details.

4 Finite element approximations

Let us introduce finite dimensional spaces $G^h \subset H^1(\Omega)$ and $R^h \subset H(\mathrm{rot};\Omega)$, and then define $G_0^h := G^h \cap H_0^1(\Omega)$ and $R_0^h := R^h \cap H_0(\mathrm{rot};\Omega)$. For these, we assume the internal existence of scalar potentials, analogously to (15):

$$\mathrm{grad}\, G^h = R^h \cap H(\mathrm{rot}^0;\Omega)\,, \quad \mathrm{grad}\, G_0^h = R_0^h \cap H(\mathrm{rot}^0;\Omega)\,. \tag{19}$$

As usual, we first consider a family of finite element triangulations $\{T_h\}_{h>0}$ of Ω, where h is the discretization parameter such that $h \downarrow 0$. Then we construct the above type of spaces G^h, R^h etc., often called *finite element spaces*, for each triangulation T_h.

Now the divergence-free condition $\vec{u} \in H(\mathrm{div}^0;\Omega)$ ($H_0(\mathrm{div}^0;\Omega)$, resp.) for $\vec{u} \in H_0(\mathrm{rot};\Omega)$ ($H(\mathrm{rot};\Omega)$, resp.) is approximated by the orthogonality condition $\vec{u}_h \perp \mathrm{grad}\, G_0^h$ ($\mathrm{grad}\, G^h$, resp.) for $\vec{u}_h \in R_0^h$ (R^h, resp.), that is,

$$(\vec{u}_h, \mathrm{grad}\ \varphi_h) = 0\,;\ \forall \varphi_h \in G_0^h\ \left(G^h,\ \mathit{resp.}\right). \tag{20}$$

We can give finite element schemes based on [E] and [H] with the divergence-free conditions omitted.

$[\mathrm{E}]_h$ *Find* $\{\lambda_h, \vec{u}_h\} \in \mathbf{R}\times R_0^h$ *such that* $\vec{u}_h \neq \vec{0}$ *and*

$$(\mathrm{rot}\, \vec{u}_h, \mathrm{rot}\, \vec{v}_h) = \lambda_h(\vec{u}_h, \vec{v}_h)\ ;\ \forall \vec{v}_h \in R_0^h\ . \tag{21}$$

$[\mathrm{H}]_h$ *Find* $\{\lambda_h, \vec{u}_h\} \in \mathbf{R}\times R^h$ *such that* $\vec{u}_h \neq \vec{0}$ *and*

$$(\mathrm{rot}\, \vec{u}_h, \mathrm{rot}\, \vec{v}_h) = \lambda_h(\vec{u}_h, \vec{v}_h)\ ;\ \forall \vec{v}_h \in R^h\ . \tag{22}$$

It is easy to see that $\vec{u}_h$ of $[\mathrm{E}]_h$ or $[\mathrm{H}]_h$ for $\lambda_h \neq 0$ satisfies the approximate divergence-free conditions (20) by taking $\vec{v}_h$ as $\mathrm{grad}\,\varphi_h$ as assured by (19).

As in the continuous cases, we can use the Lagrange multiplier to deal with the divergence-free conditions. However, under the present settings for R^h, R_0^h, G^h and G_0^h with (19), we can again see that the (approximate) multiplier essentially vanishes.

For the analysis of the above finite element schemes, we usually require:

(1) approximation capability for R^h, R_0^h, G^h and G_0^h,

(2) uniform lifting property (inf-sup condition), since our schemes are of mixed (or saddle-point) type in a sense [5],[7],

(3) discrete compactness properties, since our finite element spaces R^h and R_0^h are not necessarily contained in $H_0(\mathrm{rot};\Omega) \cap H(\mathrm{div}^0;\Omega)$ or $H(\mathrm{rot};\Omega) \cap H_0(\mathrm{div}^0;\Omega)$.

See [6] for details, where the spectral projection techniques are fully employed and the importance of the discrete compactness is emphasized.

Finally, let us show the simplest examples of edge-type finite elements for R^h introduced by Nedelec [12], which are tetrahedral or rectangular parallelepiped shape.

(i) Tetrahedral element: in each element, $\vec{u}_h = (u_1, u_2, u_3)$ is of the form $\vec{u}_h = \vec{\alpha} + \vec{\beta} \wedge \vec{x}$:

$$u_1 = \alpha_1 + \beta_2 x_3 - \beta_3 x_2\,, \quad u_2 = \alpha_2 + \beta_3 x_1 - \beta_1 x_3\,, \quad u_3 = \alpha_3 + \beta_1 x_2 - \beta_2 x_1\,, \tag{23}$$

where $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ and $\vec{\beta} = (\beta_1, \beta_2, \beta_3)$ are coefficients, and $\vec{x} = (x_1, x_2, x_3)$.

(ii) Cube-based element: $\vec{u}_h$ in each rectangular parallelepiped element is of the form

$$\begin{aligned} &u_1 = \alpha_1 + \beta_1 x_2 + \beta_2 x_3 + \beta_3 x_2 x_3\,, \quad u_2 = \alpha_2 + \beta_4 x_3 + \beta_5 x_1 + \beta_6 x_3 x_1\,, \\ &u_3 = \alpha_3 + \beta_7 x_1 + \beta_8 x_2 + \beta_9 x_1 x_2\,. \end{aligned} \tag{24}$$

In the above approximations, Ω must be an appropriate polyhedral domain so that the triangulations may be possible. In addition, it is known for the above finite element spaces that (19) hold true when Ω is simply-connected and $\partial\Omega$ is connected [1],[12].

5 Discrete compactness property

In order to perform mathematical analysis of the present FEM, it is essential to show some discrete compactness properties. For the analysis of $[\mathrm{E}]_h$, a typical example of such properties is stated as follows [2],[9].

$[\mathrm{DC}]_E$ *Let* $\{\vec{u}_h\}_{h>0}$ *be an arbitrary h-family such that*

$$\vec{u}_h \in R_0^h\,, \quad \|\vec{u}_h\|_{H(\mathrm{rot};\Omega)} = 1\,, \quad \vec{u}_h \perp \mathrm{grad}\, G_0^h\,. \tag{25}$$

Then there exist a subfamily, again denoted by $\{\vec{u}_h\}_{h>0}$, *and* $\vec{u}_0 \in H_0(\mathrm{rot};\Omega) \cap H(\mathrm{div}^0;\Omega)$ *such that* $\vec{u}_h \to \vec{u}_0$ *weakly in* $H_0(\mathrm{rot};\Omega)$ *and strongly in* $\{L_2(\Omega)\}^3$ *as* $h \downarrow 0$. *(Here the "strong-convergence" part is essential.)*

Remark 1 For comparison, let us give a statement of the original compactness for [E]: *Let* $\{\vec{u}_n\}_{n=1}^{\infty}$ *be an arbitrary sequence such that* $\vec{u}_n \in H_0(\mathrm{rot};\Omega)$, $\|\vec{u}_n\|_{H(\mathrm{rot};\Omega)} = 1$, *and satisfies* $(\vec{u}_n, \mathrm{grad}\ \varphi) = 0$ $(\forall \varphi \in H_0^1(\Omega))$ *for each* n. *Then there exist a subsequence, again denoted by* $\{\vec{u}_n\}_{n=1}^{\infty}$, *and* $\vec{u}_0 \in H_0(\mathrm{rot};\Omega) \cap H(\mathrm{div}^0;\Omega)$ *such that* $\vec{u}_n \to \vec{u}_0$ *weakly in* $H_0(\mathrm{rot};\Omega)$ *and strongly in* $L_2(\Omega)^3$ *as* $n \to \infty$.

It is also possible to give an example $[\mathrm{DC}]_H$ of such properties for $[\mathrm{H}]_h$ in a quite similar fashion. We will consider $[\mathrm{DC}]_E$ only since the analysis is also similar to that of $[\mathrm{DC}]_E$. Such a property can be effectively used to analyze finite element schemes for the present spectral problems. More specifically, we can use the spectral projection to evaluate numerical errors, cf. [3],[6]. In the present special case, it is also possible to apply the Rayleigh quotient approach based on the min-max and max-min principles [11].

Moreover, by using the orthogonal projection operator $Q_E : H_0(\mathrm{rot};\Omega) \to H_0(\mathrm{rot}^0;\Omega) = \mathrm{grad}\, H_0^1(\Omega)$, we can show that $[\mathrm{DC}]_E$ is equivalent to the condition [11]

$$\lim_{h\downarrow 0} \sup_{\vec{u}_h \in \{R_0^h \setminus \{0\}\} \cap \{\mathrm{grad}\, G_0^h\}^{\perp}} \frac{\|Q_E \vec{u}_h\|_{L_2(\Omega)^3}}{\|\vec{u}_h\|_{H(\mathrm{rot};\Omega)}} = 0 \tag{26}$$

under the approximability condition for G_0^h: $\lim_{h\downarrow 0} \inf_{\varphi_h \in G_0^h} \|\varphi_h - \varphi\|_{H^1(\Omega)} = 0\,;\ \forall \varphi \in H_0^1(\Omega)$.

Let us now assume that the domain Ω is a bounded polyhedral domain. Then we consider the regular family of triangulations $\{T^h\}_{h>0}$ by tetrahedra or rectangular parallelepipeds, where h is the discretization parameter such that $h \downarrow 0$ and denotes the maximum diameter of finite elements K's in each T^h, cf. [7]. As R^h, we consider any one in the family of edge element spaces given by Nedelec [12]. On the other hand, G^h associated with R^h is the usual node-type finite element space with the corresponding order [12]. Then we can show that (19) hold true for such G^h and R^h.

Let us introduce the interpolation operator Π_h for the considered R^h, which was defined by Nedelec [12] and plays essential roles in our analysis. More specifically, it mapps any "sufficiently smooth" vector function $\vec{v}$ to an element in R^h by specifying the degrees of freedom in the fashion stated in [12], and it is actually well-defined for some non-smooth but element-wise smooth vector functions as well. Typical examples of d. o. f. are moments of edge-direction components of vector functions in R^h. Furthermore, $\Pi_h \vec{v}$ belongs to R_0^h when $\vec{v}$ also belongs to $H_0(\mathrm{rot};\Omega)$ [12]. Then we should notice the following important properties [1],[12]:

(i) *Let* φ *be a "sufficiently smooth" scalar function in* $H^1(\Omega)$ *such that* Π_h *can operate on* $\mathrm{grad}\,\varphi \in H(\mathrm{rot}^0;\Omega)$. *Then there exists* $\varphi_h \in G^h$ *such that* $\Pi_h(\mathrm{grad}\ \varphi) = \mathrm{grad}\,\varphi_h$. *Furthermore, when* φ *belongs to* $H_0^1(\Omega)$ *as well,* φ_h *is in* G_0^h.

(ii) *sufficient conditions that Π_h is applicable to $\vec{v}$: $\vec{v} \in H(\mathrm{rot};\Omega)$ satisfies, $\forall K \in T^h$, $\vec{v}|K \in \{H^{\frac{1}{2}+\delta}(K)\}^3$ for some $\delta > 0$, and* rot *$\vec{v}|K \in \{L_p(K)\}^3$ for some $p > 2$, where $H^{\frac{1}{2}+\delta}(K)$ is the usual fractional Sobolev space over K.*

(iii) $H_0(\mathrm{rot};\Omega) \cap H(\mathrm{div}^0;\Omega) \subset \{H^{\frac{1}{2}+\delta}(\Omega)\}^3$ *(continuously) for some positive $\delta \leq \frac{1}{2}$.*

By using Π_h and the orthogonal decomposition of $\vec{u}_h$ in (25), we can obtain the following main results.

Theorem 1 *Let $\{T^h\}_{h>0}$ be a regular family of triangulations of Ω by tetrahedra or rectangular parallelepipeds, and let $\{\{R_0^h, G_0^h\}\}_{h>0}$ be the associated family of finite element spaces introduced by Nedelec in* [12]. *Then* $[\mathrm{DC}]_E$ *holds true.*

Remark 2 The present theorem is essentially due to Boffi [2] based on the Fortin operator, but here we give an alternative proof and also supplement his proof in interpolation error analysis. From the above, it is also easy to show the asymptotic *uniform coerciveness* of the bilinear form for the rotation operator in the following sense: *There exist $C > 0$ and $h_0 > 0$ such that, $0 < \forall h \leq h_0$ and $\forall \vec{v}_h \in R_0^h \cap \{\mathrm{grad}\, G_0^h\}^\perp$,*

$$\|\mathrm{rot}\, \vec{v}_h\| \geq C\|\vec{v}_h\|_{H(\mathrm{rot};\Omega)}\,. \tag{27}$$

<u>*Sketch of proof*</u>: 1° Let us consider an arbitrary h-family $\{\vec{u}_h\}_{h>0}$ such that

$$\vec{u}_h \in R_0^h\,, \quad \|\vec{u}_h\|_{H(\mathrm{rot};\Omega)} = 1\,, \quad \textit{and} \quad (\vec{u}_h, \mathrm{grad}\, \psi_h) = 0 \quad (\forall \psi_h \in G_0^h)\,.$$

Then, as in [9], let us use the orthogonal decomposition of $H_0(\mathrm{rot};\Omega)$ based on the projecition operator $Q_E : H_0(\mathrm{rot};\Omega) \to H_0(\mathrm{rot}^0;\Omega) = \mathrm{grad}\, H_0^1(\Omega)$:

$$\begin{aligned} \vec{u}_h &= Q_E\vec{u}_h + \vec{v}^h\,; \\ & Q_E\vec{u}_h = \mathrm{grad}\,\varphi^h \ \textit{for}\ \varphi^h \in H_0^1(\Omega),\ \vec{v}^h = (1 - Q_E)\vec{u}_h \in H_0(\mathrm{rot};\Omega) \cap H(\mathrm{div}^0;\Omega). \end{aligned}$$

Now by (26), what we should show is, as $h \downarrow 0$,

$$\|Q_E\vec{u}_h\|_{L_2(\Omega)^3} \to 0 \ \textit{uniformly with respect to}\ \vec{u}_h\,.$$

2° Let us consider $\Pi_h\varphi^h$ for the above φ^h. It is actually well defined [1], and so there exists $\varphi_h \in G_0^h$ such that $\Pi_h(\mathrm{grad}\,\varphi^h) = \mathrm{grad}\,\varphi_h$. Similarly, $\Pi_h\vec{v}^h$ is also well defined, thus we have from the identity $\vec{u}_h = \Pi_h\vec{u}_h$ that

$$\mathrm{grad}\,\varphi^h + \vec{v}^h = \mathrm{grad}\,\varphi_h + \Pi_h\vec{v}^h\,, \quad \textit{or} \quad \mathrm{grad}\,\varphi^h - \mathrm{grad}\,\varphi_h = \Pi_h\vec{v}^h - \vec{v}^h\,.$$

3° Let us here consider a projection $\varphi_h^* \in G_0^h$ of φ^h defined by

$$(\mathrm{grad}\,\varphi_h^*, \mathrm{grad}\,\psi_h) = (\mathrm{grad}\,\varphi^h, \mathrm{grad}\,\psi_h) \quad (\forall \psi_h \in G_0^h)\,.$$

Since the right-hand side of the above is evaluated as $(\vec{u}_h - \vec{v}^h, \operatorname{grad}\psi_h) = 0$, we find that $\operatorname{grad}\varphi_h^* = \vec{0}$. Then the best approximation property of $\operatorname{grad}\varphi_h^*$ to $\operatorname{grad}\varphi^*$ gives

$$\begin{aligned} \|Q_E\vec{u}_h\| &= \|\operatorname{grad}\varphi^h\| = \|\operatorname{grad}\varphi_h^* - \operatorname{grad}\varphi^h\| \\ &\leq \inf_{\psi_h \in G_0^h} \|\operatorname{grad}\psi_h - \operatorname{grad}\varphi^h\| \leq \|\operatorname{grad}\varphi_h - \operatorname{grad}\varphi^h\|. \end{aligned}$$

Consequently we have $\|Q_E\vec{u}_h\| \leq \|\operatorname{grad}\varphi_h - \operatorname{grad}\varphi^h\| = \|\Pi_h\vec{v}^h - \vec{v}^h\|$ by 2°, which we should show converges uniformly to 0 as $h \downarrow 0$ by using interpolation error analysis.

4° For interpolation error analysis, we should consider the affine transformation for the covariant components of vector functions, cf. Nedelec [12]. Moreover, we should note that $\operatorname{rot}\vec{v}_h \in L_p(\Omega)$ for any $p > 0$. As a result, we have the estimation

$$\|\vec{v}^h - \Pi_h\vec{v}^h\| \leq Ch\|\operatorname{rot}\vec{u}_h\| + Ch^{\frac{1}{2}+\delta}\|\vec{u}_h\|_{H(\mathrm{rot};\Omega)},$$

where δ is the parameter in (iii) of this section. By 2° and 3°, we can now show that $\|Q_E\vec{u}_h\| \to 0$ uniformly with respect to $\vec{u}_h$ as $h \downarrow 0$, and the proof is complete.

6 Concluding remarks

It appears that the fundamental difficulty of Nedelec's edge elements is now overcome at least in the simplest cases considered in this note. Although we have taken the cavity resonator problem as a model one, the established results may be effectively used for theoretical numerical analysis of various other electromagnetic problems such as

(1) Magnetostatic problems [10],

(2) Eddy current analysis,

(3) Forced vibration analysis of dielectric media required for e. g. design of microwave ovens.

Further study, however, appears to be necessary to establish numerical analysis of electromagnetics by edge elements, and a few examples of such subjects are:

(i) Discrete compactness for general (curved, covariant) edge elements given in e. g. in [14],

(ii) Discrete compactness in the case of inhomogeneous media, where even stronger singularity may appear in functions in the electromagnetic function spaces,

(iii) Development of appropriate iteration methods for solving the algebraic equations obtained by the finite element discretization, especially for 3-D problems.

References

[1] C. Amrouche, C. Bernardi, M. Dauge and V. Girault, Vector potentials in three-dimensional non-smooth domains, *Math. Meth. Appl. Sci.,* **21** (1998) 823-864.

[2] D. Boffi, Discrete compactness and Fortin operator for edge elements, *Pubbl. n. 1085/98 IAN-CNR, Pavia* (1998).

[3] D. Boffi, P. Fernandes, L. Gastaldi and I. Perugia, Computational models of electromagnetic resonators: analysis of edge element approximation, *SIAM J. Numer. Anal.,* **36** (1999) 1264-1290.

[4] A. Bossavit, Computational Electromagnetism: Variational Formulations, Complementary, Edge Elements, Academic Press, 1998.

[5] F. Brezzi and M. Fortin, Mixed and Hybrid Finite Element Methods, Springer-Verlag, 1991.

[6] J. Descloux, N. Nassif and J. Rappaz, On spectral approximation. Part 2. Error estimates for the Galerkin method, *RAIRO, Analyse Numerique,* **12** (1978) 113-119.

[7] V. Girault and P.-A. Raviart, Finite Element Methods for Navier-Stokes Equations, Springer-Verlag, 1986.

[8] F. Kikuchi, Mixed and penalty formulations for finite element analysis of an eigenvalue problem in electromagnetism, *Computer Meth. Appl. Mech. Engng,* **64** (1987) 509-521.

[9] F. Kikuchi, On a discrete compactness property for the Nedelec finite elements, *J. Fac. Sci., Univ. Tokyo, Sect. IA Math.,* **36** (1989) 479-490.

[10] F. Kikuchi, Numerical analysis of electromagnetic and magnetostatic problems, *Sugaku Expositions,* **6** (1993) 33-51.

[11] F. Kikuchi, M. Yamamoto and H. Fujio, Theoretical and computational aspects of Nedelec's edge elements for electromagnetics, *Computational Mechanics – New Trends and Applications, Eds.: E. Oñate and S. R. Idelsohn,* ©CIMNE, Barcelona, Spain (1998).

[12] J.-C. Nedelec, Mixed finite elements in $\mathbf{R}^3$, *Numer. Math.,* **35** (1980) 315-341 .

[13] J.-C. Nedelec, A new family of mixed finite elements in $\mathbf{R}^3$, *Numer. Math.,* **50** (1986) 57-81 .

[14] P. P. Silvester and R. L. Ferrari, Finite Elements for Electrical Engineers, 3rd edn., Cambridge Univ. Press, 1996.