

Theoretical and numerical analysis of fracture phenomena under dynamic loading

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1 Introduction

Let us consider the linear elasticity in the state of plate stress regarding $(x_1, x_2) \in \Omega_\Sigma = \Omega \setminus \Sigma$. Here Ω is the bounded domain in \mathbb{R}^2 with a smooth boundary containing the crack whose undeformed shape is a piecewise smooth curve $\Sigma = \sum_{j=1}^J \Sigma_j$ in \mathbb{R}^2 with the two edges γ_0, γ , which is given by

$$\begin{aligned} \Sigma &= \{(x_1, x_2) \mid x_1 = \phi_1(s), x_2 = \phi_2(s), 0 \leq s \leq a\} \\ \Sigma_j &= \{(x_1, x_2) \mid x_1 = \phi_1(s), x_2 = \phi_2(s), a_j \leq s \leq a_{j+1}\} \quad a = a_{J+1} \end{aligned}$$

with the length parameter s . The functions $\phi_l(s)$, $l = 1, 2$ are C^2 functions in the interval (a_j, a_{j+1}) , $j = 1, \dots, J$ and the edges are given by $\gamma_0 = (\phi_1(0), \phi_2(0))$, $\gamma = (\phi_1(a), \phi_2(a))$. We assume $\Sigma \subset \Omega$.

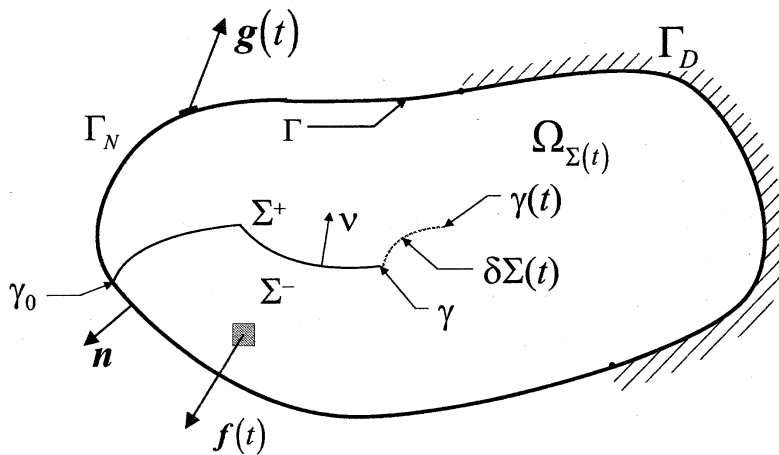


Figure 1: The elastic plate with the crack $\Sigma(t)$.

The crack extension process is considered to occur in a quasi-static manner such that inertial effects may be neglected. Therefore, when we refer to time t , we use it as a

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parameter that delineates the history of the sequence of events such as in loading or crack propagation.

Assume that the part Γ_N of the boundary $\partial\Omega$ is fixed and loads $t \mapsto \mathbf{f}(t) \in C^2([0, T], L^2(\mathbb{R}^2)^2)$ and $t \mapsto \mathbf{g}(t) \in C^2([0, T], L^2(\Gamma_N)^2)$ are given. In this situation (see Figure 1), we consider the virtual crack extension $\{\Sigma(t)\}_{0 \leq t \leq T}$

$$\begin{aligned}\Sigma(T) &= \Sigma \cup \delta\Sigma(T), \quad \delta\Sigma(T) \subset \Omega, \quad \Sigma(t) = \Sigma \cap \delta\Sigma(t), \quad 0 < t < T, \\ \delta\Sigma(t) &= \{(x_1, x_2) \mid x_1 = \phi_1(s), x_2 = \phi_2(s), a \leq s \leq a + t\},\end{aligned}$$

where $\phi_i(t)$, $i = 1, 2$ are C^2 class in $(0, T)$. For simplify, we assume the parameter t also express the length of crack extension. Throughout this paper the unit vector $\boldsymbol{\nu}(s)$, $s \in (0, T)$ denotes the normal direction from the minus side to the plus side; i.e., $\boldsymbol{\nu}(s) = (-\phi_2'(s), \phi_1'(s)) / \sqrt{\phi_1'(s)^2 + \phi_2'(s)^2}$ when x approaches to $(\phi_1(s), \phi_2(s))$ from above $\Sigma(T)$ (denoted by $\Sigma(T)^+$) and $-\boldsymbol{\nu}(s)$ is the interior normal direction from below $\Sigma(T)$ (denoted by $\Sigma(T)^-$).

Let $\mathbf{u}(t) = (u_i(t))$, $\boldsymbol{\varepsilon} = (\varepsilon_{ij}(t))$ and $\boldsymbol{\sigma} = (\sigma_{ij}(t))$ denote that displacement vector, the strain tensor and the stress tensor, respectively. The strain-displacement relation is given by

$$\varepsilon_{ij}(t) = \varepsilon_{ij}(\mathbf{u}(t)) := (u_{i,j}(t) + u_{j,i}(t))/2, \quad u_{i,j}(t) = \partial u_i(t) / \partial x_j$$

and the stress and the strain is connected by Hooke's tensor whose components c_{ijkl} ($i, j, k, l = 1, 2$) are the C^2 functions defined on \mathbb{R}^2 satisfying the conditions; $c_{ijkl} = c_{jilk} = c_{klij}$, i.e.

$$\sigma_{ij}(t) = \sigma_{ij}(\mathbf{u}(t)) := c_{ijkl} \varepsilon_{kl}(\mathbf{u}(t)).$$

For each load $\mathcal{L}(t) = (\mathbf{f}(t), \mathbf{g}(t))$, $0 \leq t \leq T$, the displacement $\mathbf{u}(t)$ satisfy the following

$$P_{\mathcal{L}(t), \Sigma(t)} : \begin{cases} -\sigma_{ij,j}(\mathbf{u}(t)) = f_i(t) & \text{in } \Omega_{\Sigma(t)} \\ \sigma_{ij}(\mathbf{u}(t))^+ \nu_j = \sigma_{ij}(\mathbf{u}(t))^- \nu_j = 0 & \text{on } \Sigma(t) \\ \sigma_{ij}(\mathbf{u}(t)) n_j = g_i(t) & \text{on } \Gamma_N \\ \mathbf{u}(t) = 0 & \text{on } \Gamma_D \end{cases}$$

where $\sigma_{ij}(\mathbf{u}(t))^\pm$ are the value of $\sigma_{ij}(\mathbf{u}(t))$ on the plus/minus side of $\Sigma(t)$, $\mathbf{n} = (n_1, n_2)$ denote the outward unit normal of $\partial\Omega$.

In next section, for virtual crack extensions, we will show what's the *crack extension force* which motivate and control the deformations associated with crack extension based on the concept written in [3] from mathematical viewpoint. In section 3, we will consider how to select the real crack extension from virtual crack extensions, especially the direction of the crack extension.

2 The crack extension force in brittle fracture

The weak solution $\mathbf{u}(t)$ of problem $P_{\mathcal{L}(t), \Sigma(t)}$ is given as the element of $V(\Omega_{\Sigma(t)})$ minimizing the potential energy functional

$$\mathcal{E}(\mathbf{v}; \mathcal{L}(t), \Omega_{\Sigma(t)}) := \int_{\Omega_{\Sigma(t)}} \{w(x, \mathbf{v}) - \mathbf{f}(t) \cdot \mathbf{v}\} dx - \int_{\Gamma_N} \mathbf{g}(t) \cdot \mathbf{v} dl$$

over $\mathbf{v} \in V(\Omega_{\Sigma(t)})$. Here $w(x, \mathbf{v}) = c_{ijkl}(x)\varepsilon_{ij}(\mathbf{v})\varepsilon_{kl}(\mathbf{v})/2$,

$$V(\Omega_{\Sigma(t)}) = \{\mathbf{v} \in H^1(\Omega_{\Sigma(t)})^2; \mathbf{v} = 0 \text{ on } \Gamma_D\}.$$

Theorem 2.1 *If the line measure of Γ_N is not zero and there is a constant $c_0 > 0$ such that*

$$c_{ijkl}\xi_{ij}\xi_{kl} \geq c_0\xi_{ij}\xi_{ij} \quad \text{for all } \xi_{ij} \in \mathbb{R}; i, j = 1, 2,$$

then the only one solution $\mathbf{u}(t)$ exists and satisfy

$$\begin{aligned} a_{\Omega_{\Sigma(t)}}(\mathbf{u}(t), \mathbf{v}) &= \int_{\Omega_{\Sigma(t)}} \mathbf{f}(t) \cdot \mathbf{v} dx + \int_{\Gamma_N} \mathbf{g}(t) \cdot \mathbf{v} dl \quad \text{for all } \mathbf{v} \in V(\Omega_{\Sigma(t)}) \quad (1) \\ \text{by } a_{\Omega_{\Sigma(t)}}(\mathbf{u}(t), \mathbf{v}) &= \int_{\Omega_{\Sigma}} \sigma_{ij}(\mathbf{u}(t))\varepsilon_{ij}(\mathbf{v}) dx, \end{aligned}$$

and satisfy the conditions in $P_{\mathcal{L}(t), \Sigma(t)}$. Here the boundary conditions in $P_{\mathcal{L}(t), \Sigma(t)}$ has the meaning on $\Sigma(t)$ by $\sigma_{ij}(\mathbf{u}(t))^{\pm} \nu_j$ belonging in the dual space of $H_{00}^{1/2}(\Sigma(t))$ with the norm

$$\|\varphi\|_{1/2, 00, \Sigma(t)} = \left(\int_{\Sigma(t)} \rho_{\partial\Sigma(t)} \varphi^2 dl \right)^{1/2},$$

where $\rho_{\partial\Sigma(t)}(x)$ is the distance between $x \in \Sigma(t)$ and $\gamma(t)$.

Proof. Refer to [8, 9, 11] for the proof.

Theorem 2.2 (see e.g. [5, 13, 4]) *At each crack tip $\gamma(t)$, let us consider the local polar coordinates (r_t, θ_t) that are oriented in such a way that the tangent half-line on the side of crack surfaces correspond to the angles $\{-\pi, +\pi\}$, that is, the plus side is on $+\pi$ and the minus side is on $-\pi$. If the elasticity is homogeneous isotropic, then we get the expansion on the neighborhood $U(\gamma(t))$ of the crack tip $\gamma(t)$, $t > 0$,*

$$u_i(x, t) = \sum_{m=1}^2 \frac{K_m(\gamma(t))}{2\mu} \sqrt{\frac{r_t}{2\pi}} S_{i,m}^C(\theta_t) + u_{i,R}(x, t)$$

for $x = (r_t, \theta_t) \in U(\gamma(t))$, where $u_i(x, t)$ are the components of the displacement $\mathbf{u}(t)$ and $u_{i,R}(t) \in H^2(U(\gamma(t)))$.

In the case of the constant load $\mathcal{L} = (\mathbf{f}, \mathbf{g})$, the most important parameter in fracture mechanics is the *energy release rate*

$$\mathcal{G}(\mathcal{L}; \Omega_{\Sigma(\cdot)}) := \lim_{t \downarrow 0} t^{-1} \left[\mathcal{E}(\mathbf{u}(0); \mathcal{L}, \Omega_{\Sigma(0)}) - \mathcal{E}(\mathbf{u}(t); \mathcal{L}, \Omega_{\Sigma(t)}) \right],$$

which is the derivative $-d\mathcal{E}(\mathbf{u}(t); \mathcal{L}(t), \Omega_{\Sigma(t)})/dt|_{t=+0}$ with respect to the crack extension. Under the constant loading, we have the following.

2.1 Generalized J -integral

Let ω be a bounded domain in \mathbb{R}^2 . We call the domain ω “regular relative to Ω_Σ ” if the identity

$$\begin{aligned} \int_{\omega \cap \Omega_\Sigma} v w_{,i} dx &= - \int_{\omega \cap \Omega} v_{,i} w dx + \int_{\partial(\omega \cap \Omega)} v w n_i ds \\ &+ \int_{\omega \cap \Sigma} (v^+ w^+ \nu_i - v^- w^- \nu_i) ds \end{aligned} \quad (2)$$

holds for all $v, w \in H^1(\Omega_\Sigma)$ and each $i = 1, 2$. If $\omega \cap \Omega_\Sigma$ has the local Lipschitz property, then (2) holds (see e.g. [6, p.121]). Therefore $\omega \cap \Omega_\Sigma$ is regular relative to Ω , if $\omega \cap \Omega_\Sigma$ can be decomposed into two disjoint domains $(\omega \cap \Omega_\Sigma)^\pm$ such that $\omega \cap \Sigma \subset \overline{(\omega \cap \Omega)^+} \cap \overline{(\omega \cap \Omega)^-}$.

For each solution \mathbf{u} of $P_{\mathcal{L}, \Sigma}$, we define the GJ -integral by

$$J_\omega(\mathbf{u}, \mathcal{X}) = P_\omega(\mathbf{u}, \mathcal{X}) + R_\omega(\mathbf{u}, \mathcal{X})$$

as a functional depending on the domain ω and a vector field $\mathcal{X} \in W^{1, \infty}(\mathbb{R}^2)^2$, where

$$\begin{aligned} P_\omega(\mathbf{u}, \mathcal{X}) &= \int_{\partial(\omega \cap \Omega)} \{w(\mathbf{u})(\mathcal{X} \cdot \mathbf{n}) - \sigma_{ij}(\mathbf{u}) n_j (\mathcal{X} \cdot \nabla u_i)\} dl, \\ R_\omega(\mathbf{u}, \mathcal{X}) &= \int_{\omega \cap \Omega_\Sigma} \{\sigma_{ij}(\mathbf{u}) \partial_j \mathcal{X}_k \partial_k u_i - \mathbf{f}(\mathcal{X} \cdot \nabla u) - \mathbf{X} \cdot \nabla_x w(x, \mathbf{u}) - w(x, \mathbf{u}) \operatorname{div} \mathcal{X}\} dx, \end{aligned} \quad (3)$$

are well-defined. Here $\mathbf{n} = (n_1, n_2)$ is the outward unit normal on $\partial(\omega \cap \Omega)$ and dl the line element of $\partial(\omega \cap \Omega)$. The integral $R_\omega(\mathbf{u}, \mathcal{X})$ is well-defined for all solution \mathbf{u} of $P_{\mathcal{L}, \Sigma}$, but $P_\omega(\mathbf{u}, \mathcal{X})$ needs the regularity of \mathbf{u} . We notice that $P_\omega(\mathbf{u}, \mathcal{X})$ contains no integral over $\omega \cap \Sigma$.

Theorem 2.3 (refer to [7, 8, 10]) *If the virtual crack extension $\{\Sigma(t)\}_{0 \leq t \leq T}$ is smooth at γ , i.e.*

$$\phi_i(s), i = 1, 2 \quad \text{are } C^2 \text{ class on } a_J < s < a + T \quad (a = a_{J+1}),$$

then we have

$$\mathcal{G}(\mathcal{L}, \Omega_{\Sigma(\cdot)}) = J_\omega(\mathbf{u}; \mathbf{X}), \quad (4)$$

where ω stands for an arbitrary small domain containing the crack tip γ and \mathbf{X} the vector field obtained from parallel extension of $\mathbf{X}_\gamma = (d\phi_1(s)/ds, d\phi_2(s)/ds)_{s=a}$ over $\bar{\omega}$. Moreover, we have

$$\begin{aligned} \mathcal{G}(\mathcal{L}, \Omega_{\Sigma(\cdot)}) &= \mathbf{X}_\gamma \cdot \mathbf{J}_\gamma(\mathbf{u}) \\ \mathbf{J}_\gamma(\mathbf{u}) &= \left(\lim_{|\omega| \rightarrow 0} P_\omega(\mathbf{u}, \mathbf{e}_1), \lim_{|\omega| \rightarrow 0} P_\omega(\mathbf{u}, \mathbf{e}_2) \right), \end{aligned} \quad (5)$$

where $|\omega|$ means the measure of ω .

If the virtual crack extension $\{\Sigma(t)\}_{0 \leq t \leq T}$ is non-smooth at γ and $\lim_{t \downarrow 0} \mathbf{J}_{\gamma(t)}(\mathbf{u}(t))$ exists, then we have

$$\mathcal{G}(\mathcal{L}, \Omega_{\Sigma(\cdot)}) = \lim_{t \downarrow 0} \mathbf{X}_{\gamma(t)} \cdot \mathbf{J}_{\gamma(t)}(\mathbf{u}(t)). \quad (6)$$

Proof. The mathematical proof of (4) is given in [7] in 2D-case, and the 3D-version of (4) is proved in [8]. Here we notice that $J_\omega(\mathbf{u}, \mathbf{X})$ are independent of ω , and $\lim_{|\omega| \rightarrow 0} R_\omega(\mathbf{u}, \mathcal{X}) = 0$ for arbitrary vector field $\mathcal{X} \in W^{1,\infty}(\mathbb{R}^2)^2$. Then (5) holds. From (5) and by the mean value theorem, we obtain

$$t^{-1} \left[\mathcal{E}(\mathbf{u}(0); \mathcal{L}, \Omega_{\Sigma(0)}) - \mathcal{E}(\mathbf{u}(t); \mathcal{L}, \Omega_{\Sigma(t)}) \right] = \mathbf{X}_{\gamma(\theta t)} \cdot \mathbf{J}_{\gamma(\theta t)}(\mathbf{u}(\theta t)) \quad \text{with } 0 < \theta < 1.$$

Since the existence of limit $\lim_{t \downarrow 0} \mathbf{J}_{\gamma(t)}(\mathbf{u}(t))$ is assumed, we can derive (6). This completes the proof of Theorem 2.3.

Theorem 2.4 *Under the constant loading, if the elasticity is homogeneous isotropic and the crack extension is smooth at γ , then the derivative $-d\mathcal{E}(\mathbf{u}(t); \mathcal{L}(t), \Omega_{\Sigma(t)})/dt|_{t=+0}$ depends only on the singularity at the crack tip γ as follows*

$$-\left. \frac{d\mathcal{E}(\mathbf{u}(t); \mathcal{L}(t), \Omega_{\Sigma(t)})}{dt} \right|_{t=+0} = \frac{1}{E} \left(K_1(\gamma)^2 + K_2(\gamma)^2 \right). \quad (7)$$

If the crack extension is non-smooth at γ and $\lim_{t \downarrow 0} K_i(\gamma(t))$, $i = 1, 2$ exists, then

$$-\left. \frac{d\mathcal{E}(\mathbf{u}(t); \mathcal{L}(t), \Omega_{\Sigma(t)})}{dt} \right|_{t=+0} = \lim_{t \downarrow 0} \frac{1}{E} \left(K_1(\gamma(t))^2 + K_2(\gamma(t))^2 \right). \quad (8)$$

However, under the varying load $\mathcal{L}(t)$, the derivative $-d\mathcal{E}(\mathbf{u}(t); \mathcal{L}(t), \Omega_{\Sigma(t)})/dt|_{t=0}$ does not depend only on the crack extension. By this reason, we introduce another definition of energy release rate

$$\mathcal{G}(\mathcal{L}(\cdot), \Omega_{\Sigma(\cdot)}) := \lim_{t \downarrow 0} \frac{1}{2t} \langle \sigma_{ij}(\mathbf{u}) \nu_j, \llbracket u_i(t) - u_i \rrbracket \rangle_{\Sigma(t)}$$

where $\langle \cdot, \cdot \rangle_{\Sigma(t)}$ is the bilinear between the dual space $(H_{00}^{1/2}(\Sigma(t)))'$ and $H_{00}^{1/2}(\Sigma(t))^2$, $\mathbf{u} = \mathbf{u}(0)$ and $\llbracket v \rrbracket = v^+ - v^-$. Since $\sigma_{ij}(\mathbf{u}) \nu_j \equiv 0$ on Σ and $\llbracket u_i \rrbracket \equiv 0$ on $\Sigma(t) \setminus \Sigma$, we think the following formulas will be valid,

$$\langle \sigma_{ij}(\mathbf{u}) \nu_j, \llbracket u_i \rrbracket \rangle_{\Sigma(t)} = 0, \quad \langle \sigma_{ij}(\mathbf{u}) \nu_j|_{\Sigma}, \llbracket u_i(t) \rrbracket|_{\Sigma} \rangle_{\Sigma(t)} = 0.$$

This will derive the integral expression

$$\lim_{t \downarrow 0} \frac{1}{2t} \langle \sigma_{ij}(\mathbf{u}) \nu_j, \llbracket u_i(t) - u_i \rrbracket \rangle_{\Sigma(t)} = \lim_{t \downarrow 0} \frac{1}{2t} \int_{\Sigma(t) \setminus \Sigma} \sigma_{ij}(\mathbf{u}) \nu_j \llbracket u_i(t) \rrbracket dl. \quad (9)$$

Theorem 2.5 *If the elasticity is homogeneous isotropic and the crack extension is smooth at γ , then*

$$\lim_{t \downarrow 0} \frac{1}{2t} \int_{\Sigma(t) \setminus \Sigma} \sigma_{ij}(\mathbf{u}) \nu_j \llbracket u_i(t) \rrbracket dl = \frac{1}{E} \left(K_1(\gamma)^2 + K_2(\gamma)^2 \right). \quad (10)$$

Proof. It will be sufficient to prove in the case when the straight crack extend straight-forward near γ . Using the localization technique by the cut-off function and the coordinate transform, we can reduce this problem to the case when the crack lie on the line $\{(x_1, 0); -\infty < x_1 < 0\}$.

By the Heaviside function $H(x)$, we obtain the expression

$$\begin{aligned} \begin{bmatrix} \sigma_{22}(x_1, \pm 0) \\ \sigma_{12}(x_1, \pm 0) \end{bmatrix} &= \frac{K_1(\gamma)}{\sqrt{2\pi x_1}} \begin{bmatrix} H(x_1) \\ 0 \end{bmatrix} + \frac{K_2(\gamma)}{\sqrt{2\pi x_1}} \begin{bmatrix} 0 \\ H(x_1) \end{bmatrix} + o(x_1^{-1/2})H(x_1), \\ \begin{bmatrix} u_1(t; x_1, 0) \\ u_2(t; x_1, 0) \end{bmatrix} &= \frac{(\kappa + 1)}{\mu} \sqrt{\frac{t - x_1}{2\pi}} \left(K_1(\gamma(t)) \begin{bmatrix} 0 \\ H(t - x_1) \end{bmatrix} + K_2(\gamma(t)) \begin{bmatrix} H(t - x_1) \\ 0 \end{bmatrix} \right) \\ &\quad + o(\sqrt{t - x_1})H(t - x_1), \end{aligned}$$

where $\kappa = (3 - \nu)/(1 + \nu)$ by Poisson's ratio ν . Because

$$\begin{aligned} \int_{\Sigma(t) \setminus \Sigma} \sigma_{ij}(\mathbf{u}) \nu_j \llbracket u_i(t) \rrbracket dl &= \int_0^t \{ \sigma_{22}(x_1, 0) \llbracket u_2(t; x_1, 0) \rrbracket + \sigma_{21}(x_1, 0) \llbracket u_1(t; x_1, 0) \rrbracket \} dx_1 \\ &= \frac{\kappa + 1}{2\mu\pi} \int_0^t (K_1(\gamma)K_1(\gamma(t)) + K_2(\gamma)K_2(\gamma(t))) \sqrt{(t - x_1)/x_1} dx_1 + o(t) \\ &= \frac{\kappa + 1}{2\mu\pi} \frac{\pi}{2} t (K_1(\gamma)^2 + K_2(\gamma)^2) + o(t). \end{aligned}$$

From this we can derive (10).

But (10) does not hold, if the virtual crack extension is non-smooth at γ .

3 The direction of crack extension

If we apply (10) to the straight initial crack Σ with $\gamma = (0, 0)$ and the virtual kinky crack extension

$$\Sigma_\alpha(t) = \{(x, y); x = l \cos \alpha, y = l \sin \alpha, 0 \leq l \leq t\},$$

we then have

$$\int_{\Sigma(t) \setminus \Sigma} \sigma_{ij}(\mathbf{u}) \nu_j \llbracket u_i(t) \rrbracket dl = \int_0^t \{ \sigma_\theta(\mathbf{u}(l, \alpha)) \llbracket u_{\alpha;\theta}(t; t - l, 0) \rrbracket + \sigma_{l\theta}(\mathbf{u}(l, \alpha)) \llbracket u_{\alpha;r}(t; t - l, 0) \rrbracket \} dl. \quad (11)$$

where

$$\begin{aligned} \sigma_\theta(\mathbf{u}(l, \alpha)) &= \sigma_{11}(\mathbf{u}(x)) \sin^2 \alpha + \sigma_{22}(\mathbf{u}(x)) \cos^2 \alpha - \sigma_{12}(\mathbf{u}(x)) \sin 2\alpha \quad (12) \\ &= (2\pi l)^{-1/2} (\mathcal{F}_{11}^\alpha K_1(\gamma) + \mathcal{F}_{12}^\alpha K_2(\gamma)) + \sigma_r^R(x), \\ \sigma_{r\theta}(\mathbf{u}(l, \alpha)) &= (\sigma_{22}(\mathbf{u}(x)) - \sigma_{11}(\mathbf{u}(x))) \sin \theta \cos \theta + \sigma_{12}(\mathbf{u}(x)) \cos 2\theta \\ &= (2\pi l)^{-1/2} (\mathcal{F}_{21}^\alpha K_1(\gamma) + \mathcal{F}_{22}^\alpha K_2(\gamma)) + \sigma_{r\theta}^R(x), \\ \mathcal{F}_{11}^\alpha &= \frac{3}{4} \cos(\alpha/2) + \frac{1}{4} \cos(3\alpha/2), \quad \mathcal{F}_{12}^\alpha = -\frac{3}{4} \sin(\alpha/2) - \frac{3}{4} \sin(3\alpha/2), \\ \mathcal{F}_{21}^\alpha &= \frac{1}{4} \sin(\alpha/2) + \frac{1}{4} \sin(3\alpha/2), \quad \mathcal{F}_{22}^\alpha = \frac{1}{4} \cos(\alpha/2) + \frac{3}{4} \cos(3\alpha/2), \end{aligned}$$

$u_{\alpha;r}(t; r_t, \theta_t)$, $u_{\alpha;\theta}(t; r_t, \theta_t)$ the component of $\mathbf{u}(t)$ in the polar coordinate (r_t, θ_t) of the center $\gamma_\alpha(t)$ and $\sigma_\theta^R, \sigma_{r\theta}^R \in H^1(\text{near } \gamma)$. The jumps $\llbracket u_{\alpha;\theta}(t; r_t, 0) \rrbracket, \llbracket u_{\alpha;r}(t; r_t, 0) \rrbracket$ express

the opening and sliding displacements of the crack surface, respectively. Then, near the crack tip $\gamma_\alpha(t) = (t \cos \alpha, t \sin \alpha)$, we have the expression

$$\begin{aligned} \llbracket u_{\alpha;\theta}(t; t-l, 0) \rrbracket &= K_1(\gamma_\alpha(t))(2\mu)^{-1} \sqrt{(t-l)/(2\pi)} + o(\sqrt{t-l}), \\ \llbracket u_{\alpha;r}(t; t-l, 0) \rrbracket &= K_2(\gamma_\alpha(t))(2\mu)^{-1} \sqrt{(t-l)/(2\pi)} + o(\sqrt{t-l}) \end{aligned} \quad (13)$$

on $\Sigma_\alpha(t)$. If $K_{i,\alpha}(\gamma) = \lim_{t \downarrow 0} K_i(\gamma_\alpha(t))$ exist for $i = 1, 2$, then by the mean value theorem, we obtain

$$K_1(\gamma_\alpha(t)) = K_{i,\alpha}(\gamma) + K'_i(\gamma_\alpha(\tau))t, \quad \text{for } 0 < \tau < t. \quad (14)$$

Combining (11)–(14), we can derive that the left-hand side of (11) is

$$\frac{1}{E} \{ (\mathcal{F}_{11}^\alpha K_1(\gamma) + \mathcal{F}_{21}^\alpha K_2(\gamma)) K_{1,\alpha}(\gamma) + (\mathcal{F}_{21}^\alpha K_1(\gamma) + \mathcal{F}_{22}^\alpha K_2(\gamma)) K_{2,\alpha}(\gamma) \}. \quad (15)$$

On the other hand, the left-hand side of (11) becomes by (8)

$$\frac{1}{E} (K_{1,\alpha}(\gamma)^2 + K_{2,\alpha}(\gamma)^2). \quad (16)$$

Therefore, it is possible that

$$K_{i,\alpha}(\gamma) = \mathcal{F}_{i1}^\alpha K_1(\gamma) + \mathcal{F}_{i2}^\alpha K_2(\gamma) \quad (17)$$

for $i = 1, 2$. However, (17) is not valid by the papers [14, 12, 1] they have been shown in engineering general situations

$$\begin{aligned} K_{i,\alpha}(\gamma) &= F_{i1}^\alpha K_1(\gamma) + F_{i2}^\alpha K_2(\gamma) \quad \text{for } i = 1, 2, \\ F_{11}^\alpha &= 1 - \frac{3\pi^2}{8} m^2 + \left(\pi^2 - \frac{5\pi^4}{128} \right) m^4 + \left(\frac{\pi^2}{9} - \frac{11\pi^4}{72} + \frac{119\pi^6}{15360} \right) m^6 + O(m^8), \\ F_{12}^\alpha &= -\frac{3\pi}{2} m + \left(\frac{10\pi}{3} + \frac{\pi^3}{16} \right) m^3 + \left(-2\pi - \frac{133\pi^3}{180} + \frac{59\pi^5}{1280} \right) m^5 + O(m^7), \\ F_{21}^\alpha &= \frac{\pi}{2} m - \left(\frac{4\pi}{3} + \frac{\pi^3}{48} \right) m^3 + \left(-\frac{2\pi}{3} + \frac{13\pi^3}{30} - \frac{59\pi^5}{3840} \right) m^5 + O(m^7), \\ F_{22}^\alpha &= 1 - \left(4 + \frac{3\pi^2}{8} \right) m^2 + \left(\frac{8}{3} + \frac{29\pi^2}{18} - \frac{5\pi^4}{128} \right) m^4 \\ &\quad + \left(-\frac{32}{15} - \frac{4\pi^2}{9} - \frac{1159\pi^4}{7200} + \frac{119\pi^6}{15360} \right) m^6 + O(m^8), \end{aligned} \quad (18)$$

with $m = \alpha/\pi$. The Taylor's expansions of \mathcal{F}_{ij}^α ($i, j = 1, 2$) are

$$\begin{aligned} \mathcal{F}_{11}^\alpha &= 1 - \frac{3m^2\pi^2}{8} + \frac{7m^4\pi^4}{128} - \frac{61m^6\pi^6}{15360} + O(m^7) \\ \mathcal{F}_{21}^\alpha &= \frac{-3m\pi}{2} + \frac{7m^3\pi^3}{16} - \frac{61m^5\pi^5}{1280} + O(m^7), \\ \mathcal{F}_{12}^\alpha &= \frac{m\pi}{2} - \frac{7m^3\pi^3}{48} + \frac{61m^5\pi^5}{3840} + O(m^7), \\ \mathcal{F}_{22}^\alpha &= 1 - \frac{7m^2\pi^2}{8} + \frac{61m^4\pi^4}{384} - \frac{547m^6\pi^6}{46080} + O(m^7). \end{aligned}$$

This leads the following modification of (17)

$$K_{i,\alpha}(\gamma) = \mathcal{F}_{i1}^\alpha K_1(\gamma) + \mathcal{F}_{i2}^\alpha K_2(\gamma) + O(\alpha^2). \quad (19)$$

There are many criterions which determine the crack direction α^* . We only show the famous three criterions in homogeneous isotropic elastic plate.

Definition 3.1 (maximum stress criterion [2]) Find the angle α^* that is the maximum value of $\mathcal{F}_{11}^\alpha K_1(\gamma) + \mathcal{F}_{12}^\alpha K_2(\gamma)$ over $-\pi < \alpha < \pi$.

Also α^* satisfy $\mathcal{F}_{21}^{\alpha^*} K_1(\gamma) + \mathcal{F}_{22}^{\alpha^*} K_2(\gamma) = 0$.

Definition 3.2 (maximum energy release rate (see for example [14, 1])) Find the angle α^* that is the maximum value of the energy release rate equivalent to $\frac{1}{E} (K_{1,\alpha}(\gamma)^2 + K_{2,\alpha}(\gamma)^2)$.

Definition 3.3 (local symmetry (see for example [14, 12, 1])) Find the angle α^* that satisfy the condition $K_{2,\alpha^*}(\gamma) = 0$.

If (17) is true, we can prove the following; If the crack extends smoothly, then $\alpha^* = 0$ and all criterions are valid. But each criterions make the difference when α^* far from 0 (see Figure 2).

We now apply the various criterions of the direction for an infinite body loaded by uniform forces at infinity with the angle β . By the experiment, the crack extends straight-forward at $\beta = \pi/2 \sim 1.57$. The Figure 2 indicates that all criterions give the similar angles near $\beta = \pi/2$, but they differ near $\beta = 0$. The curve with the label $J_\alpha^2(\gamma)$ is the same one by the maximum energy release rate criterion, and the curve with $J_\alpha^1(\gamma)$ is obtained by the formal application of J -integral for non-smooth crack extension. In the calculation, we used (18).

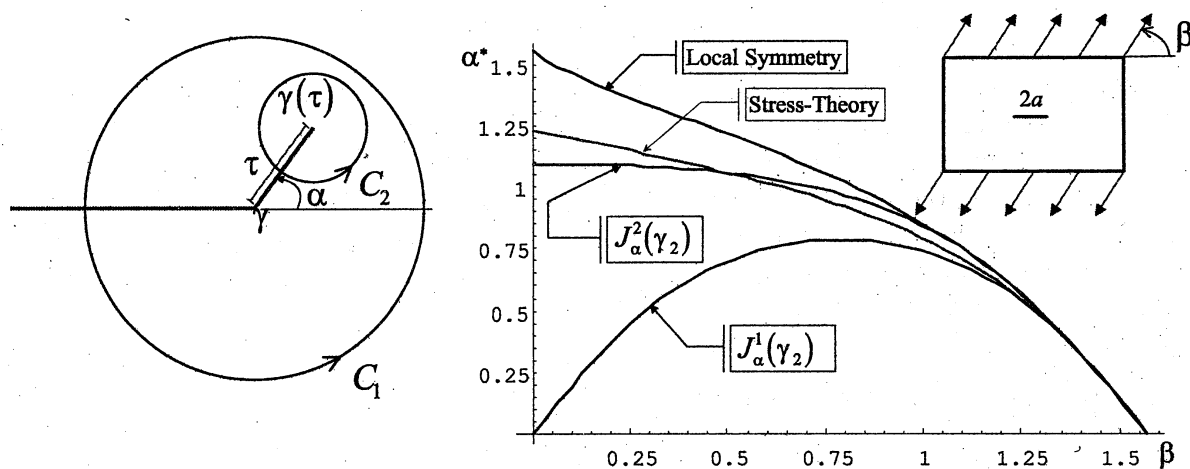
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References

- [1] M. Amestoy and J.B. Leblond. Crack paths in plane situation – II. Detailed form of the expansion of the stress intensity factors, *Int. J. Solids Structures*, 29(1992), 465–501.
- [2] F. Erdogan, and G. E. Sih, On the crack extension in plates under plane loading and transverse shear, *J. Basic Engng*, 85(1963) 519–527.
- [3] G. R. Irwin, Fracture mechanics, in the book “Structural mechanics”, Pergamon Press, 1958, 557–594.
- [4] A. Friedman, B. Hu, J. J. L. Velazquez, Asymptotics for the biharmonic equation near the tip of a crack, *Indiana Univ. Math. J.*, 47(1998), 1327–1395.
- [5] V. A. Kondrat’ev, Boundary problems for elliptic equations in domains with conical or angular points. *Transactions Moscow Mathematical Society* 16(1967), 227–313.
- [6] J. Nečas, Méthodes directes en théorie des équations elliptiques, Masson Éditeur, Paris, 1967

- [7] K. Ohtsuka, J-integral and two-dimensional fracture mechanics, RIMS Kokyuroku, Kyoto Univ., 386(1980), 231–248.
- [8] K. Ohtsuka, Generalized J-integral and three dimensional fracture mechanics I, Hiroshima Math. J., 11(1981), 21–52.
- [9] K. Ohtsuka, Generalized J-integral and three-dimensional fracture mechanics. II. Surface crack problems, Hiroshima Math. J., 16(1986), 327–352.
- [10] K. Ohtsuka, Mathematical aspects of fracture mechanics, Lecture Notes in Num. Appl. Anal., 13, 39–59, 1994.
- [11] K. Ohtsuka, What's the crack extension force in three-dimensional quasi-static brittle fracture, to appear.
- [12] Y. Sumi, A note on the first order perturbation solution of a straight crack with slightly branched and curved extension under a general geometric and loading condition, Engineering Fracture Mechanics 24 (1986), 479–481.
- [13] W.L.Wendland, E.P.Stephan, A hypersingular boundary integral method for two-dimensional screen and crack problems, Arch. Rational Mech. Anal. 112 (1990) 363–390.
- [14] Chien H. Wu, Maximum-energy-release-rate criterion applied to a tension-compression specimen with crack, J. Elasticity, 8(1978), 235–257.



$$J_{\alpha}^1(\gamma) = J_{C_1}(\mathbf{u}, \mathbf{e}_1) \cos \alpha + J_{C_1}(\mathbf{u}, \mathbf{e}_2) \sin \alpha,$$

$$J_{\alpha}^2(\gamma) = \lim_{\tau \rightarrow 0} \lim_{C_2 \rightarrow 0} [J_{C_2}(\mathbf{u}(\tau), \mathbf{e}_1) \cos \alpha + J_{C_2}(\mathbf{u}(\tau), \mathbf{e}_2) \sin \alpha].$$

Figure 2: The direction of crack extension by various criterion.