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Numerical Computation to the Advection in the Field of Some Point Vortices

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1 Introduction

The analysis of the motion of point vortices is a classical problem, which is treated in some detail in many textbooks on fluid mechanics. The point vortex is defined on the two dimensional perfect fluid occupying the whole of a plane. When several vortices are in the fluid, an interaction occurs; every vortex drifts away with the flow due to the other vortices. Such a phenomenon is described by the ordinary differential equation

$$\frac{d}{dt}\overline{z_j(t)} = \sum_{k\neq j} \frac{\Gamma_k}{2\pi i} \frac{1}{z_j(t) - z_k(t)}, \quad j = 1, 2, \ldots, n, \tag{1}$$

where n is a number of vortices, $z_j(t)$ is the complex position of the jth point vortex of strength Γ_j, and $i = \sqrt{-1}$. Since the strength of vortex in the perfect fluid is constant in time t, we treat Γ_j as a given constant.

We summarize some results on the solution of (1). When $n = 2$, it is easy to determine the motion of two point vortices; see textbooks on fluid mechanics (Marchioro and Pulvirenti [5], for example). A qualitative analysis of the motion of three point vortices is done by Aref [1]. When $n \geq 4$, (1) is not solved yet in *general* cases. However some *special* cases are investigated. For examples, Morikawa and Swenson [6] analyze the case where identical point vortices are on the vertices of a regular polygon. Chaotic motion of four point vortices is investigated by Aref and Pomphrey [2]. Nakaki [7] shows that five point vortices, which satisfy

$$(z_j(0), \Gamma_j) = \begin{cases} \left(\exp \frac{j-1}{2}\pi i, 1\right), & j = 1, 2, 3, 4; \\ (0, -1.5), & j = 5, \end{cases} \tag{2}$$

exhibit relaxation oscillation (see Section 2).

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In this paper, taking (2) into considerations, we treat five point vortices such that

$$z_1(0) = \frac{\alpha + i}{2},\ z_2(0) = \frac{-\alpha + i}{2},\ z_3(0) = \frac{-\alpha - i}{2},\ z_4(0) = \frac{-\alpha - i}{2},\ z_5(0) = 0 \qquad (3)$$

and

$$\Gamma_j = \begin{cases} 1, & j = 1, 2, 3, 4; \\ -\kappa, & j = 5, \end{cases} \qquad (4)$$

where $\alpha\,(\geq 1)$ and $\kappa \in \boldsymbol{R}$ are parameters. When $\alpha = 1$ and $\kappa = 1.5$, (3)–(4) is consistent with (2). Aref and Pomphrey's results correspond to $\alpha = 1.75$ or $\alpha = 2$ and $\kappa = 0$, for example.

Our goal in this paper is as follows: We show the existence of periodic or quasi-periodic motion of the five vortices. Some periodic motions are demonstrated by numerical simulations. Following Aref [3], Babiano et al. [4] and Neufeld and Tél [8], we analyze the dynamics of passively advected particles in the field of the periodic point vortices from numerical points of view.

2 Relaxation oscillation

In this section, we describe the motion of five vortices when the parameters in (3)–(4) are $\alpha = 1$ and $\kappa \in \boldsymbol{R}$. When $\kappa = 1.5$, this problem is analyzed by [7].

In this problem, it is easily found that

$$z_j(t) = z_j(0) \exp(i\Omega t), \quad t \in \boldsymbol{R} \qquad (5)$$

is a trivial solution of (1), where

$$\Omega = \frac{\kappa - 1.5}{2\pi} \qquad (6)$$

is the angular velocity of the trivial solution. By the standard linearized analysis, the trivial solution is unstable for $\kappa > 0.5$, and the real part of linearized eigenvalues are all zero for $\kappa < 0.5$. For the latter case, we have

Theorem 1. *Let $\kappa < 0.5$. Then the trivial solution (5) is stable against the perturbation satisfying*

$$z_1(t) = -z_3(t), \quad z_2(t) = -z_4(t) \quad \text{and} \quad z_5(t) = 0 \quad (t \in \boldsymbol{R}). \qquad (7)$$

By our numerical simulations, the solution (5) with $\kappa < 0.5$ seems to be stable against the perturbation which does not satisfy (7).

In the rest of this section, we consider the case of $\kappa > 0.5$, that is, the trivial solution is unstable in the linearized sense. First of all, we demonstrate the numerical simulation. Figure 1 shows the numerical solution when $\kappa = 1.5$. In this case, we have $\Omega = 0$, which means that the trivial solution is steady state in time. The numerical solution does

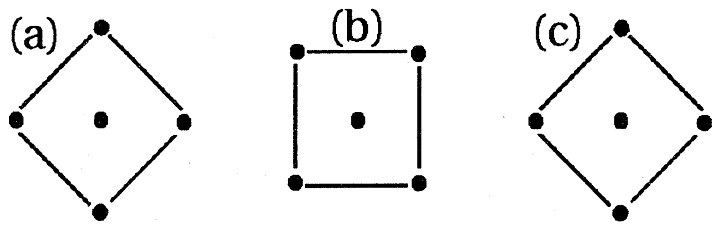

Figure 1: The numerical solution when $\alpha = 1$ and $\kappa = 1.5$. The relaxation oscillation occurs. (a):$0 \le t \le 30$, (b):$53 \le t \le 100$, (c):$123 \le t \le 160$.

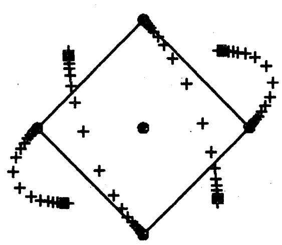

Figure 2: The time evolution of vortices between (a) and (b) in Figure 1.

not move for a while, however, it suddenly begin to move toward another configuration (Figure 1: (a)→(b)). After taking some rest, the solution resumes moving ((b)→(c)). The motion of vortices from Figure 1 (a) to (b) is shown in Figure 2. The reason why such a relaxation oscillation occurs can be explained by the existence of a heteroclinic orbit which connects two steady state solutions. If the heteroclinic orbit exists, the solution of (1), the initial configuration of which is close to a steady state solution, travels near the orbits. The motion of solution is slow when the solution stays near the steady state solution, and becomes fast when it goes away from the steady state solution. By the influence of the heteroclinic orbit, the solution approaches to another steady state solution and the motion becomes slow down. As a result, we observe the relaxation oscillation. For the existence of the heteroclinic orbits, we obtain

Theorem 2. *Let $\kappa > 0.5$, and define $w_j(t) = z(t)\exp(-i\Omega t)$ $(j = 1, 2, \ldots, 5)$. Then there exist a solution $\{z_j(t)\}$ of (1) and constants $\eta_1^+, \eta_2^+ \in \boldsymbol{C}$ which satisfy*

$$w_j(\pm\infty) = \eta_j^\pm, \quad j = 1, 2, \ldots, 5, \tag{8}$$

where

$$\eta_1^- = 1, \quad \eta_2^- = i, \quad \eta_{j+2}^\pm = -\eta_j^\pm (j = 1, 2) \quad \eta_5^\pm = 0. \tag{9}$$

We note that $\{w_j(t)\}$ in Theorem 2 indicates the solution $\{z_j(t)\}$ observed in the corotating coordinate with angular velocity Ω. When $\kappa = 1.5$, we find that $\eta_1^+ = \frac{1+i}{\sqrt{2}}$ and $\eta_2^+ = \frac{1-i}{\sqrt{2}}$,

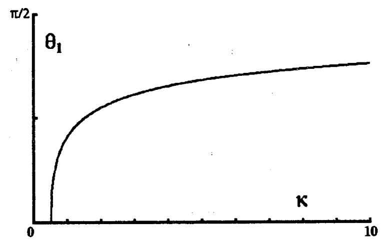

Figure 3: The numerical values of η_1^+ and η_2^+ in Theorem 2. Using the value of θ_1 in this figure, we have $\eta_1^+ = \exp i\theta_1$ and $\eta_2^+ = \eta_1^+ \exp(-\pi/2)$.

and the steady state solutions $\{\eta_j^-\}$ and $\{\eta_j^+\}$ correspond to (a) and (b) in Figure 1, respectively. For another value of κ, we do not know the values of η_1^+ and η_2^+, however, we can compute approximate values by numerical simulations (see Figure 3).

3 Periodic and quasi-periodic motion

Let us treat the case of $\alpha > 1$. In this case, Aref and Pomphrey [2] shows that the motion of vortices is chaotic for $\alpha = 2$ and $\kappa = 0$. They also demonstrate the motion for $\alpha = 1.75$ and $\kappa = 0$ (see Figure 4). The chaotic motion occurs because the (quasi-)periodic motion

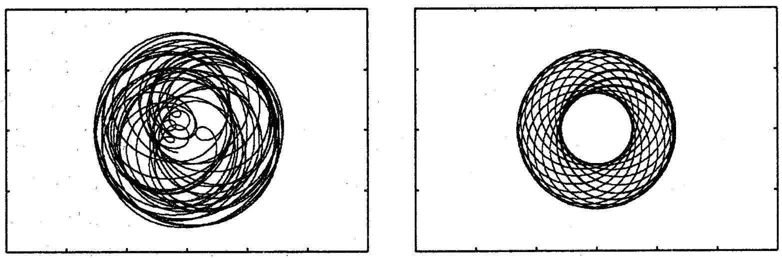

Figure 4: Time trajectory of the point vortex $z_1(t)$ with $\kappa = 0$. Aref and Pomphrey [2] demonstrate the cases of $\alpha = 2$ (left) and $\alpha = 1.75$ (right). In the former case, chaotic motion occurs. A regular precession motion appears in the latter case.

becomes unstable. In fact, the motion of numerical solution of (1) under the condition

$$z_1(t) = -z_3(t), \quad z_2(t) = -z_4(t) \quad \text{and} \quad z_5(t) = 0 \quad (t \in \boldsymbol{R}) \tag{10}$$

is not chaotic as shown in Figure 5. The existence of periodic or quasi-periodic motion is shown by the following.

Theorem 3. *Let $\alpha > 1$ and $\kappa \in \boldsymbol{R}$. Then there exists a periodic or quasi-periodic solution of (1), (3) and (4).*

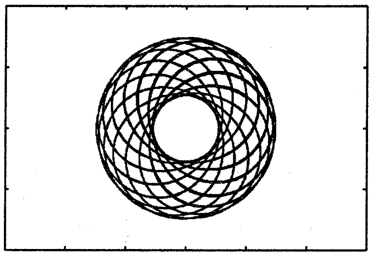

Figure 5: Numerical solution with $\alpha = 2$ and $\kappa = 0$ under the condition (10). The motion is not chaotic anymore.

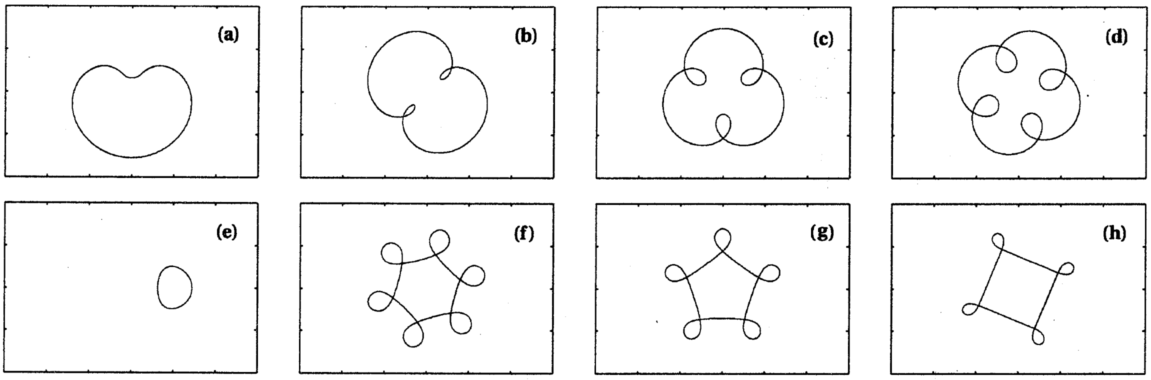

Figure 6: The time evolution of the point vortex $z_1(t)$ with $\alpha = 2$. (a):$\kappa = 0.3506\cdots$, (b):$\kappa = 0.4051\cdots$, (c):$\kappa = 0.5$, (d):$\kappa = 0.5817\cdots$, (e):$\kappa = 1.036\cdots$, (f):$\kappa = 1.571\cdots$, (g):$\kappa = 1.705\cdots$, (h):$\kappa = 1.923\cdots$.

The stability of the solution is not proved yet. The numerical simulations in Figures 4 and 5 suggest that the solution with $\alpha = 2$ and $\kappa = 0$ is unstable, however, is stable against the perturbation satisfying (10).

Next let us show numerical simulations with $\kappa \neq 0$. The aspect ratio α is taken to $\alpha = 2$, for simplicity. Figure 6 displays the numerical solution with $\alpha = 2$. For special value of κ, periodic solutions appears. Mathematically we have

Theorem 4. *Let $\alpha = 2$ and $\kappa = 0.5$. Then the solution $\{z_j(t)\}$ of (1), (3) and (4) is a periodic solution.*

Theorem 4 shows that the solution (c) in Figure 6 is periodic. Until now, we do not know whether or not other solutions in Figure 6 is periodic.

We also make numerical simulations for another value of α. The solutions with a similar shape of (c) in Figure 6 are shown in Figure 7. We can observe that the vortices move on the wide range and the value of κ is small when α becomes large.

On numerical methods, it is not easy to specify the value of two parameters under which a periodic solution appears. We try to compute such a value on a narrow region of parameters (see Figure 8). We remark that they are *not* all value of parameters under which a periodic

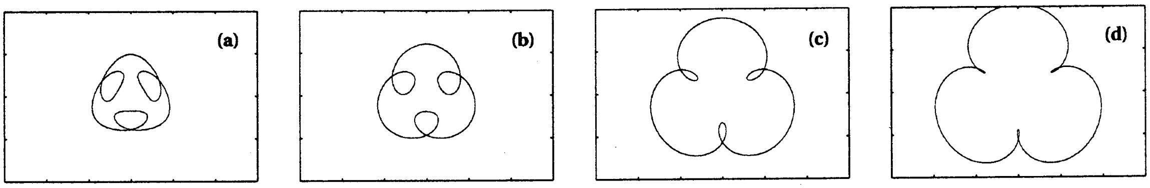

Figure 7: Trajectory of numerical solution $z_1(t)$. (a):$\alpha = 1.1$ and $\kappa = 0.9917\cdots$, (b):$\alpha = 1.5$ and $\kappa = 0.6984\cdots$, (c):$\alpha = 2.5$ and $\kappa = 0.3095\cdots$, (d):$\alpha = 3$ and $\kappa = 0.08712\cdots$.

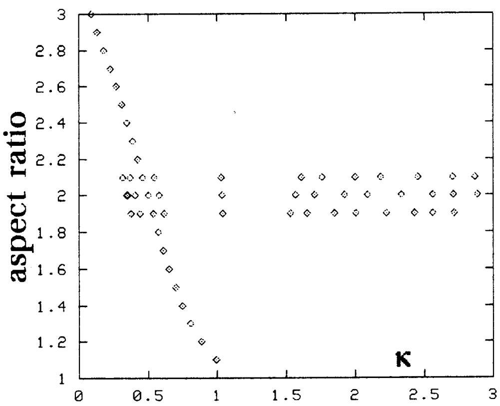

Figure 8: The value of parameters under which a periodic solution appears.

solution appears. We numerically check the periodic solution with $0 < T \leq 100$, where T is the period of the solution on some parameter region. To find the law between such parameters is one of our further problems.

4 Motion of passively advected particles

In this section, following Aref [3], Babiano et al. [4] and Neufeld and Tél [8], we analyze the dynamics of passively advected particles in the field of the periodic point vortices from numerical points of view. We use the periodic point vortices in Theorem 4. Let $x(t)$ be the complex position of the particle. Then we have

$$\frac{d}{dt}\overline{x(t)} = \sum_{j=1}^{5} \frac{\Gamma_j}{2\pi i} \frac{1}{x(t) - z_j(t)}, \tag{11}$$

where $\{z_j(t)\}$ is the periodic solution in Theorem 4. We numerically solve (11) under the initial condition

$$x(0) = x_0, \tag{12}$$

where $x_0 \in \boldsymbol{C}$ is the initial position of particle.

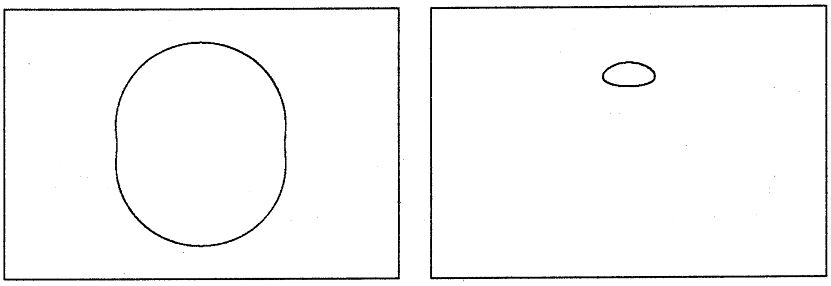

Figure 9: The position of particle at $t = 1000T, 10001T, \ldots, 10000T$, where T is the period of periodic solution in Theorem 4. The initial position of particle is $x_0 = 3$ (left) and $x_0 = 2i$ (right).

If the initial particle is located far away from vortices, it stays there and rotates around the vortices. To see this, we draw some positions of particle on the Poincaré section, that is,

$$P = \{x(nT);\ n = 1000, 1001, \ldots, 10000\}, \tag{13}$$

where T is the period of periodic solution $\{z_j(t)\}$ (see Figure 9). As shown in this figure, there are two types of trajectory P of the particle.

If the particle is some position at the initial time, chaotic behavior of the particle appears as shown in Figure 10. Until now, we do not succeeded in proving that the motion in Figure 10 is chaos, however, the figure suggests that it is chaotic.

References

[1] H. Aref, Motion of three vortices, *Phys. Fluids*, **22** (1979), 393–400.

[2] H. Aref and N. Pomphrey, Integrable and chaotic motions of four vortices I. The case of identical vortices, *Proc. Roy. Soc. London Ser. A*, **380** (1982), 359–387.

[3] H. Aref, Stirring by chaotic advection, *J. Fluid Mech.*, **143** (1983), 1–21.

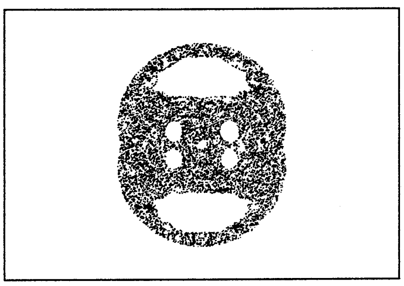

Figure 10: The position of particle at $t = 1000T, 10001T, \ldots, 10000T$ with $x_0 = 2.2$.

[4] A. Babiano, G. Boffetta, A. Provenzale and A. Vulpiani, Chaotic advection in point vortex models and two-dimensional turbulence, *Phys. Fluids*, **6** (1994), 2465–2474.

[5] C. Marchioro and M. Pulvirenti, *Mathematical Theory of Incompressible Nonviscous Fluids*, Springer–Verlag, 1994.

[6] G. K. Morikawa and E. V. Swenson, Interacting motion of rectilinear geostrophic vortices, *Phys, Fluids*, **14** (1971), 1058–1073.

[7] T. Nakaki, Behavior of point vortices in a plane and existence of heteroclinic orbits, *Dynam. Contin. Discrete Impuls. Systems*, **5** (1999), 159–169.

[8] Z. Neufeld and T. Tél, The vortex dynamics analogue of the restricted three-body problem: advection in the field of three identical point vortices, *J. Phys. A: Math. Gen.*, **30** (1997), 2263–2280.