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Continuous and Discrete Fourier Coefficients of Equi-distant Piecewise Linear Continuous Periodic Functions

- Application to Mathematical Analysis of An FEM-CSM Combined Method for 2D Exterior Laplace Problems -

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Abstract

The author has investigated an FEM-CSM combined method for 2D exterior Laplace problems during these years ([2], [3]). Here the abbreviation of CSM is employed for the charge simulation method (See [1]). In the mathematical analysis for the method, especially in the proof of an a priori error estimate for the approximate solutions obtained by the method, a relation between continuous and discrete Fourier coefficients of equi-distant piecewise linear continuous 2π-periodic function plays a key role. In this paper, the relation is introduced with illustrative examples of application to the mathematical analysis mentioned above.

1. Relation between continuous and discrete Fourier coefficients for equi-distant piecewise linear continuous 2π-periodic functions

Let $f(\theta)$ be a complex valued continuous 2π-periodic function of θ. For $n \in \boldsymbol{Z}$, a continuous Fourier coefficient f_n of the function $f(\theta)$ is defined through

$$f_n = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)e^{-in\theta}d\theta.$$

Fix a positive integer N. Set

$$\theta_1 = \frac{2\pi}{N}, \qquad \theta_j = j\theta_1 \quad \text{for} \quad j \in \boldsymbol{Z}.$$

For $n \in \boldsymbol{Z}$, a discrete Fourier coefficient $f_n^{(N)}$ of the function $f(\theta)$ is defined through

$$f_n^{(N)} = \frac{1}{N}\sum_{j=0}^{N-1} f(\theta_j)e^{-in\theta_j}.$$

It is to be noted that we have for any continuous 2π-periodic function $f(\theta)$,

$$f_{n+Nr}^{(N)} = f_n^{(N)}, \quad n \in \boldsymbol{Z}, \quad r \in \boldsymbol{Z} - \{0\}. \tag{1}$$

Let $\hat{w}(\theta)$ be the reference roof function defined through

$$\hat{w}(\theta) = \begin{cases} 1 - |\theta| & : \quad |\theta| \le 1, \\ 0 & : \quad |\theta| \ge 1. \end{cases}$$

For any $j \in \boldsymbol{Z}$, define a piecewise linear basis function $w_j^{(N)}(\theta)$ through the following formula:

$$w_j^{(N)}(\theta) = \hat{w}(\frac{\theta - \theta_j}{\theta_1}), \quad -\infty < \theta < \infty.$$

A complex valued function $f(\theta)$ is said to be an **equi-distant piecewise linear continuous 2π-periodic function** (with N nodal points) in this paper if $f(\theta)$ is represented as

$$f(\theta) = \sum_{j=0}^{N} f(\theta_j) w_j^{(N)}(\theta), \quad 0 \le \theta \le 2\pi,$$

with

$$f(2\pi) = f(0).$$

Introduce a function $\alpha(\theta)$ through the formula:

$$\alpha(\theta) = \frac{2(1 - \cos\theta)}{\theta^2} \quad \text{for} \quad \theta \neq 0, \quad \text{with} \quad \alpha(0) = 1.$$

Theorem 1 *We have the following relation for any equi-distant piecewise linear continuous 2π-periodic function (with N nodal points) $f(\theta)$,*

$$f_n = \alpha(\theta_n) f_n^{(N)}, \quad n \in \boldsymbol{Z}. \tag{2}$$

Proof A straightforward calculus leads the relation. □

Corollary *We have the following identity for any equi-distant piecewise linear continuous 2π-periodic function (with N nodal points) $f(\theta)$,*

$$f_{n+Nr} = \left(\frac{n}{n+Nr}\right)^2 f_n, \quad n \in \boldsymbol{Z}, \quad r \in \boldsymbol{Z} - \{0\}. \tag{3}$$

Proof Since we have

$$\alpha(\theta_{n+Nr}) = \left(\frac{n}{n+Nr}\right)^2 \alpha(\theta_n), \quad n \in \boldsymbol{Z}, \quad r \in \boldsymbol{Z} - \{0\},$$

Theorem 1 together with Equality (1) implies Equality (3). □

2. Boundary bilinear forms of Steklov type for exterior Laplace problems and its CSM-approximation forms

Let D_a be the interior of the disc with radius a being the origin as its center, and let Γ_a be the boundary of D_a. Let $\Omega_e = (D_a \cup \Gamma_a)^C$, which is said to be the exterior domain. We use the notation $\mathbf{r} = \mathbf{r}(\theta)$ for the point in the plane corresponding to the complex number $re^{i\theta}$ with $r = |\mathbf{r}|$ where $|\mathbf{r}|$ is the Euclidean norm of $\mathbf{r} \in \boldsymbol{R}^2$. Similarly we use $\mathbf{a} = \mathbf{a}(\theta)$, and $\vec{\rho} = \vec{\rho}(\theta)$, corresponding to $ae^{i\theta}$ with $a = |\mathbf{a}|$, and $\rho e^{i\theta}$ with $\rho = |\vec{\rho}|$, respectively.

For functions $u(\mathbf{a}(\theta))$ and $v(\mathbf{a}(\theta))$ of $H^{1/2}(\Gamma_a)$, let us introduce the boundary bilinear form of Steklov type for exterior Laplace problem through the following formula:

$$b(u,v) = 2\pi \sum_{n=-\infty}^{\infty} |n| f_n \overline{g_n}, \tag{4}$$

where f_n, and g_n, are continuous Fourier coefficients of $u(\mathbf{a}(\theta))$, and $v(\mathbf{a}(\theta))$, respectively.

It is to be noted that the following fact:

If $u(\mathbf{a}(\theta))$ is the boundary value on Γ_a of the function $u(\mathbf{r})$ satisfying the following boundary value problem (E)*:*

$$\text{(E)} \qquad \begin{cases} -\Delta u &= 0 \quad \text{in} \quad \Omega_e, \\ u &= \varphi \quad \text{on} \quad \Gamma_a, \\ \sup\limits_{\Omega_e} & |u| < \infty, \end{cases}$$

with

$$\varphi = u(\mathbf{a}(\theta)),$$

then

$$b(u,v) = -\int_{\Gamma_a} \frac{\partial u}{\partial r} v d\Gamma. \tag{5}$$

The CSM approximate form for $b(u,v)$ of the first type, which is denoted by $b^{(N)}(u,v)$, is represented through the following formula (6):

$$b^{(N)}(u,v) = -\int_{\Gamma_a} \frac{\partial u^{(N)}}{\partial r} v^{(N)} d\Gamma, \tag{6}$$

where $u^{(N)}(\mathbf{r})$ is a CSM-approximate solution for $u(\mathbf{r})$ satisfing (E) with $\varphi = u(\mathbf{a}(\theta))$. Namely $u^{(N)}(\mathbf{r})$ is determined through the following problem ($\mathrm{E}^{(N)}$):

$$(\mathrm{E}^{(N)}) \qquad \begin{cases} u^{(N)}(\mathbf{r}) &= \sum\limits_{j=0}^{N-1} q_j G_j(\mathbf{r}) + q_N, \\ u^{(N)}(\mathbf{a}_j) &= u(\mathbf{a}_j), \quad 0 \le j \le N-1, \\ \sum\limits_{j=0}^{N-1} q_j &= 0, \end{cases}$$

where

$$\mathbf{a}_j = \mathbf{a}(\theta_j), \qquad \vec{\rho}_j = \vec{\rho}(\theta_j) \quad \text{with} \quad 0 < \rho < a,$$

$$G_j(\mathbf{r}) = E(\mathbf{r} - \vec{\rho}_j) - E(\mathbf{r}), \qquad E(\mathbf{r}) = -\frac{1}{2\pi}\log r.$$

Problem ($E^{(N)}$) is to find $N+1$ unkmowns q_j, $0 \le j \le N$, and it is uniquely solvable for any fixed $\rho \in (0, a)$.

The CSM approximate form for $b(u,v)$ of the second type, which is denoted by $\overline{b}^{(N)}(u,v)$, is represented through the following formula (7):

$$\overline{b}^{(N)}(u,v) = -\frac{2\pi a}{N}\sum_{j=0}^{N-1}\frac{\partial u^{(N)}(\mathbf{a}_j)}{\partial r}v^{(N)}(\mathbf{a}_j), \tag{7}$$

which is the quadrature formula for $b^{(N)}(u,v)$ with the use of trapezoidal rule.

We use the following notations:

$$b(v) = b(v,v)^{1/2}, \quad b^{(N)}(v) = b(v,v)^{1/2}, \quad \overline{b}^{(N)}(v) = \overline{b}^{(N)}(v,v)^{1/2}.$$

Denote the totality of equi-distant piecewise linear continuous 2π-periodic functions (with N nodal points) $v(\mathbf{a}(\theta))$ by V_N:

$$V_N = \{v(\mathbf{a}(\theta)) = \sum_{j=0}^{N} v(\mathbf{a}_j)w_j^{(N)}(\theta)\}.$$

Let

$$N(\gamma) = \frac{\log 2}{-\log\gamma} \qquad \text{with} \qquad \gamma = \frac{\rho}{a}.$$

Theorem 2 *We have the following inequalities for any $v \in V_N$.*

$$\frac{1}{4\sqrt{1+2\zeta(3)}}b(v) \le b^{(N)}(v) \le \frac{\pi^2}{2}b(v)$$

provided that $N \ge N(\gamma)$, where

$$\zeta(3) = \sum_{r=1}^{\infty}\frac{1}{r^3}.$$

Theorem 3 *For $u, v \in V_N$, we have*

$$|b^{(N)}(u,v) - \overline{b}^{(N)}(u,v)| \le 8\gamma^{2N}b^{(N)}(u)b^{(N)}(v)$$

provided that $N \ge N(\gamma)$.

3. Proof of Theorem 2

For a fixed positive integer N, introduce sets of integers $\mathcal{N}_r$ through

$$\mathcal{N}_r = \{n : -\frac{N}{2} \leq n - Nr < \frac{N}{2}, \quad n \neq Nr\}$$

with

$$r = 0, \pm 1, \pm 2, \cdots.$$

For any integer n $\in [1, N-1]$, define a function $s_n^{(N)}(\gamma)$ of $\gamma \in (0,1)$, numbers $\Lambda_n^{(N)}$ and $\overline{\Lambda}_n^{(N)}$ as follows.

$$s_n^{(N)}(\gamma) = \int_0^\gamma \frac{x^{n-1} + x^{N-n-1}}{1 - x^N} dx,$$

$$\Lambda_n^{(N)} = \frac{s_n^{(N)}(\gamma^2)}{\{s_n^{(N)}(\gamma)\}^2}, \qquad \overline{\Lambda}_n^{(N)} = \frac{\gamma \frac{d}{d\gamma} s_n^{(N)}(\gamma)}{s_n^{(N)}(\gamma)}.$$

We admit the validity of the following Proposition 1 without proof.

Proposition 1 *For $u, v \in V_N$, we have*

$$b^{(N)}(u,v) = 2\pi \sum_{n \in \mathcal{N}_0} \Lambda_{|n|}^{(N)} f_n^{(N)} \overline{g_n^{(N)}}$$

and

$$\overline{b}^{(N)}(u,v) = 2\pi \sum_{n \in \mathcal{N}_0} \overline{\Lambda}_{|n|}^{(N)} f_n^{(N)} \overline{g_n^{(N)}},$$

where $f_n^{(N)}$, and $g_n^{(N)}$, are discrete Fourier coefficients of $u(\mathbf{a}(\theta))$, and $v(\mathbf{a}(\theta))$, respectively.

Using the representation of $\Lambda_n^{(N)}$, we obtain

Proposition 2 *If $N \geq N(\gamma)$, then*

$$\frac{n}{16} \leq \Lambda_n^{(N)} \leq 4n, \qquad 1 \leq n \leq \frac{N}{2}.$$

An elemental calculus leads

Proposition 3 *It holds*

$$\frac{4}{\pi^2} \leq \alpha(\theta) \leq 1, \qquad -\pi \leq \theta \leq \pi.$$

Proposition 4 *For $v \in V_N$, we have*

$$\frac{1}{16}\left\{2\pi \sum_{n \in \mathcal{N}_0} |n||g_n|^2\right\} \leq b^{(N)}(v,v) \leq \frac{\pi^4}{4}\left\{2\pi \sum_{n \in \mathcal{N}_0} |n||g_n|^2\right\}$$

provided that $N \geq N(\gamma)$.

Proof Due to Theorem 1 and Proposition 1, we have

$$b^{(N)}(v,v) = 2\pi \sum_{n\in\mathcal{N}_0} \Lambda_{|n|}^{(N)} \frac{1}{|\alpha(\theta_n)|^2}|g_n|^2.$$

Propositions 2 and 3 imply the conclusion of Proposition 4. □

Proposition 5 *For* $v \in V_N$, *we have*

$$\left\{2\pi \sum_{n\in\mathcal{N}_0} |n||g_n|^2\right\} \leq b(v,v) \leq (1+2\zeta(3))\left\{2\pi \sum_{n\in\mathcal{N}_0} |n||g_n|^2\right\}.$$

Proof Due to Corollary of Theorem 1, we have

$$\frac{1}{2\pi}b(v,v) = \sum_{r\in\boldsymbol{Z}}\sum_{n\in\mathcal{N}_0}\left|\frac{n}{n+Nr}\right|^3 |n||g_n|^2.$$

For $r \in \boldsymbol{Z} - \{0\}$, we have

$$\left|\frac{n}{n+Nr}\right| \leq \frac{1}{|r|}, \qquad n \in \mathcal{N}_0.$$

Therefore

$$b(v,v) \leq \left(1+2\sum_{r=1}^{\infty}\frac{1}{r^3}\right)\left\{2\pi \sum_{n\in\mathcal{N}_0} |n||g_n|^2\right\}.$$

Hence the second inequality of the conclusion is valid, while the first one is trivial by definition of $b(u,v)$. □

Propositions 4 and 5 complete the proof of Theorem 2.

4. Proof of Theorem 3

Proposition 6 *For an integer* $n \in [1, N-1]$, *define* B_n *through the following formula:*

$$B_n = \sum_{p\in\boldsymbol{Z}}\sum_{q\in\boldsymbol{Z}}\left\{\gamma^{|n+Np|}\frac{\gamma^{|n+Nq|}}{|n+Nq|}\right\}.$$

Then we have

$$s_n^{(N)}(\gamma^2) \leq B_n \leq (1+8\gamma^2)s_n^{(N)}(\gamma^2)$$

provided that $N \geq N(\gamma)$.

Proof A lengthy but straightforward calculus leads the conclusion. □

Proposition 7 *For* $N \geq N(\gamma)$, *we have*

$$\Lambda_n^{(N)} \leq \overline{\Lambda}_n^{(N)} \leq (1+8\gamma^2)\Lambda_n^{(N)}.$$

Proof Let

$$\Gamma_n = s_n^{(N)}(\gamma).$$

Then we have

$$\Lambda_n^{(N)} = \frac{s_n^{(N)}(\gamma^2)}{{\Gamma_n}^2},$$

and

$$\overline{\Lambda}_n^{(N)} = \frac{B_n}{{\Gamma_n}^2}.$$

Hence Proposition 6 implies the conclusion. □

The proof of Theorem 3 is now straightforward. In fact, we have

$$b^{(N)}(u,v) - \overline{b}^{(N)}(u,v) = 2\pi \sum_{n\in\mathcal{N}_0} (\Lambda_{|n|}^{(N)} - \overline{\Lambda}_{|n|}^{(N)}) f_n^{(N)} \overline{g_n^{(N)}}.$$

Hence it holds

$$|b^{(N)}(u,v) - \overline{b}^{(N)}(u,v)| \leq 2\pi \left\{ \sum_{n\in\mathcal{N}_0} |\Lambda_{|n|}^{(N)} - \overline{\Lambda}_{|n|}^{(N)}| |f_n^{(N)}|^2 \right\}^{1/2} \times \left\{ \sum_{n\in\mathcal{N}_0} |\Lambda_{|n|}^{(N)} - \overline{\Lambda}_{|n|}^{(N)}| |g_n^{(N)}|^2 \right\}^{1/2}.$$

Let $N \geq N(\gamma)$. Proposition 7 implies

$$0 \leq |\Lambda_{|n|}^{(N)} - \overline{\Lambda}_{|n|}^{(N)}| \leq 8\gamma^2 \Lambda_{|n|}^{(N)}, \qquad n \in \mathcal{N}_0.$$

Therefore we get

$$|b^{(N)}(u,v) - \overline{b}^{(N)}(u,v)| \leq 8\gamma^2 \times \left\{ 2\pi \sum_{n\in\mathcal{N}_0} \Lambda_{|n|}^{(N)} |f_n^{(N)}|^2 \right\}^{1/2} \times \left\{ 2\pi \sum_{n\in\mathcal{N}_0} \Lambda_{|n|}^{(N)} |g_n^{(N)}|^2 \right\}^{1/2}$$

provided that $N \geq N(\gamma)$. Due to Proposition 1 we have the conclusion of Theorem 3.

5. Application to mathematical analysis of an FEM-CSM combined method for exterior Laplace problems

Fix a simply connected bounded domain $\mathcal{O}$ in the plane. Assume that the boundary $\mathcal{C}$ of $\mathcal{O}$ is sufficiently smooth. The exterior domain of $\mathcal{C}$ is denoted by Ω.

Fix a function $f \in L^2(\Omega)$. Assume that the support of f, supp(f), is compact.

Choose a so large that the open disc D_a may contain the union $\mathcal{O} \cup \text{supp}(f)$ in its interior.

As a model problem the following Poisson equation (E) is employed.

$$\text{(E)} \qquad \begin{cases} -\Delta u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \mathcal{C}, \\ \sup_{|\mathbf{r}|>a} |u| &< \infty. \end{cases}$$

The intersection of the domain Ω and the disc D_a is said to be the interior domain, denoted by Ω_i :

$$\Omega_i = \Omega \cap D_a.$$

Consider the Dirichret inner product $a(u,v)$ for $u, v \in H^1(\Omega_i)$:

$$a(u,v) = \int_{\Omega_i} \mathrm{grad}u \ \mathrm{grad}v \ d\Omega.$$

Since the trace $\gamma_a v$ on Γ_a is an element of $H^{1/2}(\Gamma_a)$ for any $v \in H^1(\Omega_i)$, the boundary bilinear form of Steklov type $b(u,v)$ is well defined for $u, v \in H^1(\Omega_i)$. Therefore we can define a continuous symmetric bilinear form :

$$t(u,v) = a(u,v) + b(u,v)$$

for $u, v \in H^1(\Omega_i)$.

Let $F(v)$ be a continuous linear functional on $H^1(\Omega_i)$ defined through the following formula:

$$F(v) = \int_{\Omega_i} fv \ d\Omega.$$

A function space V is defined as follows:

$$V = \left\{ v \in H^1(\Omega_i) : v = 0 \quad \text{on} \quad \mathcal{C} \right\}.$$

Using these notations, the following weak formulation problem (Π) is defined.

$$(\Pi) \qquad \begin{cases} t(u,v) = F(v), & v \in V, \\ u \in V. \end{cases}$$

Admitting the equivalence between the equation (E) and the problem (Π), we consider the problem (Π) and its approximate ones.

Fix a positive number ρ so as to satisfy $0 < \rho < a$. For a fixed positive integer N, set the points $\vec{\rho}_j, \mathbf{a}_j, 0 \le j \le N-1$, as is defined in Section 2.

A family of finite dimensional subspaces of V :

$$\{V_N : N = N_0, N_0 + 1, \cdots\}$$

is supposed to have the following properties:

$$(\mathrm{V_N} - 1) \qquad V_N \subset C(\overline{\Omega_i}).$$

$$(\mathrm{V_N} - 2) \qquad \begin{cases} \textit{For any } v \in V_N,\ v(\mathbf{a}(\theta)) \textit{ is an equi}-\textit{distant piecewise linear} \\ \textit{continuous } 2\pi-\textit{periodic function with respect to } \theta. \end{cases}$$

$$(\mathrm{V_N} - 3) \qquad \min_{v \in V_N} a(v - v_N) \le \frac{C}{N} ||v||_{H^2(\Omega_i)}, \qquad v \in V \cap H^2(\Omega_i).$$

In the property $(\mathrm{V_N}-3)$, C is a constant independent of N and v, and

$$a(v)=a(v,v)^{1/2},\quad v\in V.$$

For $u,v\in H^1(\Omega_i)\cap C(\overline{\Omega_i})$, we define bilinear forms $t^{(N)}(u,v)$ and $\overline{t}^{(N)}(u,v)$ as follows.

$$t^{(N)}(u,v)=a(u,v)+b^{(N)}(u,v),$$

and

$$\overline{t}^{(N)}(u,v)=a(u,v)+\overline{b}^{(N)}(u,v).$$

Now two approximate problems $(\Pi^{(N)})$ and $(\overline{\Pi}^{(N)})$ are stated as follows.

$$(\Pi^{(N)})\qquad \begin{cases} t^{(N)}(u_N,v)=F(v), & v\in V_N,\\ u_N\in V_N. \end{cases}$$

$$(\overline{\Pi}^{(N)})\qquad \begin{cases} \overline{t}^{(N)}(\overline{u}_N,v)=F(v), & v\in V_N,\\ \overline{u}_N\in V_N. \end{cases}$$

With the aide of Theorems 2 and 3 and other necessary discussions, we can show the following error estimate.

Theorem 4 *For a constant C, we have the following estimate.*

$$\left.\begin{array}{l} \|u-u_N\|_{H^1(\Omega_i)} \\[2ex] \|u-\overline{u}_N\|_{H^1(\Omega_i)} \end{array}\right\}\le \frac{C}{N}\|u\|_{H^2(\Omega_i)}.$$

In the above, the constant C is independent of the solution u of (Π) *and N.*

References

[1] Katsurada, M. and Okamoto, H., A mathematical study of the charge simulation method I, J. Fac. Sci. Univ. Tokyo, Sect. IA, Math, Vol. 35, pp. 507-518 (1988).
[2] Ushijima, T., Some remarks on CSM approximate solutions of bounded harmonic function in a domain exterior to a circle, in Japanese, Abstract of 1998 Annual Meeting of Japan Society for Industrial and Applied Mathematics, pp. 60-61 (1998.9.12).
[3] Ushijima, T., An FEM-CSM combined method for 2D exterior Laplace problems, in Japanese, Abstract of Applied Mathematics Branch in Fall Joint Meeting of Mathematical Society of Japan, pp.126-129 (1998.10.3).