数理解析研究所講究録
1145 巻 2000 年 220-228

Numerical computation of attractors in two-phase Stefan problems

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1 Introduction

Free boundary problems are boundary value problems defined on domains whose boundaries are unknown and must be determined as the solution. Due to nonlinearity they easily involve chaotic phenomena. They are very important from the practical view point, so investigation of chaotic phenomena is very important. It is carried out via analysis of bifurcation and attractors. Bifurcation phenomena in a free boundary problem related to natural convection were analyzed numerically[6]. Attractors in free boundary porblems were analyzed theoretically[1].

In the paper we consider a one-dimensional free boundary problem with some parameters. This problem is of the type of a two-phase Stefan problem. The spectral collocation method in space and time is used. However, the spectral collocation method is not directly applicable to free boundary problems without additional techniques, due to the unknown shape of domains. So, the fixed domain method is adopted[3].

2 Our free boundary problem

We consider the following one-dimensional free boundary problem with some parameters.

Problem 1. Find $u^{\pm}(x,t)$ and $s(t)$ such that

$$\begin{aligned}
u_t^{\pm}(x,t) &= u_{xx}^{\pm}(x,t) + g^{\pm}(x,t), \qquad 0 < t,\ 0 < x < s(t), \\
u^{\pm}(\mp 1, t) &= h^{\pm}(t)\ , \qquad 0 \le t, \\
u^{\pm}(s(t),t) &= 0, \qquad 0 \le t, \\
u^{+}(x,0) &= u^{+}(x), \qquad -1 < x < s_0, \\
u^{-}(x,0) &= u^{-}(x), \qquad s_0 < x < 1, \\
\frac{d}{dt}s(t) &= -k^{+}(t)u_x^{+}(s(t),t) + k^{-}(t)u_x^{-}(s(t),t), \qquad 0 < t, \\
s(0) &= s_0
\end{aligned}$$

where $|\alpha^{\pm}|,\ |\beta|,\ |s_0| < 1,\ 0 \le r \le 1,$

$$\begin{aligned}
k^{\pm}(t) &= r + (1-r)\,\frac{1}{2}\,\frac{1 \pm \beta \sin t}{\pm 1 + \alpha^{\pm}\sin t}\beta \cos t, \\
h^{\pm}(t) &= \pm 1 + \alpha^{\pm}\sin t, \\
g^{\pm}(x,t) &= \pm\frac{(\beta - \alpha^{\pm})\cos t}{(1 \pm \beta \sin t)^2}(x - \beta \sin t) \pm \frac{\pm 1 + \alpha^{\pm}\sin t}{1 \pm \beta \sin t}\beta \cos t, \\
u^{+}(x) &= a(x - s_0)^2 + a(s_0 + 1)(x - s_0) - \frac{x - s_0}{s_0 + 1}, \\
u^{-}(x) &= b(x - s_0)^2 + b(s_0 - 1)(x - s_0) + \frac{x - s_0}{s_0 - 1}.
\end{aligned}$$

Parameters a, b are determined such that $u^{+}(x) \ge 0$, $u^{-}(x) \le 0$.

Remark. For $a = b = s_0 = r = 0$, there are exact solutions as follows:

$$\begin{aligned}
s(t) &= s_p(t) \equiv \beta \sin t, \\
u^{\pm}(x,t) &= \frac{\mp h^{\pm}(t)}{1 \pm s_p(t)}(x - s_p(t)) = \mp\frac{\pm 1 + \alpha^{\pm}\sin t}{1 \pm \beta \sin t}(x - \beta \sin t).
\end{aligned}$$

3 Fixed domain method

This section presents methods for numerical simulation.

The spectral methods are superior in accuracy[2]. In particular, the spectral collocation method is preferable for the application to nonlinear problems. However, it can not be applied directly to free boundary problems due to the unknown shape of the domain. To avoid this difficulty, we use the fixed domain method[3],[5]. Mapping functions are introduced for mapping the unknown domain to the fixed rectangular domain.

We use the following variable transformation : $(x,t) \to (\xi,\tilde{t})$ such that

$$
\begin{aligned}
t &= t(\tilde{t}) = \tilde{t}, \qquad 0 \le t, \\
x &= x(\xi,\tilde{t}) = \begin{cases} \dfrac{\tilde{s}(\tilde{t})+1}{2}(\xi+1)-1, & 0 \le t,\ -1 \le x \le s(t), \\ \dfrac{1-\tilde{s}(\tilde{t})}{2}(\xi-1)+1, & 0 \le t,\ s(t) \le x \le 1. \end{cases}
\end{aligned}
$$

Using these mapping functions, we define

$$
\begin{aligned}
\tilde{s}(\tilde{t}) &= s(t(\tilde{t})), \\
\tilde{u}^{+}(\xi,\tilde{t}) &= u^{+}(x(\xi,\tilde{t}),t(\tilde{t})), \\
\tilde{u}^{-}(\xi,\tilde{t}) &= u^{-}(x(\xi,\tilde{t}),t(\tilde{t})).
\end{aligned}
$$

Then, Problem 1 is transformed in the following fixed boundary problem.

Problem 2. Find $\tilde{u}^{\pm}(\xi,\tilde{t})$ and $\tilde{s}(\tilde{t})$ such that

$$
\begin{aligned}
\tilde{u}^{+}_{\tilde{t}}(\xi,\tilde{t}) &= -k^{+}(\tilde{t})\frac{2(\xi+1)}{\{\tilde{s}(\tilde{t})+1\}^2}\tilde{u}^{+}_{\xi}(1,\tilde{t})\tilde{u}^{+}_{\xi}(\xi,\tilde{t}) \\
&\quad -k^{-}(\tilde{t})\frac{2(\xi+1)}{\{\tilde{s}(\tilde{t})\}^2-1}\tilde{u}^{-}_{\xi}(-1,\tilde{t})\tilde{u}^{+}_{\xi}(\xi,\tilde{t}) + \frac{4}{\{\tilde{s}(\tilde{t})+1\}^2}\tilde{u}^{+}_{\xi\xi}(\xi,\tilde{t}) \\
&\quad +\frac{(\beta-\alpha^{+})\cos\tilde{t}}{(1+\beta\sin\tilde{t})^2}\left(\frac{\tilde{s}(\tilde{t})+1}{2}(\xi+1)-1-\beta\sin\tilde{t}\right) \\
&\quad +\frac{(1+\alpha^{+}\sin\tilde{t})\beta\cos\tilde{t}}{1+\beta\sin\tilde{t}}, \qquad 0<\tilde{t}, \quad -1<\xi<1, \\
\tilde{u}^{+}(-1,\tilde{t}) &= 1+\alpha^{+}\sin\tilde{t}, \qquad 0 \le \tilde{t}, \\
\tilde{u}^{+}(1,\tilde{t}) &= 0, \qquad 0 \le \tilde{t}, \\
\tilde{u}^{+}(\xi,0) &= \left(\frac{a}{4}(s_0+1)(\xi+1)-\frac{1}{2(s_0+1)}\right)(s_0+1)(\xi-1), \qquad -1<\xi<1, \\
\tilde{u}^{-}_{\tilde{t}}(\xi,\tilde{t}) &= -k^{+}(\tilde{t})\frac{2(\xi-1)}{\{\tilde{s}(\tilde{t})\}^2-1}\tilde{u}^{+}_{\xi}(1,\tilde{t})\tilde{u}^{-}_{\xi}(\xi,\tilde{t}) \\
&\quad -k^{-}(\tilde{t})\frac{2(\xi-1)}{\{\tilde{s}(\tilde{t})-1\}^2}\tilde{u}^{-}_{\xi}(-1,\tilde{t})\tilde{u}^{-}_{\xi}(\xi,\tilde{t}) + \frac{4}{\{\tilde{s}(\tilde{t})-1\}^2}\tilde{u}^{-}_{\xi\xi}(\xi,\tilde{t}) \\
&\quad -\frac{(\beta-\alpha^{-})\cos\tilde{t}}{(1-\beta\sin\tilde{t})^2}\left(\frac{1-\tilde{s}(\tilde{t})}{2}(\xi-1)+1-\beta\sin\tilde{t}\right)
\end{aligned}
$$

$$
\begin{aligned}
& +\frac{(1-\alpha^- \sin\tilde{t})\beta\cos\tilde{t}}{1-\beta\sin\tilde{t}}, \qquad 0<\tilde{t}, \quad -1<\xi<1, \\
\tilde{u}^-(-1,\tilde{t}) &= 0, \qquad 0\le\tilde{t}, \\
\tilde{u}^-(1,\tilde{t}) &= -1+\alpha^-\sin\tilde{t}, \qquad 0\le\tilde{t}, \\
\tilde{u}^-(\xi,0) &= \left(\frac{b}{4}(s_0-1)(\xi-1)-\frac{1}{2(s_0-1)}\right)(s_0-1)(\xi+1), \quad -1<\xi<1, \\
\frac{d}{d\tilde{t}}\tilde{s}(\tilde{t}) &= -k^+(\tilde{t})\frac{2}{\tilde{s}(\tilde{t})+1}\tilde{u}^+_\xi(1,\tilde{t})-k^-(\tilde{t})\frac{2}{\tilde{s}(\tilde{t})-1}\tilde{u}^-_\xi(-1,\tilde{t}), \ 0<\tilde{t}, \\
\tilde{s}(0) &= s_0.
\end{aligned}
$$

For application of the spectral collocation method in time, the time axis is divided into intervals. In each interval the initial and boundary value problem is solved. This procedure is executed iteratively[3]. In the interval $[\tilde{t}_s,\tilde{t}_e]$ we consider the following variable transformation : $\tilde{t}\to\tau$ such that

$$
\tilde{t}=\tilde{t}(\tau)=\frac{\Delta\tilde{t}}{2}\tau+\frac{1}{2}(\tilde{t}_s+\tilde{t}_e), \qquad \tilde{t}_s\le\tilde{t}\le\tilde{t}_e
$$

where

$$
\Delta\tilde{t}=\tilde{t}_e-\tilde{t}_s.
$$

Using this variable transformation, we define

$$
\begin{aligned}
\bar{s}(\tau) &= \tilde{s}(\tilde{t}(\tau)), \\
\bar{u}^+(\xi,\tau) &= \tilde{u}^+(\xi,\tilde{t}(\tau)), \\
\bar{u}^-(\xi,\tau) &= \tilde{u}^-(\xi,\tilde{t}(\tau)).
\end{aligned}
$$

Then, Problem 2 is transformed in the following Proglem 3.

Problem 3. For the interval $[\tilde{t}_s,\tilde{t}_e]$ after the interval $[\tilde{t}'_s,\tilde{t}'_e]$, find $\bar{u}^\pm(\xi,\tau)$ and $\bar{s}(\tau)$ such that

$$
\begin{aligned}
\frac{2}{\Delta\tilde{t}}\bar{u}^+_\tau(\xi,\tau) &= -\bar{k}^+(\tau)\frac{2(\xi+1)}{\{\bar{s}(\tau)+1\}^2}\bar{u}^+_\xi(1,\tau)\bar{u}^+_\xi(\xi,\tau) \\
& -\bar{k}^-(\tau)\frac{2(\xi+1)}{\{\bar{s}(\tau)\}^2-1}\bar{u}^-_\xi(-1,\tau)\bar{u}^+_\xi(\xi,\tau)+\frac{4}{\{\bar{s}(\tau)+1\}^2}\bar{u}^+_{\xi\xi}(\xi,\tau) \\
& +\frac{(\beta-\alpha^+)\cos\{\tilde{t}(\tau)\}}{(1+\beta\sin\{\tilde{t}(\tau)\})^2}\left(\frac{\bar{s}(\tau)+1}{2}(\xi+1)-1-\beta\sin\{\tilde{t}(\tau)\}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{(1+\alpha^+\sin\{\tilde{t}(\tau)\})\beta\cos\{\tilde{t}(\tau)\}}{1+\beta\sin\{\tilde{t}(\tau)\}}, \qquad -1<\tau\le 1, \quad -1<\xi<1,\\
\bar{u}^+(-1,\tau) &= 1+\alpha^+\sin\{\tilde{t}(\tau)\}, \qquad -1\le\tau\le 1,\\
\bar{u}^+(1,\tau) &= 0, \qquad -1\le\tau\le 1,\\
\bar{u}^+(\xi,-1) &= \begin{cases} \left(\dfrac{a}{4}(s_0+1)(\xi+1)-\dfrac{1}{2(s_0+1)}\right)(s_0+1)(\xi-1), & \tilde{t}_s=0,\\ \tilde{u}^+(\xi,\tilde{t}'_e), & \text{otherwise,}\end{cases}\\
&\qquad -1<\xi<1,\\
\frac{2}{\Delta\tilde{t}}\bar{u}^-_\tau(\xi,\tau) &= -\bar{k}^+(\tau)\frac{2(\xi-1)}{\{\bar{s}(\tau)\}^2-1}\bar{u}^+_\xi(1,\tau)\bar{u}^-_\xi(\xi,\tau)\\
&\quad -\bar{k}^-(\tau)\frac{2(\xi-1)}{\{\bar{s}(\tau)-1\}^2}\bar{u}^-_\xi(-1,\tau)\bar{u}^-_\xi(\xi,\tau)+\frac{4}{\{\bar{s}(\tau)-1\}^2}\bar{u}^-_{\xi\xi}(\xi,\tau)\\
&\quad -\frac{(\beta-\alpha^-)\cos\{\tilde{t}(\tau)\}}{(1-\beta\sin\{\tilde{t}(\tau)\})^2}\left(\frac{1-\bar{s}(\tau)}{2}(\xi-1)+1-\beta\sin\{\tilde{t}(\tau)\}\right)\\
&\quad +\frac{(1-\alpha^-\sin\{\tilde{t}(\tau)\})\beta\cos\{\tilde{t}(\tau)\}}{1-\beta\sin\{\tilde{t}(\tau)\}}, \qquad -1<\tau\le 1, \quad -1<\xi<1,\\
\bar{u}^-(-1,\tau) &= 0, \qquad -1\le\tau\le 1,\\
\bar{u}^-(1,\tau) &= -1+\alpha^-\sin\{\tilde{t}(\tau)\}, \qquad -1\le\tau\le 1,\\
\bar{u}^-(\xi,-1) &= \begin{cases} \left(\dfrac{b}{4}(s_0-1)(\xi-1)-\dfrac{1}{2(s_0-1)}\right)(s_0-1)(\xi+1), & \tilde{t}_s=0,\\ \tilde{u}^-(\xi,\tilde{t}'_e), & \text{otherwise,}\end{cases}\\
&\qquad -1<\xi<1,\\
\frac{2}{\Delta\tilde{t}}\frac{d}{d\tau}\bar{s}(\tau) &= -\bar{k}^+(\tau)\frac{2}{\bar{s}(\tau)+1}\bar{u}^+_\xi(1,\tau)-\bar{k}^-(\tau)\frac{2}{\bar{s}(\tau)-1}\bar{u}^-_\xi(-1,\tau), \quad -1<\tau\le 1,\\
\bar{s}(-1) &= \begin{cases} s_0, & \tilde{t}_s=0,\\ \tilde{s}(\tilde{t}'_e), & \text{otherwise.}\end{cases}
\end{aligned}
$$

Then the spectral collocation method in space and time is applied[3].

4 Numerical results

In this section, numerical results are shown.

Fig. 1 shows a numerical result for $r=0$, $\alpha^\pm=\beta=0.5$, $s_0=0$, $a=b=0$. In this case exact solutions are known as in Remark. They are periodic. Numerical solutions are also periodic. Numerical simulation is very satisfactory in accuracy.

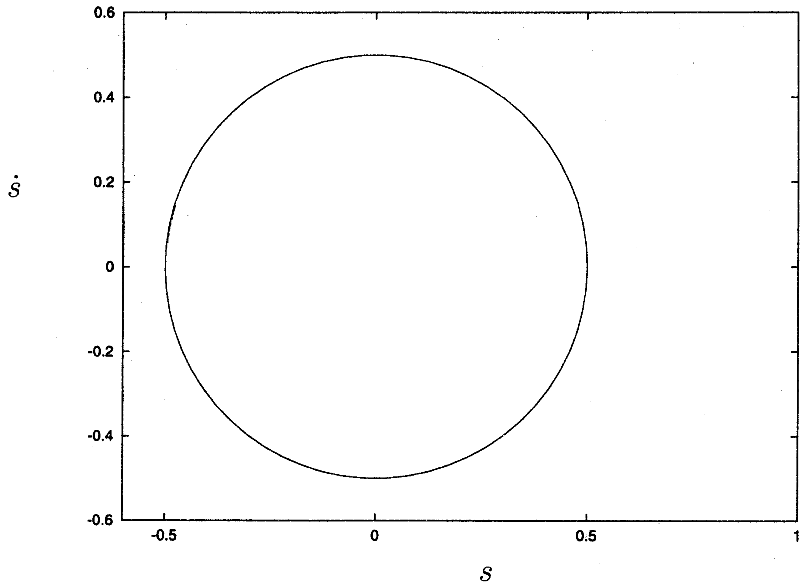

Fig.1. Numerical solution for $r = 0,\ \alpha^{\pm} = \beta = 0.5,\ s_0 = 0,\ a = b = 0.$

Fig. 2 shows these periodic solutions are not stable. For slightly different initial conditions, numerical solutions evolve separately from the periodic solutions. This means the periodic solutions are not stable. In this case there seems to be no attractor.

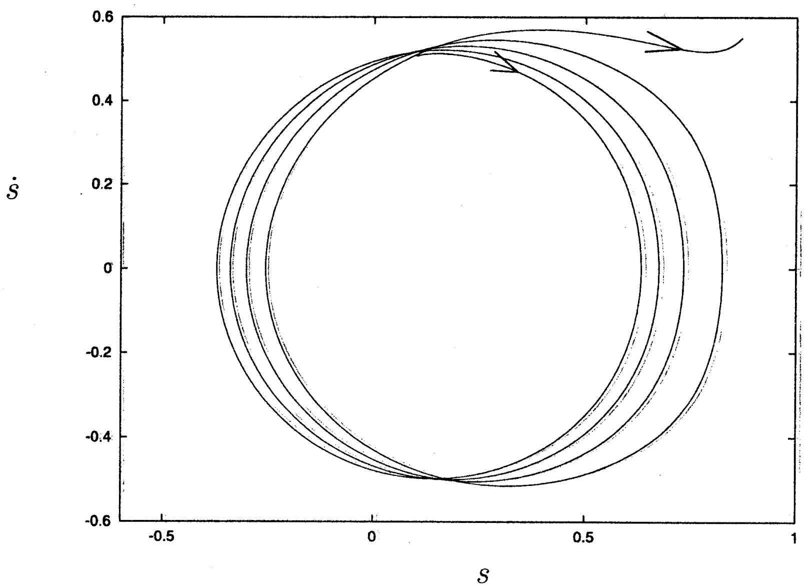

Fig.2. Numerical solution for $r = 0,\ \alpha^{\pm} = \beta = 0.5,\ s_0 = 0.1,\ a = b = 0.$

Fig.3 shows a numerical result for $r = 1,\ \alpha^{\pm} = \beta = 0.5,\ s_0 = 0,\ a = b = 0$. In this case there are no exact solutions. Numerical solutions converge to the attractor in Fig.3 which is a closed curve. This means periodic solutions are stable.

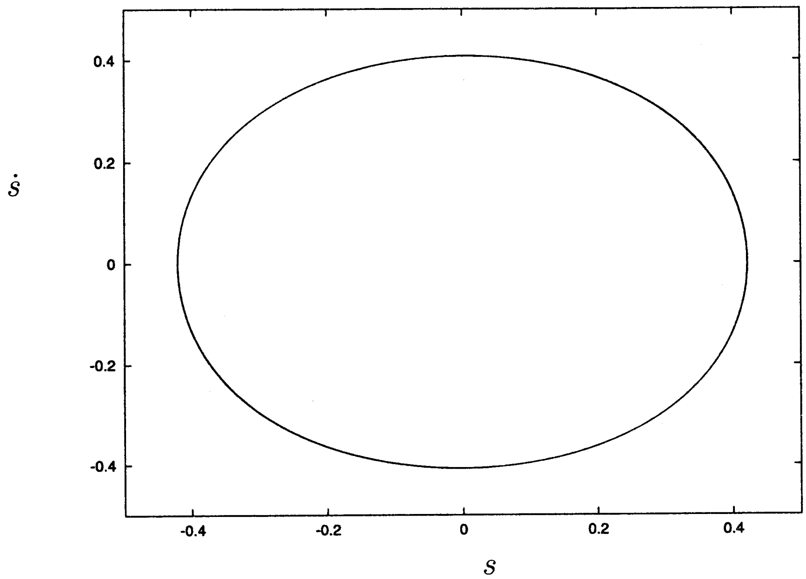

Fig.3. Attractor for $r = 1$, $\alpha^{\pm} = \beta = 0.5$, $a = b = 0$.

Fig. 4 shows numerical results for $r = 1$, $\alpha^{\pm} = \beta = 0$ and several initial conditions. Solution curves converge to a point. This means the fixed point is the attractor. The steady state is stable.

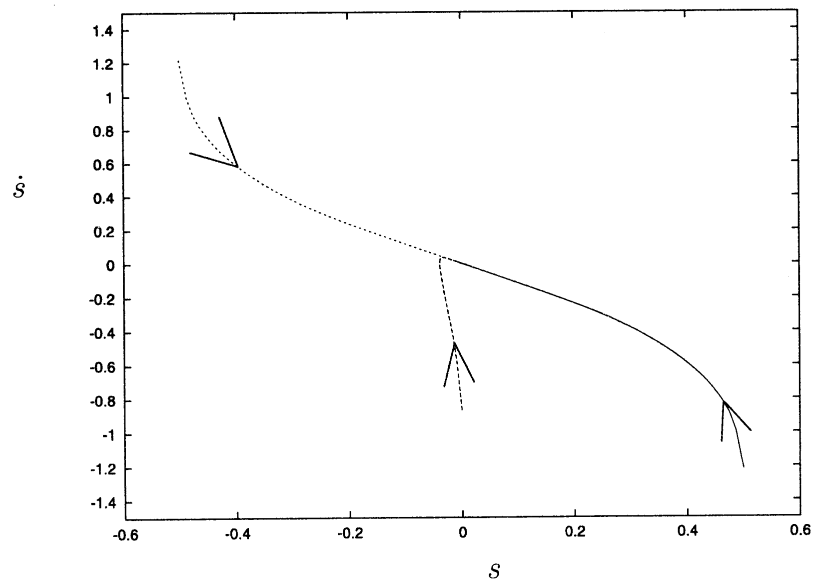

Fig.4. Numerical solution for $r = 1$, $\alpha^{\pm} = \beta = 0$.

Fig. 5 shows the attractor for $r = 1$, $\alpha^{\pm} = \beta = 0.5$. It is a closed curve in the three-dimensional space. u^+ and u^- represents $u^+(\frac{s(t)-1}{2}, t)$ and $u^-(\frac{1-s(t)}{2}, t)$, respectively. s, u^+ and u^- are all unknowns of the ODE system which is derived by the discretization of the PDE system. This means our approach enables to approximate arbitrarily the original attractor of the PDE system in the functional space.

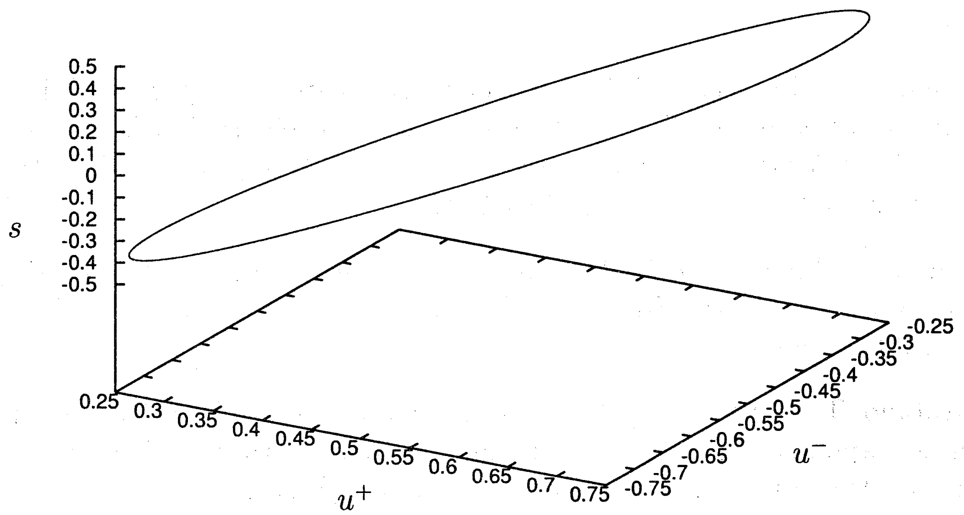

Fig.5. Attractor in the solution space for $r = 1$, $\alpha^{\pm} = \beta = 0.5$, $a = b = 0$.

5 Conclusion

In the paper numerical computation of attractors to a free boundary problems is carried out. The problem considered here is a one-dimensional free boundary problem with some parameters. This problem is of the type of a two-phase Stefan problem. It is transformed into a fixed boundary problem by the fixed domain method. Then, the spectral collocation method in space and time is applied for numerical computation. From numerical results, attractors are found numerically for some values of parameters. Our next goal is investigation of Lyapunov exponents of the attractors[4].

Acknowledgement. This work is partially supported by Grant-in-Aid for Scientific Research(Nos. 09440080 and 10354001). This work is also a collaboration with CCSE of Japan Atomic Energy Research Institute.

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