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# Numerical computations for Ginzburg-Landau equation with a variable coefficient by using the discrete Morse semiflow

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## 1 Introduction

The following Ginzburg-Landau equation with a variable coefficient in a bounded domain $D\subset \mathbf{R}^{2}$ subject to Neumann boundary condition are considered:

$$(P)\quad \begin{cases} a(x,y)^{-1}\mathrm{div}(a(x,y)\nabla\Phi)+\lambda(1-|\Phi|^{2})\Phi=0 & (x,y)\in D,\\ \dfrac{\partial\Phi}{\partial\nu}=0 \qquad x\in\partial D, \end{cases}$$

where $a(x,y)$ is a positive smooth function, $\partial/\partial\nu$ denotes the outer normal derivative on the boundary $\partial D$ and $\Phi(x,y)$ is a complex valued function, say $\Phi(x,y)=u(x,y)+iv(x,y)$. $\Phi(x,y)$, which is called order parameter describing a superconducting state, is always identified with the two-component real vector function $(u(x,y),v(x,y))$. This equation $(P)$ is a simplified model to describe a superconducting phenomenon in a thin material with a variable thickness. The thickness of the material with the bottom $D$ is denoted by $a(x,y)$.

For type II super conductors a third state exists, which is known as the mixed state. The mixed state is neither wholly superconducting nor wholly normal but consists of many normal filaments embedded in a superconducting material. These filaments are often known as vortices. Each of these filaments carries with it a quantized amount of magnetic flux and is circled by a vortex of superconducting current; thus these filaments are often known as vortices. From an industrial perspective, it is interesting to know the behavior of vortices, especially their stable condition. If vortices move, electromagnetic induction occurs, causing voltage drop, and therefore loss of energy. Moreover, to apply a superconducting phenomenon (for example " pinning effect"), the position where the vortices appear should be investigated.

Stable solutions of $(P)$ with zero are called vortex solutions. Mathematically, the vortices are considered zero points of $\Phi(x,y)$. For constant $a(x,y)$, there is no stable nonconstant solution to the Ginzburg-Landau equation in any convex domain with Neumann boundary condition [3]. The objective is therefore to investigate the relation between the thickness $a(x,y)$ and the stable vortex solution. Such interest acts as a catalyst for the proposition of the following numerical method. By applying the discrete Morse semiflow (time discretized functional method) to this problem, numerical experiments are carried out.

## 2 Mathematical results

$(P)$ is the Euler-Lagrange equation for the energy functional

$$E(\Phi) = \int_D \{|\nabla\Phi|^2 + \frac{\lambda}{2}(1 - |\Phi|^2)^2\} a(x, y) dxdy. \tag{2.1}$$

We call the solution of $(P)$ is stable if it is a local minimizer of (2.1) (cf. [6]).

Let $h > 0$ and let $D$ be a bounded domain in $\mathbf{R^2}$ with smooth boundary. Let $a(x, y) > 0$ be a bounded function on $\bar{D}$. Here, smoothness of $a(x, y)$ is not assumed.

Now the sequence of functionals will be defined,

$$\begin{aligned} E_n^h(\Phi) &= \int_D \frac{|\Phi - \Phi_{n-1}^h|^2}{h} a(x, y) dxdy + E(\Phi), \qquad (2.2) \\ \Phi \in \mathrm{K} &= W_\psi^{1,2}(D; \mathbf{R^2}) \cap L^4(D; \mathbf{R^2}). \end{aligned}$$

If suitable $h$ and $\lambda$ are chosen, the minimizer is uniquely determined for each functional $E_n^h(\Phi)$.

**Lemma 2.1.** *For any $m \in \mathbf{N}$, if $1/h > \lambda$ holds, the minimizer of $E_m^h$ is uniquely determined.*

*proof)* It holds that

$$E_m^h(\Phi) = \int_D \left\{ \left(\frac{1}{h} - \lambda\right) |\Phi|^2 - \frac{2}{h}\Phi \cdot \Phi_{m-1}^h + \frac{1}{h}|\Phi_{m-1}^h|^2 + |\nabla\Phi|^2 + \frac{\lambda}{2}(1 + |\Phi|^4) \right\} a(x, y) dxdy.$$

Then we have, for $1 \le \theta \le 1$,

$$\begin{aligned} &(1 - \theta)\ E_m^h(\Phi) + \theta E_m^h(\Psi) - E_m^h(\Phi + \theta(\Psi - \Phi)) \\ &\ge (1 - \theta)\theta \int_D \left\{ \left(\frac{1}{h} - \lambda\right) |\Phi - \Psi|^2 + |\nabla(\Phi - \Psi)|^2 \right\} a(x, y) dxdy. \end{aligned}$$

If $1/h - \lambda$ is positive, the functional $E_m^h(\Phi)$ is convex. Therefore its minimizer is unique. □

The sequence of functions $\{\Phi_n^h\}_{n=1}^\infty$ is called discrete Morse semiflow (see [7], [8], [9] and [10]). The boundedness of $\Phi_m^h$ for each $m$ is given.

**Lemma 2.2.** *If $\|\Phi_0\|_\infty \le 1$, $\|\Phi_m^h\|_\infty \le 1$ for all $m \in \mathbf{N}$ holds.*

*proof)* We suppose that our assertion holds for any $n \le m - 1$. If $\Phi_m^h$ is a minimizer of $E_m^h$ and $\{x; |\Phi_m^h| > 0\}$ has a positive Lebesgue measure, the following comparison function is chosen: $\Psi := \Phi_m^h/|\Phi_m^h|$ in $\{x; |\Phi_m^h| > 1\}$, $:= \Phi_m^h$ in $\{x; |\Phi_m^h| \leq 1\}$. Then $E_m^h(\Psi) < E_m^h(\Phi_m^h)$ is obtained by direct calculation. It contradicts $\Phi_m^h$ is a minimizer. $\square$

Throughout this paper, an initial data $\Phi_0$ satisfies $\|\Phi_0\|_\infty \leq 1$ is supposed.

**Lemma 2.3.** *It holds that*

$$E(\Phi_M^h) + \sum_{m=1}^{M} \int_D \frac{|\Phi_m^h - \Phi_{m-1}^h|^2}{h} a(x,y)dxdy \leq E(\Phi_0).$$

*proof)* Because $\Phi_m^h$ is the minimizer of $E_m^h$, it holds the following inequality,

$$\begin{aligned} E_m^h(\Phi_m^h) &\equiv \int_D \frac{|\Phi_m^h - \Phi_{m-1}^h|^2}{h} a(x,y)dxdy + E(\Phi_m^h) \\ &\leq E_m^h(\Phi_{m-1}^h) = E(\Phi_{m-1}^h). \end{aligned} \tag{2.3}$$

By summing up the both sides of (2.3), Lemma 2.3 can be shown. $\square$

Now, the existence of the limit function $\Phi_\infty^h$ of the subsequence $\{\Phi_m^h\}$ will be shown.

**Lemma 2.4.** *For any subsequence $\{\Phi_{m_j}^h\} \subset \{\Phi_m^h\}$, there exists a subsequence $\{\Phi_{m_{j_\nu}}^h\} \subset \{\Phi_{m_j}^h\}$ and a function $\Phi_\infty^h$ on $D$ such that*

$$\Phi_{m_{j_\nu}}^h \to \Phi_\infty^h \qquad \textit{weakly in} \quad W^{1,2}, \tag{2.4}$$

$$\Phi_{m_{j_\nu}}^h \to \Phi_\infty^h \qquad \textit{strongly in} \quad L^2, \tag{2.5}$$

$$\Phi_{m_{j_\nu}}^h \to \Phi_\infty^h \qquad \textit{weakly in} \quad L^p, \quad \forall p > 1, \tag{2.6}$$

*as $\nu \to \infty$. Moreover, we have*

$$|\Phi_\infty^h| \leq 1 \quad \textit{a.e. in} \quad D. \tag{2.7}$$

*proof)* By Lemma 2.3, $\{\Phi_m^h\}$ is weakly compact in $W^{1,2}$. Therefore it holds (2.4) by use of a weak compactness argument and by Rellich's theorem (2.5) is obtained. We readily get (2.6) and (2.7) by Lemma 2.2. $\square$

**Theorem 2.1.** *The limit function $\Phi_\infty^h$ is a minimizer of the functional*

$$E_\infty^h(\Phi) = \int_D \frac{|\Phi - \Phi_\infty^h|^2}{h} a(x,y)dxdy + \int_D \{|\nabla\Phi|^2 + \frac{\lambda}{2}(1 - |\Phi|^2)^2\} a(x,y)dxdy$$

*in* K, *hence,* $\Phi_\infty^h$ *satisfies*

$$\int_D \nabla\Phi_\infty^h \nabla\phi a(x,y)dxdy - \int_D \lambda\Phi_\infty^h(1 - |\Phi_\infty^h|^2)\phi a(x,y)dxdy = 0$$

*for any* $\phi \in C_0^\infty(D)$.

*proof)* We assume that there exists $v \in \mathrm{K}$ such that

$$E_\infty^h(\Phi_\infty^h) - E_\infty^h(v) = 3d > 0.$$

It is easy to see

$$\begin{aligned}
&|E_{m_j}^h(v) - E_\infty^h(v)| \\
&= \frac{1}{h}\int_D \Big\{2v \cdot (\Phi_\infty^h - \Phi_{m_j-1}^h) + (|\Phi_{m_j-1}^h|^2 - |\Phi_\infty^h|^2)\Big\} a(x,y)dxdy \\
&\le \frac{1}{h}\|a\|_\infty \cdot \|\Phi_\infty^h - \Phi_{m_j-1}^h\|_{L^2}\Big\{2\|v\|_{L^2} + \|\Phi_{m_j-1}^h\|_{L^2} + \|\Phi_\infty^h\|_{L^2}\Big\}.
\end{aligned}$$

Thus, there exists a positive number $M$ such that for all $j \ge M$

$$|E_{m_j}^h(v) - E_\infty^h(v)| \le d$$

holds.

On the other hand, by Lemma 2.4, it holds that

$$\begin{aligned}
\int_D |\nabla \Phi_\infty^h|^2 dx \quad &\le \quad \liminf_{j\to\infty} \int_D |\nabla \Phi_{m_j}^h|^2 dx, \\
\int_D \frac{1}{\delta}(1 - |\Phi_\infty^h|^2)^2 dx \quad &\le \quad \liminf_{j\to\infty} \int_D \frac{1}{\delta}(1 - |\Phi_{m_j}^h|^2)^2 dx.
\end{aligned}$$

Therefore there exists $M \in \mathbf{N}$ such that for $j \ge M$ we have

$$E_\infty^h(\Phi_\infty^h) \le E_{m_j}^h(\Phi_{m_j}^h) + d.$$

Combining these estimates with a minimality of $E_{m_j}^h(\Phi_{m_j}^h)$, we have

$$\begin{aligned}
E_{m_j}^h(\Phi_{m_j}^h) \ &\le E_{m_j}^h(v) \\
&\le E_\infty^h(v) + d \\
&= E_\infty^h(\Phi_\infty^h) - 2d \\
&\le E_{m_j}^h(\Phi_{m_j}^h) - d.
\end{aligned}$$

This is a contradiction. $\square$

# 3 Numerical results

Here, the following some numerical experiments are introduced. These results are obtained by minimizing method to the functional (2.2).

The numerical scheme used here is the usual finite element method for elliptic variational problems. A minimizer for each step is sought by use of a gradient method (see [9] and [10] for examples). Note that, each minimizer is uniquely determined, if $h$ and $\delta$ are chosen suitably by Lemma 2.1. The parameters chosen are $\delta = 1.0 \times 10^{-5}$ and $\lambda = 1/0.05$.

Let $D = \{|x| < 1\}$ and $a(x,y) = a(r)$ be a radially symmetric function. The thickness $a(x,y)$ is defined

$$a(x,y) = a(r) = \begin{cases} 1 & 0.5 < r \le 1, \\ d & 0 \le r \le 0.5. \end{cases}$$

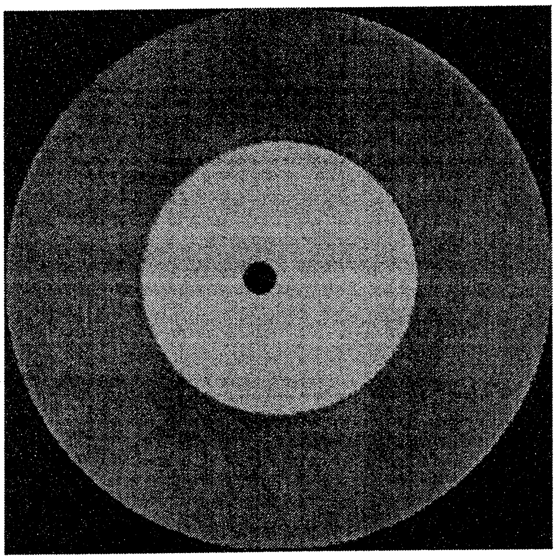
We may consider the $d$ plays an important role in the position of vortex. Numerical computations were tested in the three cases; $d = 0.01$, $d = 0.5$ and $d = 0.4$. All of the cases, the following function is chosen

$$\Phi_0(x,y) = \begin{cases} 0 & \text{if } \rho = 0 \\ ((x+0.1)/\rho, y/\rho) & \text{otherwise} \end{cases}$$

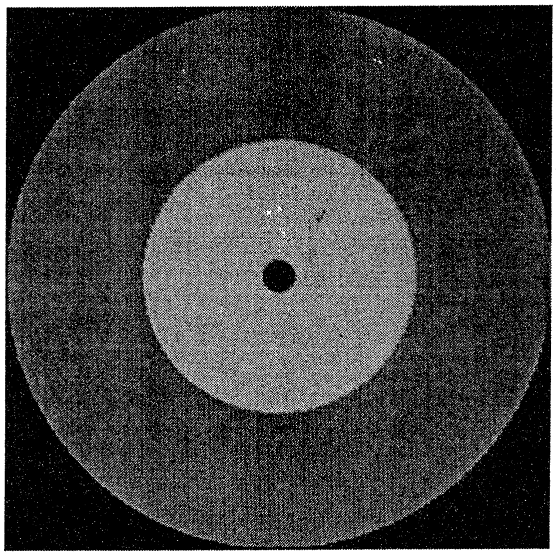
as the initial condition, where $\rho = \sqrt{(x+0.1)^2 + y^2}$.

**Case 1** d=0.01

The vortex solution whose vortex is at the center is unstable for constant $a(x,y)$. However, the vortex of $\Phi_\infty$ is at the center of the domain. For the result of [5], the vortex solution whose vortex is at the center is known. This fact is ascertained numerically.



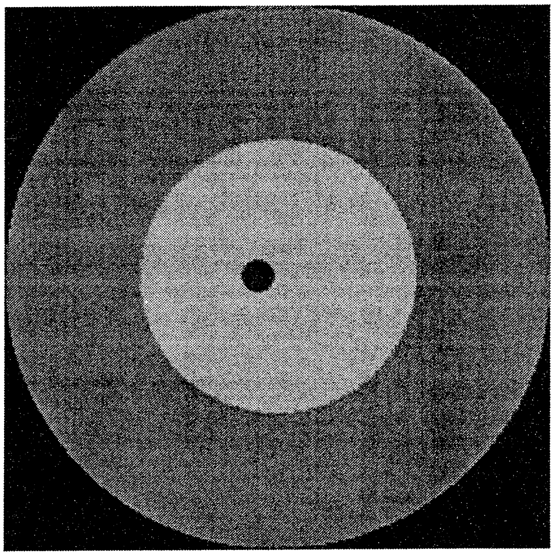
**The profile of $\Phi_0$**



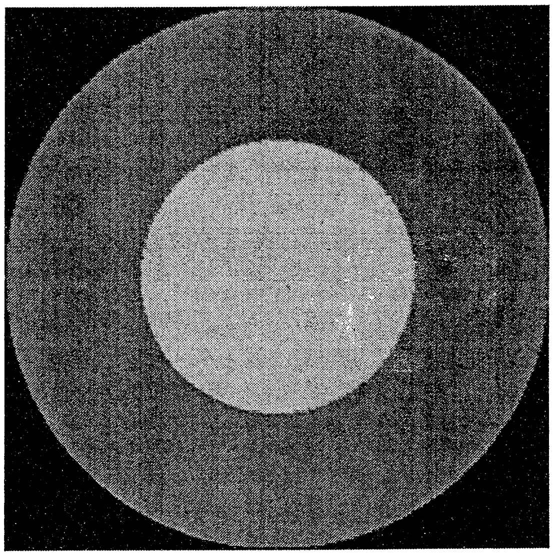
**The profile of $\Phi_\infty$**

**Case 2** d=0.5

The vortex goes out from the domain. It is the same as $a(x, y)$ is constant.



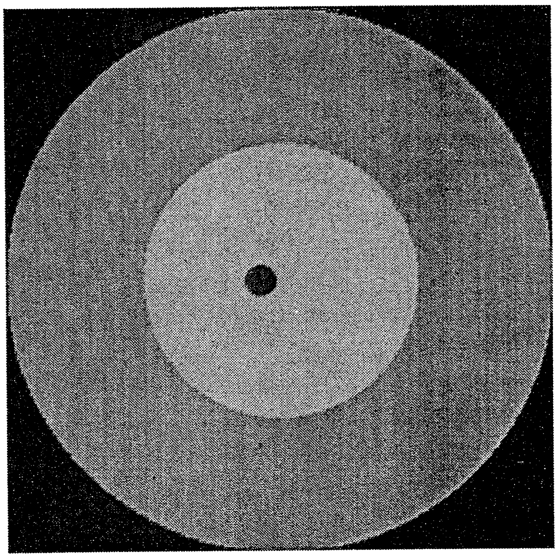
**The profile of $\Phi_0$**



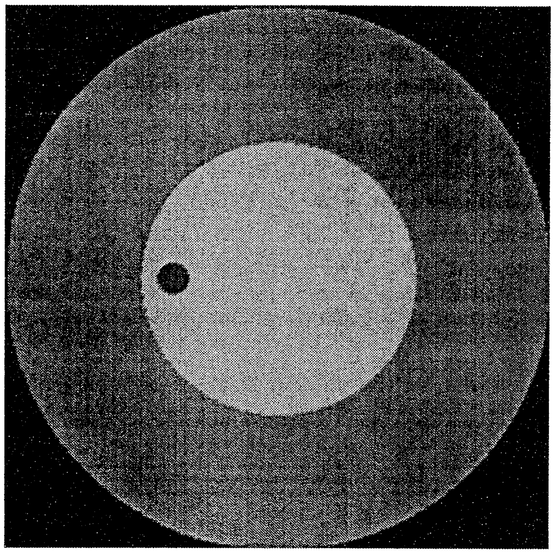
**The profile of $\Phi_\infty$**

**Case 3** d=0.4

The vortex is trapped in the domain. It can not go over the layer at $r = 0.5$.



**The profile of $\Phi_0$**



**The profile of $\Phi_\infty$**

# 4 Conclusion

Here, the Ginzburg-Landau system was treated and its weak solutions were constructed by use of a notion of discrete Morse semiflow. At the same time, numerical computations were also carried out. The numerical scheme used here was the usual finite element method for elliptic variational problems. A minimizer for each step was sought by use of a gradient method. These minimizers were uniquely determined, and located relatively quickly.

Numerical experiments were carried out on a special shape of the domain. The stability of the solution and the position of the vortex were affected by the thickness of the domain. For the result of [5], the vortex solution whose vortex is at the center was known. This fact was ascertained numerically, and our results suggested the existence of another vortex solution exists.

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# References

[1] F. Bethuel, J. M. Coron, J. M. Ghidaglia, and A. Soyeur, Heat flows and relaxed energies for harmonic maps, in Nonlinear diffusion equations and their equilibrium states, 3. (Progress in nonlinear differential equations and their applications, 7.) Birkhäuser, Boston, Basel, Berlin (1992), 99–109.

[2] S. Jimbo and Y. Morita, Stable solutions with zeros to the Ginzburg-Landau equation with Neumann boundary condition, *Journal of Differential Equations,* **128**(1994), 596–613.

[3] S. Jimbo and Y. Morita, Stablity of nonconstant steady-state solutions to a Ginzburg-Landau equation in higher space dimensions *Nonlinear Analysis, Theory, Methods & Applications,* **22**(1994), 753–770.

[4] S. Jimbo, Y. Morita and J. Zhai, Ginzburg-Landau equation and stable steady state solutions in a non-trivial domain, *Commun. In Partial Differential Equations,* **20**(1995), 2093–2112.

[5] S. Jimbo and Y. Morita, Stable vortex solutions to the Ginzburg-Landau equation with a variable coefficient in a disk, *Journal of Differential Equations,* **155**(1999), 153-176.

[6] L. Simon, Asymptotics for a class of non-linear evolution equations, with applications to geometric problems, *Annals of Mathematics,* **118**(1983), 525–571.

[7] T. Nagasawa and S. Omata, Discrete Morse semiflows of a functional with free boundary, *Adv. Math. Sci. Appl.* **2**(1993), 147–187.

[8] T. Nagasawa and S. Omata, Discrete Morse semiflows and their convergence of a functional with free boundary, *in Nonlinear Partial Differential Equations - Proceedings of the International Conference Zhejiang University, June 1992,* ed: Dong Guangchang and Lin Fanghua, International Academic Publishers, Beijing (1993), 205–213

[9] S. Omata, T. Okamura and K. Nakane, Numercal analysis for the discrete Morse semiflow related to the Ginzburg Landau functional, *Nonlinear Analysis* **vol.37, No.5,** (1999), 589-602.

[10] T. Nagasawa, K. Nakane and S. Omata, Numerical computations for a hyperbolic Ginzburg-Landau system, *in Proceedings of the Eighth International Colloquium on Differential Equations Plovdiv*, Bulgaria, August 18-23, (1997).