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ON THE VELOCITY-VORTICITY FORMULATION OF THE INCOMPRESSIBLE NAVIER-STOKES EQUATIONS WITH NUMERICAL APPLICATIONS

Vitoriano RUAS

Département de Mathématiques, Université de Saint-Etienne &
Laboratoire de Modélisation en Mécanique, Université Paris VI
8, rue du Capitaine Scott, 10ème étage, 75015 Paris, FRANCE
E-mail: ruas@ccr.jussieu.fr

Abstract

The purpose of this work is two-fold: On the one hand it is aimed at presenting some arguments that justify the use of certain spaces of functions or vector fields in which the vorticity is to be searched for, in the framework of weak formulations of the incompressible Navier-Stokes equations expressed as a second order system in terms of this variable together with the velocity. On the other hand it is shown how the adopted formulations, when combined with a boundary condition uncoupling technique of the so-called Glowinski-Pironneau type, can be approximated by finite element methods having similar convergence properties to those of methods previously proposed by the author (cf. [9]), to discretize the classical stream function-vorticity formulation of these equations. Throughout the work, an emphasis will be given to the case of the velocity-vorticity Stokes system, in which the main difficulties to overcome are encountered. More precisely, we mean the equivalence with the standard velocity-pressure formulation of the system of equations, and coupling boundary conditions.

1 Introduction

Let us consider the system of equations that govern the stationary flow of a viscous incompressible fluid, in terms of the primitive variables velocity $\mathbf{u}$ and pressure p, in a bounded domain Ω of $\mathbb{R}^N$, for $N = 2, 3$. Let us denote the boundary of Ω by Γ, and the unit outer normal vector to Γ by $\mathbf{n}$.

The presentation that follows can be considerably simplified, by considering as a model simultaneously the following cases:

- The velocity is fully prescribed on the boundary;
- Ω is a simply connected domain.

Notice that these assumptions are by no means essential, as it is not really much harder to treat more general cases.

The equations under consideration are the classical incompressible Navier-Stokes equations, namely,

$$\left\{\begin{array}{ll} \text{Find } \mathbf{u} \in W^{1,q}(\Omega)^N \text{ and } p \in L^2_0(\Omega) \text{ such that :} & \\ -\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\mathbf{grad}\,)\mathbf{u} + \frac{1}{\rho}\mathbf{grad}\; p = \mathbf{f} & \text{in } \Omega \\ \mathbf{div}\;\mathbf{u} = 0 & \text{in } \Omega \\ \mathbf{u} = \mathbf{g} & \text{on } \Gamma, \end{array}\right. \tag{1}$$

where all the above relations hold in an appropriate sense, and

- ν is the kinematic viscosity of the fluid;
- ρ is the density of the fluid;
- $\mathbf{f}$ is a given field of body foreces per mass unit;
- $\mathbf{g}$ is the given velocity on Γ satisfying the condition $\oint_\Gamma \mathbf{g}\cdot\mathbf{n} = 0$;
- $q \geq 2$ is suitably chosen in terms of N (cf [15]).

Denoting by **curl** $(\cdot)$ either the scalar or the vector curl of a vector field over $\mathbf{R}^N$ or yet of a scalar function of two space variables, according to the case being considered, our study will be conducted under the following hypotheses:

1. Γ is lipschitzian;
2. $\mathbf{f} \in H^{-1}(\Omega)^N$ and $\mathbf{curl}\,\mathbf{f} \in H^{-1}(\Omega)^{2N-3}$;
3. $\mathbf{g} \in W^{1/2,q}(\Gamma)^N$.

Our answers to the questions to be addressed here can essentially be given in the framework of the linearized form of equations (1), namely the Stokes system:

$$\left\{\begin{array}{ll} \text{Find } \mathbf{u} \in H^1(\Omega)^N \text{ and } p \in L^2_0(\Omega) \text{ such that :} & \\ -\nu\Delta\mathbf{u} + \frac{1}{\rho}\mathbf{grad}\; p = \mathbf{f} & (\text{in } H^{-1}(\Omega)^N), \\ \mathbf{div}\;\mathbf{u} = 0 & \text{a.e. in } \Omega \\ \mathbf{u} = \mathbf{g} & \text{a.e. on } \Gamma. \end{array}\right. \tag{2}$$

Now we define the vorticity $\boldsymbol{\omega}$ to be **curl u** (regarded as a scalar function if $N = 2$). Next, after having applied the curl operator on both sides of the first equation of (2), we may rewrite this system in the form of the following velocity-vorticity system of the Cauchy-Riemann type:

$$\left\{\begin{array}{ll} \text{Find } \mathbf{u} \in H^1(\Omega)^N \text{ and } \boldsymbol{\omega} \in L^2(\Omega)^{2N-3} \text{ such that :} & \\ -\nu\Delta\boldsymbol{\omega} = \mathbf{curl}\,\mathbf{f} & (\text{in } H^{-1}(\Omega)^{2N-3}), \\ \mathbf{curl}\;\mathbf{u} = \boldsymbol{\omega} & \text{a.e. in } \Omega \\ \mathbf{div}\;\mathbf{u} = 0 & \text{a.e. in } \Omega \\ \mathbf{u} = \mathbf{g} & \text{a.e. on } \Gamma. \end{array}\right. \tag{3}$$

Since Ω is simply connected by assumption, one can readily establish that system (3) is equivalent to (2). However the two first order equations of the Cauchy-Riemann system cannot be handled so easily, at least in the framework of a numerical solution. That is probably why most authors prefer to combine both equations, in order to derive a single second order equation to replace them.

More specifically, applying the curl operator on both sides of the second equation of (3), and then exploiting the continuity equation $\mathbf{div}\ \mathbf{u} = 0$, together with the well-known identity $-\Delta(\cdot) = \mathbf{curl}\ \mathbf{curl}\ (\cdot) - \mathbf{grad}\ \mathbf{div}\ (\cdot)$, we derive from (3), the following second order velocity-vorticity system:

$$\begin{cases} -\nu\Delta\omega = \mathbf{curl}\ \mathbf{f} & \text{in the sense of } H^{-1}(\Omega)^{2N-3} \\ -\Delta\mathbf{u} = \mathbf{curl}\ \omega & \text{"in } \Omega\text{"} \\ \mathbf{u} = \mathbf{g} & \text{on } \Gamma, \end{cases} \tag{4}$$

where the quotation marks in the second equation above, mean that its sense remains to be specified.

As a matter of fact, one of the the main problems that we intend to treat here can be stated as follows:

Under our minimum regularity assumptions on $\mathbf{f}$, $\mathbf{g}$ and Γ, in which space should the vorticity be searched for, and what kind of boundary conditions should be added to (4), in such a way that the resulting problem is well-posed and equivalent to (2)?

In the next two sections we attempt to bring about appropriate answers to such question.

2 The two-dimensional case

In order to study the case $N = 2$, we first note that, since we search for $\mathbf{u}$ in $H^1(\Omega)^2$, ω belongs "at least" to $L^2(\Omega)$. It follows that the second equation of (4) must hold in $H^{-1}(\Omega)^2$. Incidentally, the first equation of (4) implies that $\Delta\omega \in H^{-1}(\Omega)$. Hence applying the curl operator on both sides of the second equation of (4), we trivially derive $\Delta\mathbf{curl}\ \mathbf{u} = \Delta\omega$ in $H^{-1}(\Omega)$. Now we assume temporarily that Γ is of the $\mathcal{C}^\infty$-class, and that we prescribe the following additional boundary condition:

$$\omega = \mathbf{curl}\ \mathbf{u} \text{ "on } \Gamma\text{"}. \tag{5}$$

Owing to the fact that $\mathbf{curl}\ \mathbf{u} \in L^2(\Omega)$ and $\Delta\mathbf{curl}\ \mathbf{u} \in H^{-1}(\Omega)$ condition (5) holds in the sense of $H^{-1/2}(\Gamma)$ (cf. [6]), and this implies the equivalence between (4)-(5) and (2). Indeed, $\zeta = \mathbf{curl}\ \mathbf{u} - \omega$ is a harmonic function of $L^2(\Omega)$. Moreover, since its trace on Γ vanishes in the sense of $H^{-1/2}(\Gamma)$, ζ is the solution of a Laplace equation in $L^2(\Omega)$ with homogeneous Dirichlet boundary conditions. Then taking into account the assumed smoothness of Γ, according to [6], we must have $\zeta = 0$. Otherwise stated, the fundamental relation,

$$\mathbf{curl}\ \mathbf{u} = \omega \text{ a.e. in } \Omega \tag{6}$$

is satisfied. Furthermore, since in this case we have $\mathbf{curl}\ \mathbf{curl}\ \mathbf{u} = \mathbf{curl}\ \omega$, we immediately derive $\mathbf{grad}\ \mathbf{div}\ \mathbf{u} = \mathbf{0}$ a.e. in Ω. Finally recalling that $\oint_\Gamma \mathbf{u}\cdot\mathbf{n} = 0$, this immediately yields the continuity equation. It follows that the Cauchy-Riemann system (3) can be derived from (4)-(5) and conversely, which implies the claimed equivalence between (2) and the latter system.

Nevertheless it turns out that the arguments we just employed cannot be applied if Ω is not smooth, and in this case such equivalence becomes doubtful. For instance, assume that Ω is the domain expressed in polar coordinates by $\Omega = \{(r,\theta)/\ 0 < r < 1,\ 0 < \theta < \alpha\}$, where α is an angle strictly comprised between π and 2π. The function $w(r,\theta) = r^{-\pi/\alpha} sin(\pi\theta/\alpha)$ is harmonic and belongs to $H^s(\Omega)$, for some s, $0 \leq s < 1/2$, but not to $H^1(\Omega)$, as one can easily

check. In spite of this. its trace on Γ does belong to $H^{1/2}(\Gamma)$, and hence the non homogeneous Dirichlet problem of finding $v \in H^1(\Omega)$ such that $\Delta v = 0$ a.e. in Ω and $v = w$ a.e. on Γ, has a unique solution. Since necessarily $v \neq w$, it follows that the Laplace equation with homogeneous boundary conditions in such domain Ω, has solutions in the space $H^s(\Omega)$ for a certain s, $0 \leq s < 1/2$, other than the trivial one. The same situation would occur for other types of domains such as non convex polygons. As a consequence, in certain cases it is not possible to assert that (4)-(5) implies (2), at least by means of the arguments employed in the case of a smooth domain. This conclusion applies, even assuming that $\mathbf{g} \in H^t(\Gamma)^2$ with $t \geq 1$, and that boundary condition (5) holds in a fairly strong sense (in $L^2(\Gamma)$, for example), as long as it is not possible to guarantee that $\mathbf{curl}\ \mathbf{u} - \omega$ belongs to a space of functions sufficiently smooth, and in any case strictly contained in $L^2(\Omega)$.
Actually, in order to be more conclusive about this point, let us consider again the case of the domain Ω defined in polar coordinates as above. Since the function $v - w$ is harmonic in Ω, $\mathbf{curl}\ (v-w)$ is a distribution in $H^{-1}(\Omega)^2$ whose curl vanishes. Therefore, according to a result proven by ZHU (cf [16]), there exists a function $y \in L^2_0(\Omega)$ such that $\mathbf{grad}\ y = \mathbf{curl}\ (v-w)$. Let now ξ and η be the restrictions to Ω of the solutions to two homogeneous Dirichlet problems for the laplacian operator in the unit disk centered at the origin, and whose right hand sides are respectively the extensions by zero outside of Ω, of $-y$ and $v - w$. Then setting $\mathbf{u} = \mathbf{grad}\ \xi + \mathbf{curl}\ \eta$, from classical regularity results (cf. [6]), we know that $\mathbf{u} \in H^1(\Omega)^2$. Moreover, by construction, $\mathbf{u}$ satisfies simultaneouly the relations $\mathbf{div}\ \mathbf{u} = y$ and $\mathbf{curl}\ \mathbf{u} = v - w$ a.e. in Ω. As a consequence (for an appropriate datum $\mathbf{g}$ and $\mathbf{f}=\mathbf{0}$), if we set $\omega = 0$, we do have $-\Delta\mathbf{u} = \mathbf{curl}\ \omega$ a.e. in Ω and $\mathbf{curl}\ \mathbf{u} - \omega = 0$ a.e. on Γ, although $\mathbf{curl}\ \mathbf{u} \neq \omega$.

In view of the above arguments, second order velocity-vorticity systems in strong form such as (4)-(5) must be handled with care, whenever sufficient regularity properties of both the geometry and the data are lacking, otherwise one might determine a wrong solution.

In order to avoid such inconvenience we will rewrite the velocity-vorticity system (4)-(5) in a well-posed weak form, whose equivalence with (2) under our minimum regularity assumptions will be established directly, that is, without resorting to (4). In order to do this, let us first introduce some notations and spaces:

- $(\,(\cdot),(\cdot)\,)$ and $\|\ (\cdot)\ \|$ represent respectively, the standard inner product of $L^2(\Omega)$ and the corresponding norm;
- $\langle\,(\cdot),(\cdot)\,\rangle$ denotes the duality product between $H^{-1}(\Omega)$ and $H^1_0(\Omega)$;
- $\mathbf{H}_{n0}(\Omega) = \{\mathbf{v}/\ \mathbf{v} \in L^2(\Omega)^2,\ \mathbf{curl}\ \mathbf{v} \in L^2(\Omega),\ \mathbf{div}\ \mathbf{v} \in L^2(\Omega),\ \mathbf{v}\cdot\mathbf{n} = 0 \text{ a.e. on } \Gamma\}$;
- $\mathbf{V}_{\mathrm{g}} = \{\mathbf{v}/\ \mathbf{v} \in H^1(\Omega)^2,\ \mathbf{v} = \mathbf{g} \text{ on } \Gamma\}$.

We recall that (cf. [4]) $\mathbf{H}_{n0}(\Omega)$ is a Hilbert space, when equipped with the inner product $(\,(\cdot),(\cdot)\,)_1$ defined by $(\ \mathbf{curl}\ (\cdot), \mathbf{curl}\ (\cdot)\ )\ +\ (\ \mathbf{div}\ (\cdot), \mathbf{div}\ (\cdot)\ )$.

Naturally enough the velocity will be searched for in the linear manifold $\mathbf{V}_{\mathrm{g}}$, whereas the vorticity will be assumed to belong to the space $X(\Omega)$ introduced by the author in (cf. [7]), whose definition we recall:

$$X(\Omega) = \{\chi/\ \chi \in L^2(\Omega),\ \Delta\chi \in H^{-1}(\Omega)\}.$$

Now we set the problem to solve in weak form, namely,

$$\begin{cases} \text{Find } \mathbf{u} \in \mathbf{V_g} \text{ and } \boldsymbol{\omega} \in X(\Omega) \text{ such that} & \\ -\nu\langle \Delta\boldsymbol{\omega}, \varphi\rangle = \langle \mathbf{curl\,f}, \varphi\rangle & \forall \varphi \in H_0^1(\Omega) \\ (\mathbf{u}, \mathbf{v})_1 = (\boldsymbol{\omega}, \mathbf{curl\,v}) & \forall \mathbf{v} \in \mathbf{H}_{n0}(\Omega). \end{cases} \tag{7}$$

As proven in [7], using the classical theory on linear variational (cf. [1] and [2]), (7) has a unique solution. Moreover this solution is nothing but $\mathbf{u}$ and $\mathbf{curl\,u}$, where $\mathbf{u}$ is the velocity that, together with the pressure p, solves the Stokes problem (2). In order to verify the above assertion we procede as follows:

First we note that $\boldsymbol{\omega} = \mathbf{curl\,u}$ belongs indeed to $X(\Omega)$, for $\Delta\mathbf{curl\,u} = \mathbf{curl\,f} \in H^{-1}(\Omega)$ by assumption. Next, since $\mathbf{div\,u} = 0$ in Ω, and $(\mathbf{curl\,u}, \mathbf{curl\,v}) = (\boldsymbol{\omega}, \mathbf{curl\,v})$ for every $\mathbf{v} \in \mathbf{H}_{n0}(\Omega)$, the second equation of (7) does hold. Finally the first equation ot the latter problem trivially follows from the first equation of (2).

Incidentally, as one can easily check, the solution of (7) necessarily satisfies all the relations of (4) together with the boundary conditions (5) in the natural sense. However, whenever the latter only holds in the sense of the traces of functions of $X(\Omega)$, system (4)-(5) does not necessarily imply (7), as pointed out before.

Remark 2.1 *Weak formulations of the velocity-vorticity system incorporating the treatment of multiple connectedness may be found in [10].* ■

To conclude this section, we briefly illustrate a wide class of finite element methods to solve the velocity-vorticity system (7), through the presentation of two particular ones. This class of methods is based on the decomposition of the vorticity space $X(\Omega)$ into the direct sum of $H_0^1(\Omega)$ and the subspace $X_H(\Omega) = \{\chi/\ \chi \in X(\Omega), \Delta\chi = 0\}$. This means that, we determine separately approximations of the components of $\boldsymbol{\omega}$, namely, $\omega_0 \in H_0^1(\Omega)$ and $\omega_H \in X_H(\Omega)$, by using discrete harmonic functions to represent the latter. Actually these methods employ a technique that generalizes the so-called Glowinski-Pironneau method (cf [5]) for the stream function-vorticity formulation. A detailed description of them in the context of this formulation can be found in [9]. For their complete study in the case of the velocity-vorticity formulation, the author refers to [13]. Here, for the sake of brevity, we only treat the case where $\mathbf{g} = \mathbf{0}$ and Ω is a polygon.

Let $\{\mathcal{T}^h\}^h$ be a quasiuniform family of triangulation of $\bar{\Omega}$, h denoting as usual, the maximum diameter of the triangles belonging to $\mathcal{T}^h$. Let also P_k be the space of polynomials of degree less than or equal to k.

Defining Σ^h to be the set of segments contained in Γ, which are edges of an element of $\mathcal{T}^h$, we introduce the following spaces for $k \geq 1$:

$$S^{h,k} = \left\{v^h/\ v^h_{/T} \in P_k,\ \forall T \in \mathcal{T}^h\right\} \tag{8}$$

$$S_0^{h,k} = \left\{v^h \in S^{h,k},\ v^h = 0 \text{ on } \Gamma\right\} \tag{9}$$

$$X^{h,k} = \left\{v^h \in H^1(\Omega);\ v^h \in C^0(\Gamma) \text{ and } v^h|_{\mathcal{S}} \in P_k,\ \forall\, \mathcal{S} \in \Sigma^h\right\}. \tag{10}$$

In order to approximate ω_H, we introduce the following discrete harmonic function space (cf. [9]), indexed by a second integer $l \in \{0, 1\}$, whose value characterizes a particular method, out of the two ones to be considered hereafter:

$$X_H^{h,k,l} = \left\{ v^h \in S^{h,k+l+1} \bigcap X^{h,k};\ (v^h, \varphi^h)_1 = 0,\ \forall\, \varphi^h \in S_0^{h,k+l+1} \right\}, \tag{11}$$

In so doing, the finite element approximation of (7) is as follows:

$$\left\{ \begin{array}{ll} \text{Find } \mathbf{u}^h \text{ and } \omega^h = \omega_0^h + \omega_H^h & \\ \text{where } (\mathbf{u}^h, \omega_0^h, \omega_H^h) \in [S_0^{h,k+l}]^2 \times S_0^{h,k} \times X_H^{h,k,l} & \text{such that} \\ \text{(a) } (\mathbf{u}^h, \mathbf{v}^h)_1 = (\omega_0^h + \omega_H^h, \mathbf{curl}\, \mathbf{v}^h)_0\ , & \forall\, \mathbf{v}^h \in [S_0^{h,k+l}]^2 \\ \text{(b) } (\omega_H^h, \chi_H^h)_0 = -(\omega_0^h, \chi_H^h)_0\ , & \forall\, \chi_H^h \in X_H^{h,k,l} \\ \text{(c) } (\omega_0^h, \phi^h)_1 = \nu^{-1}(\mathbf{f}, \mathbf{curl}\, \phi^h)_0\ , & \forall\, \phi^h \in S_0^{h,k}. \end{array} \right. \tag{12}$$

The proof of the following result can be found in [13]:

Theorem 2.1 *Problem* (12) *has a unique solution.* ■

As for the convergence results for the above defined approximation methods, we denote below by $\|\cdot\|_s$ and $|\cdot|_s$ the standard norm and seminorm of the Sobolev space $H^s(\Omega)$, $s > 0$. They are stated below assuming that Γ is such that the solution of the homogeneous Dirichlet problem for the laplacian operator with a right hand side in $L^2(\Omega)$, belongs at least to $H^2(\Omega)$. More precisely we have (cf. [13]):

Theorem 2.2 *Assume that Ω is convex and that the solution $(\mathbf{u}, \omega)$, of system (7) belongs to $[H^{k+l+1}(\Omega)]^2 \times H^{k+l+s}(\Omega)$, $s \in [0,1]$. Then there exists a constant C independent of h and ν such that the following error estimates hold for the finite element solution $(\mathbf{u}^h, \omega^h)$ of (12), with $j = min\{k+1, k+l+s\}$:*

$$\|\omega - \omega^h\|_0 \ \leq\ C \left\{ h^j |\omega|_{k+l+s} + h^{k+l} \|\mathbf{u}\|_{k+l+1} \right\} \tag{13}$$

$$\|\mathbf{u} - \mathbf{u}^h\|_1 \ \leq\ C \left\{ h^j |\omega|_{k+l+s} + h^{k+l} |\mathbf{u}|_{k+l+1} \right\}. \tag{14}$$

■

Remark 2.2 *Whenever Ω is not convex the error estimates of Theorem 2.2 must be adjusted in terms of the angles of the re-entrant corners of Γ. Typically, if these are such that every solution of the homogeneous Dirichlet problem in Ω for a right hand side in $L^2(\Omega)$ belongs to $H^r(\Omega)$, for $r \in (3/2, 2]$, one should replace j in estimates (13) and (14) with $j + r - 2$.*

Remark 2.3 *The finite element methods considered above are certainly expensive to solve a single Stokes problem, but become reasonably competitive in the framework of an iterative solution of the Navier-Stokes equations. For numerical results illustrating the behavior of these methods, among others of the same class, the author refers to [14].*

3 An extension to the three-dimensional case

The results presented in the previous Section can be extended in a natural though far from trivial way, to the case of problems posed in bounded simply connected domains of $\mathbb{R}^3$. Here we further assume for simplicity, that the boundary Γ of Ω is connected. Moreover we also make the assumption that $\mathbf{f} \in L^2(\Omega)^3$ (cf. [11]), which is rather reasonable from the physical point of view.

In so doing, among other possibilities (cf. [8] and [12]), we give below a set of boundary conditions (15), that one may add to the Stokes system (4) in terms of the velocity and vorticity fields, which can be viewed as the three-dimensional analogue of (5).

$$\begin{cases} \mathbf{curl}\,\mathbf{u} \times \mathbf{n} = \boldsymbol{\omega} \times \mathbf{n} & \text{a.e. on } \Gamma \\ \mathbf{div}\,\boldsymbol{\omega} = 0 & \text{a.e. on } \Gamma \end{cases} \tag{15}$$

Likewise the case of (4)-(5), the resulting system (4)-(15) introduced in [11], is equivalent to (2), under suitable regularity hypotheses.

However, like in the two-dimensional case, if one wishes to ensure the validity of such equivalence results, under our rather weak regularity assumptions, we consider again a suitable variational formulation of the velocity-vorticity system (4)-(15). In order to do so, let us first introduce the proposed three-dimensional analogue of space $X(\Omega)$, namely, the Hilbert space denoted by $\mathbf{X}(\Omega, div)$, in which $\boldsymbol{\omega}$ is to be searched for. We define such space using the duality with the space $\mathbf{H}_{t0}(\Omega)$ (cf. [4]), used as test field space as seen below. We recall that

$$\mathbf{H}_{t0}(\Omega) = \{\mathbf{v}/\ \mathbf{v} \in L^2(\Omega)^2,\ \mathbf{curl}\,\mathbf{v} \in L^2(\Omega),\ \mathbf{div}\,\mathbf{v} \in L^2(\Omega),\ \mathbf{v} \times \mathbf{n} = \mathbf{0} \text{ a.e. on } \Gamma\},$$

which equipped with the inner product $(\,(\cdot),(\cdot)\,)_1$, is also a Hilbert space (cf. [4]).

Now for a given field $\boldsymbol{\chi} \in C^\infty(\bar{\Omega})^3$, we define the linear functional $\mathcal{L}^{\boldsymbol{\chi}}_{curl}$ in the topological dual of $[\mathbf{H}_{t0}(\Omega)]'$, equipped with the standard norm $\|\,(\cdot)\,\|_*$, by

$$\mathcal{L}^{\boldsymbol{\chi}}_{curl}(\mathbf{v}) \;=\; (\mathbf{curl}\,\boldsymbol{\chi}, \mathbf{curl}\,\mathbf{v})\ \forall \mathbf{v} \in \mathbf{H}_{t0}(\Omega).$$

Equipping $C^\infty(\bar{\Omega})^3$ with the norm $\|\cdot\|_X$, where

$$\|\,\boldsymbol{\chi}\,\|_X \;=\; [\,\|\mathcal{L}^{\boldsymbol{\chi}}_{curl}\|_*^2 \;+\; \|\,\mathbf{div}\,\boldsymbol{\chi}\,\|^2 \;+\; \|\,\boldsymbol{\chi}\,\|^2]^{1/2},$$

we define the space $\mathbf{X}(\Omega, div)$ to be the completion of $C^\infty(\bar{\Omega})^3$ for the topology induced by $\|\cdot\|_X$.

Remark 3.1 *The easy-to-check inclusion $\mathbf{H}(\Omega, div) \cap \mathbf{H}(\Omega, curl) \subset \mathbf{X}(\Omega, div)$, is strict. Moreover $\forall \boldsymbol{\chi} \in \mathbf{H}(\Omega, div) \cap \mathbf{H}(\Omega, curl)$ the uniquely defined functional $\mathcal{L}^{\boldsymbol{\chi}}_{curl}$ in the topological dual space of $\mathbf{H}_{t0}(\Omega)$ associated with any element of $\mathbf{X}(\Omega, div)$, through the completion process, is simply given by $\mathcal{L}^{\boldsymbol{\chi}}_{curl}(\mathbf{v}) \;=\; (\mathbf{curl}\,\boldsymbol{\chi}, \mathbf{curl}\,\mathbf{v})$, $\forall \mathbf{v} \in \mathbf{H}_{t0}(\Omega)$ [11].* ■

With the obvious definitions of $\mathbf{V}_g$ and $\mathbf{H}_{n0}(\Omega)$ extended to a three-dimensional domain, the weak formulation of the velocity-vorticity system that we wish to solve is then,

$$\begin{cases} \text{Find } \mathbf{u} \in \mathbf{V}_g \text{ and } \boldsymbol{\omega} \in \mathbf{X}(\Omega, div) \text{ such that} \\ \nu\,[\mathcal{L}^{\boldsymbol{\omega}}_{curl}(\boldsymbol{\varphi}) + (\mathbf{div}\,\boldsymbol{\omega}, \mathbf{div}\,\boldsymbol{\varphi})] \;=\; (\mathbf{f}, \mathbf{curl}\,\boldsymbol{\varphi}) & \forall \boldsymbol{\varphi} \in \mathbf{H}_{t0}(\Omega) \\ (\mathbf{u}, \mathbf{v})_1 \;=\; (\boldsymbol{\omega}, \mathbf{curl}\,\mathbf{v}) & \forall \mathbf{v} \in \mathbf{H}_{n0}(\Omega) \end{cases} \tag{16}$$

According to [11], (16) has a unique solution, which is precisely **u** and **curl u**, where **u** is the velocity field that, together with the pressure p, solves the Stokes system (2). Furthermore, this pair of fields is also a solution of (4)-(15), while the converse statement is true provided one can ensure a sufficient regularity of both the data and the domain. For instance, if Γ is of the $\mathcal{C}^{1,1}$-class or if Ω is a convex polyhedron, and moreover $\mathbf{g}$ belongs to $H^{3/2}(\Gamma)^3$, we may assert that the solution of (4)- (15) belongs to $H^2(\Omega)^3 \times H^1(\Omega)^3$, and in this case this system is equivalent to (2) (cf. [12]).

Remark 3.2 *In [12]) the author and collaborator considered another extension to the three-dimensional case of the variational form (7) of the velocity-vorticity system, in which the vorticity components are searched for in the same space $X(\Omega)^3$. However in that paper, well-posedness and equivalence with the standard velocity-pressure system, were only demonstrated in the case where the velocity lies in $H^2(\Omega)^3$. Notice that this formulation can be derived from (16), if we take $\varphi \in H^1_0(\Omega)^3$ and we further require that ω lies in the linear manifold of $X(\Omega)^3$, consisting of those fields whose normal trace on Γ is equal to a function $\eta = curl\mathbf{v} \cdot \mathbf{n}$ (actually η depends only on the tangential components of $\mathbf{g}$).*

Remark 3.3 *More recently the analysis of the formulation mentioned in Remark 3.2, was extended to the case of arbitrary lipschitzian domains by Ern, Guermond and Quartapelle in the promptly processed and published paper [3]. Incidentally, in the same paper the authors endeavoured to rewrite in terms of distributions spaces the formulation (16). In so doing they slightly improved this author's study of this formulation performed in [11], as the non restrictive assumption $\mathbf{f} \in L^2(\Omega)^3$ could be slightly weakened. However they seem to have overlooked this author's paper, published even before submission of theirs, for no reference to the former can be found in the latter.* ■

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