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Numerical Conformal Mapping onto the Unit Disk with Concentric Circular Slits by the Charge Simulation Method

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1 Introduction

Conformal mappings are familiar in science and engineering. However exact mapping functions are not known except for some special domains. The numerical conformal mapping has been an attractive subject in numerical analysis [5, 6, 13].

We here present a method of numerical conformal mapping of multiply-connected domains with closed boundary Jordan curves onto the unit disk with concentric circular slits. It is a basic problem of conformal mapping of multiply-connected domains. If the domain is bounded by a single closed Jordan curve, the problem is identified as Riemann's mapping theorem. We reduce the mapping problem to the Dirichlet problem with a pair of conjugate harmonic functions and employ the charge simulation method [7, 8, 10], where the conjugate harmonic functions are approximated by a linear combination of complex logarithmic potentials. We give an explicit form of approximate mapping function which is continuous with the principal value of logarithmic function.

2 Mapping Theorem

Let D be a multiply-connected domain with the closed boundary Jordan curves $C_1, C_2, \ldots, C_n$ in the z-plane. Consider the conformal mappings $w = f_l(z; z_0)$ ($z_0 \in D$; $l = 1, 2, \ldots, n$) of D onto the unit disk with concentric circular slits in the w-plane, where C_l is mapped onto the unit circle. They are uniquely determined by the

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normalization conditions $f_l(z_0; z_0) = 0$ and $f_l'(z_0; z_0) > 0$ [9]. We take $z_0 = 0$ and abbreviate $f_l(z; 0)$ as $f_l(z)$.

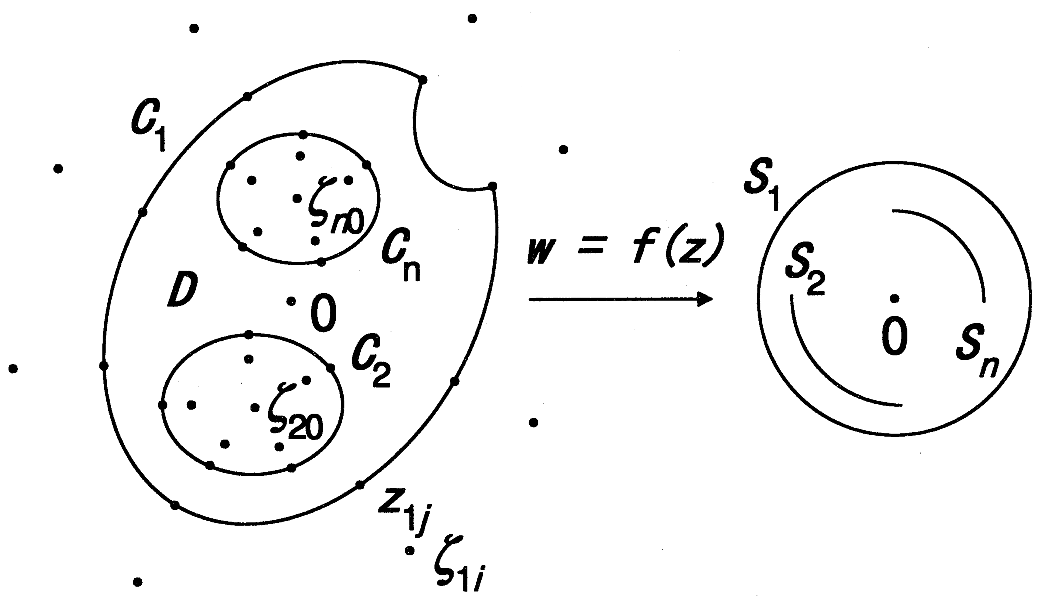

Figure 1: Conformal mapping $w = f_1(z)$ onto the unit disk with concentric circular slits by the charge simulation method.

Problem 1 *Let D be bounded, and C_1 and $C_2, \ldots, C_n$ be the outer and inner boundary curves as shown in Figure 1. Our problem is to construct an approximate mapping function of $w = f_1(z)$, which normalization conditions are $f_1(0) = 0$ and $f_1'(0) > 0$. As a result, C_1 and $C_2, \ldots, C_n$ are mapped onto the unit circle and the concentric circular slits $S_2, \ldots, S_n$ with the radii $r_2, \ldots, r_n$.*

We express the mapping function as

$$f_1(z) = \frac{z}{r_D} \exp(g(z) + \mathrm{i}h(z)) \tag{1}$$

where $g(z)$ and $h(z)$ are conjugate harmonic functions in D, and r_D is a positive constant. The boundary condition $|f_1(z)| = r_l$ $(z \in C_l)$ requires

$$\begin{aligned} g(z) + \log|z| - \log r_D &= \log r_l \quad (z \in C_l;\ l = 1, 2, \ldots, n), && (2) \\ r_1 &= 1, && (3) \end{aligned}$$

and the normalization condition $f_1'(0) = 1/r_D$ requires

$$g(0) + \mathrm{i}h(0) = 0. \tag{4}$$

Conversely, if (2), (3) and (4) are satisfied, (1) is the mapping function of the problem. From the uniqueness of the solution, the problem is now reduced to finding the conjugate harmonic functions $g(z)$ and $h(z)$ together with the radii $R_1, R_2, \ldots, R_n$ and the constant R_D.

The conformal mappings $w = f_l(z)$ $(l = 2, \ldots, n)$ for the same bounded domain D and $w = f_l(z)$ $(l = 1, 2, \ldots, n)$ for the unbounded domain D' exterior to the closed Jordan curves $C_1, C_2, \ldots, C_n$ are reduced to Problem 1 by the pre-mappings

$$z^*(z) = \frac{1}{z - \zeta_{l0}} + \frac{1}{\zeta_{l0}}, \tag{5}$$

where ζ_{l0} is a point inside C_l, $(l = 2, \ldots, n)$ for D and $(l = 1, 2, \ldots, n)$ for D', respectively. The solutions are given by

$$f_l(z) = \exp\left\{ \mathrm{i} \arg\left(\frac{1}{\zeta_{l0}^2}\right)\right\} f_l^*(z^*). \tag{6}$$

We abbreviate $f_1(z)$ as $f(z)$.

3 Numerical Method

We approximate $g(z)$ and $h(z)$ by a linear combination of complex logarithmic potentials and have an approximate mapping function

$$F(z) = \frac{z}{R_D} \exp(G(z) + \mathrm{i}H(z)), \tag{7}$$

$$G(z) + \mathrm{i}H(z) = Q_0 + \sum_{l=1}^{n} \sum_{i=1}^{N_l} Q_{li} \log(z - \zeta_{li}) \tag{8}$$

where N_l charge points $\zeta_{l1}, \zeta_{l2}, \ldots, \zeta_{lN_l}$ are placed outside C_1 or inside C_l $(l = 2, \ldots, n)$. The complex constant Q_0 and the real charges Q_{li} are determined to satisfy the requirement for the outer charges [3],

$$\sum_{i=1}^{N_1} Q_{1i} = -1, \tag{9}$$

the requirement for $H(z)$ to be single-valued in D,

$$\sum_{i=1}^{N_l} Q_{li} = 0 \quad (l = 2, \ldots, n), \tag{10}$$

and the boundary conditions (2) and (3) at the same number of collocation points $z_{l1}, z_{l2}, \ldots, z_{lN_l}$ on C_l $(l = 1, 2, \ldots, n)$, i.e., the linear equations called collocation condition,

$$G(z_{mj}) - \log R_m - \log R_D = -\log|z_{mj}| \tag{11}$$
$$(z_{mj} \in C_m;\ j = 1, 2, \ldots, N_m;\ m = 1, 2, \ldots, n)$$

$$R_1 = 1. \tag{12}$$

If C_1 and $C_2, \ldots, C_n$ are starlike with respect to the origin and $\zeta_{20}, \ldots, \zeta_{n0}$ inside $C_2, \ldots, C_n$, using (10), we can rewrite (8) to

$$\begin{aligned} G(z) + \mathrm{i}H(z) &= Q_0 + \sum_{l=1}^{n}\sum_{i=1}^{N_l} Q_{li}\log(z-\zeta_{li}) - \sum_{l=2}^{n}\sum_{i=1}^{N_l} Q_{li}\log(z-\zeta_{l0}) \\ &= Q_0 + \sum_{i=1}^{N_1} Q_{1i}\left\{\log\left(1-\frac{z}{\zeta_{1i}}\right) + \log(-\zeta_{1i})\right\} \\ &\quad + \sum_{l=2}^{n}\sum_{i=1}^{N_l} Q_{li}\log\left(\frac{z-\zeta_{li}}{z-\zeta_{l0}}\right) \end{aligned} \tag{13}$$

for $H(z)$ to be continuous in D with the principal value of complex logaritmic function. From the normalization condition (4),

$$G(0) + \mathrm{i}H(0) = Q_0 + \sum_{i=1}^{N_1} Q_{1i}\log(-\zeta_{1i}) + \sum_{l=2}^{n}\sum_{i=1}^{N_l} Q_{li}\log\left(\frac{\zeta_{li}}{\zeta_{l0}}\right) = 0. \tag{14}$$

We eliminate Q_0 from (13) and (14), and obtain the following algorithm.

Algorithm 1 *If $C_1, C_2, \ldots, C_n$ are starlike with respect to the origin and $\zeta_{20}, \ldots, \zeta_{n0}$ inside $C_2, \ldots, C_n$, the approximate mapping function is given by*

$$\begin{aligned} F(z) &= \frac{z}{R_D}\exp(G(z)+\mathrm{i}H(z)), \\ G(z) + \mathrm{i}H(z) &= \sum_{i=1}^{N_1} Q_{1i}\log\left(1-\frac{z}{\zeta_{1i}}\right) + \sum_{l=2}^{n}\sum_{i=1}^{N_l} Q_{li}\left\{\log\left(\frac{z-\zeta_{li}}{z-\zeta_{l0}}\right) - \log\left(\frac{\zeta_{li}}{\zeta_{l0}}\right)\right\} \end{aligned}$$

where the charges $Q_{11}, Q_{12}, \ldots, Q_{nN_n}$, the radii $R_1, R_2, \ldots, R_n$ and the constant R_D are solutions of the $N_1 + N_2 + \cdots + N_n + n + 1$ simultaneous linear equations

$$\begin{aligned} \sum_{i=1}^{N_1} Q_{1i}\log\left|1-\frac{z_{mj}}{\zeta_{1i}}\right| + \sum_{l=2}^{n}\sum_{i=1}^{N_l} Q_{li}\left(\log\left|\frac{z_{mj}-\zeta_{li}}{z_{mj}-\zeta_{l0}}\right| - \log\left|\frac{\zeta_{li}}{\zeta_{l0}}\right|\right)& \\ -\log R_m - \log R_D &= -\log|z_{mj}| \\ (z_{mj} \in C_m;\ j = 1, 2, \ldots, N_m;\ m = 1, 2, \ldots, n),& \end{aligned}$$

$$\begin{aligned} R_1 &= 1, \\ \sum_{i=1}^{N_1} Q_{1i} &= -1, \\ \sum_{i=1}^{N_l} Q_{li} &= 0 \quad (l = 2, \ldots, n). \end{aligned}$$

The algorithm gives an approximate mapping function in the case of Riemann's mapping theorem [1, 3] if $n = 1$.

In general cases, using (9) and (10), we should rewrite (8) to

$$\begin{aligned} G(z) + \mathrm{i}H(z) &= Q_0 + \sum_{l=1}^{n} \left\{ Q_{l1} \log(z - \zeta_{l1}) + \sum_{i=2}^{N_l} \left(\sum_{k=1}^{i} Q_{lk} - \sum_{k=1}^{i-1} Q_{lk} \right) \log(z - \zeta_{li}) \right\} \\ &= Q_0 + \sum_{l=1}^{n} \left\{ \sum_{i=1}^{N_l - 1} \left(\sum_{k=1}^{i} Q_{lk} \right) (\log(z - \zeta_{li}) - \log(z - \zeta_{li+1})) \right. \\ &\quad \left. + \left(\sum_{k=1}^{N_l} Q_{lk} \right) \log(z - \zeta_{lN_l}) \right\} \\ &= Q_0 + \sum_{l=1}^{n} \sum_{i=1}^{N_l - 1} \left(\sum_{k=1}^{i} Q_{lk} \right) \log \left(\frac{z - \zeta_{li}}{z - \zeta_{li+1}} \right) - \log(z - \zeta_{1N_1}) \end{aligned} \tag{15}$$

for $H(z)$ to be continuous in D with the principal value of complex logaritmic function. From the normalization condition (4),

$$G(0) + \mathrm{i}H(0) = Q_0 + \sum_{l=1}^{n} \sum_{i=1}^{N_l - 1} \left(\sum_{k=1}^{i} Q_{lk} \right) \log \left(\frac{\zeta_{li}}{\zeta_{li+1}} \right) - \log(-\zeta_{1N_1}) = 0. \tag{16}$$

We eliminate Q_0 from (15) and (16), and obtain the following algorithm.

Algorithm 2 *The approximate mapping function is given by*

$$\begin{aligned} F(z) &= \frac{z}{R_D} \exp(G(z) + \mathrm{i}(H(z)), \\ G(z) + \mathrm{i}H(z) &= \sum_{l=1}^{n} \sum_{i=1}^{N_l - 1} Q_l^i \left\{ \log \left(\frac{z - \zeta_{li}}{z - \zeta_{li+1}} \right) - \log \left(\frac{\zeta_{li}}{\zeta_{li+1}} \right) \right\} - \log \left(1 - \frac{z}{\zeta_{1N_1}} \right) \end{aligned}$$

where the unknown constants, the partial sums of the charges

$$Q_l^i = \sum_{k=1}^{i} Q_{lk} \quad (i = 1, 2, \ldots, N_l - 1;\ l = 1, 2, \ldots, n),$$

the radii $R_1, R_2, \ldots, R_n$ and the constant R_D are solutions of the $N_1 + N_2 + \cdots + N_n + 1$ simulataneous linear equations

$$\begin{aligned} \sum_{l=1}^{n} \sum_{i=1}^{N_l - 1} Q_l^i &\left\{ \log \left| \frac{z_{mj} - \zeta_{li}}{z_{mj} - \zeta_{li+1}} \right| - \log \left| \frac{\zeta_{li}}{\zeta_{li+1}} \right| \right\} \\ &- \log R_m - \log R_D = -\log |z_{mj}| + \log \left| 1 - \frac{z_{mj}}{\zeta_{1N_1}} \right| \\ &(z_{mj} \in C_m;\ j = 1, 2, \ldots, N_m;\ m = 1, 2, \ldots, n), \end{aligned}$$

$$R_1 = 1.$$

The charge point ζ_{1N_1} should be placed for the discontinuity of $\mathrm{Arg}(1-z/\zeta_{1N_1})$ not to intersect D.

From the maximum modulus theorem for analytic functions, the error takes its maximum value somewhere on $C_1, C_2, \ldots, C_n$ and is estimated as

$$E_F(z) = |F(z) - f(z)| \leq \max_{z \in C_1 \cup C_2 \cup \cdots \cup C_n} |F(z) - f(z)| = E_F. \tag{17}$$

4 An Example

We use Algorithm 1 and compute

$$E_{Ml} = \max_{1 \leq j \leq N_l} ||F(z_{lj+1/2})| - R_l|, \quad E_{Rl} = |R_l - R_{ld}| \quad (l = 1, 2, \ldots, n),$$
$$E_{RD} = |R_D - R_{Dd}|$$

for error estimation, where $z_{lj+1/2}$ is the middle point on C_l between z_{lj} and z_{lj+1}, and R_{ld} and R_{Dd} are obtained by doubling the number of simulation charges.

Example 1 *A triply-connected domain,*

$$C_1 : x^2/4^2 + y^2 = 1, \quad C_2 : |z - 1.2| = 0.3, \quad C_3 : |z + 1| = 0.6,$$
$$\zeta_{20} = 1.2, \quad \zeta_{30} = -1.$$

Collocation points and charge points are

$$z_{1j} = z\left(\sqrt{\frac{5}{3}}\mathrm{e}^{\mathrm{i}\theta_j}\right), \quad \zeta_{1j} = z\left(\sqrt{\frac{5}{3}}q^{-1}\mathrm{e}^{\mathrm{i}\theta_j}\right),$$
$$z_{2j} = 0.3\mathrm{e}^{\mathrm{i}\theta_j} + 1.2, \quad \zeta_{2j} = 0.3q\mathrm{e}^{\mathrm{i}\theta_j} + 1.2,$$
$$z_{3j} = 0.6\mathrm{e}^{\mathrm{i}\theta_j} - 1, \quad \zeta_{3j} = 0.6q\mathrm{e}^{\mathrm{i}\theta_j} - 1, \quad \theta_j = \frac{2\pi(j-1)}{N} \quad (j = 1, 2, \ldots, N)$$

using Joukowski's transformation

$$z(t) = \frac{\sqrt{a^2 - b^2}}{2}\left(t + \frac{1}{t}\right) \quad (a = 4, b = 1),$$

where $0 < q < 1$ is a parameter for charge arrangement.

Figure 2 and Table 1 show the results. The values of R_l or R_D are shown until a nonzero digit appears in $|R_l - R_{ld}|$ or $|R_D - R_{Dd}|$, and *cond* is the L_1 condition number of the coefficient matrix to be solved. If $q = 0.8$ for $N = 128$, then the results are $E_{M1} = 9.1 \times 10^{-8}$, $E_{M2} = 4.9 \times 10^{-14}$, $E_{M3} = 6.9 \times 10^{-14}$ and $cond = 3.0 \times 10^8$.

Reichel [11] applied a first kind integral equation method [12] to the same problem and obtained E_{M1} =4.3E-3, R_1 =2.5000001, E_{M2} =5.5E-4, R_2 =1.9555848, E_{M3} =3.8E-3, R_3 =1.744207 for, roughly speaking, $N = 63$, where $r_1 = 2.5$ is the capacity of C_1. The accuracy of the charge simulation method is an order of magnitude higher though the values of Table 1 should be multiplied by 2.5 for comparison.

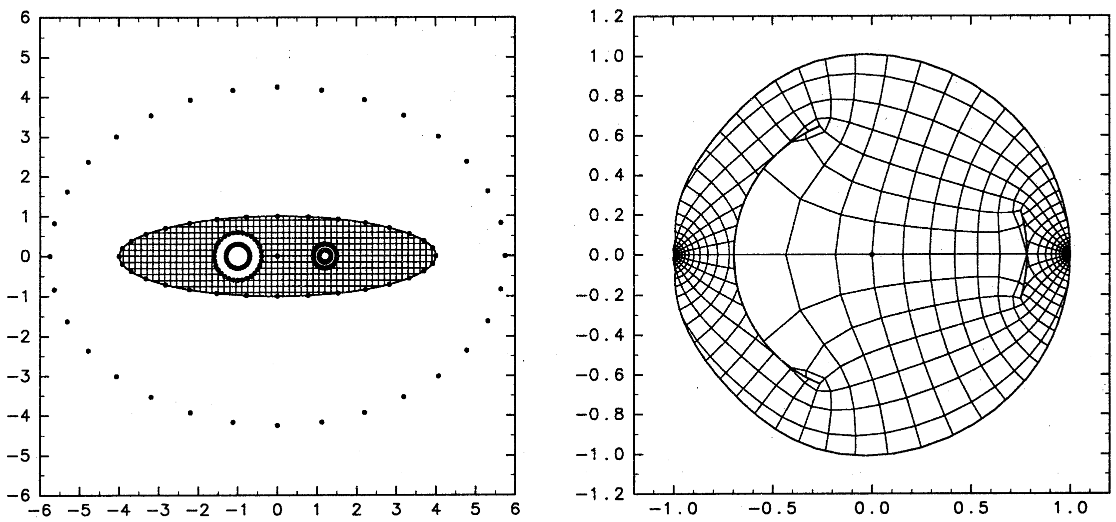

Figure 2: Numerical conformal mapping of the domain bounded by an ellipse and two circles ($N = 32$, $q = 0.5$).

Table 1: Numerical results ($q = 0.5$), where * shows the case of ill-conditioning.

N		E_{Ml}	E_{Rl}	R_l	E_{RD}	R_D	*cond*
	C_1	2.7E-02	0.	1.			
16	C_2	7.8E-06	2.8E-03	0.785	7.5E-03	0.890	5.9E03
	C_3	9.9E-04	7.6E-03	0.705			
	C_1	1.4E-02	0.	1.			
32	C_2	1.3E-10	1.1E-04	0.7821	1.9E-04	0.8980	3.1E06
	C_3	9.9E-06	6.4E-04	0.6970			
	C_1	2.0E-04	0.	1.			
64	C_2	1.1E-10	4.2E-07	0.7822338	1.8E-06	0.897771	4.0E11
	C_3	1.6E-09	2.2E-06	0.697682			
	C_1	6.1E-05	0.	1.			
128	C_2	4.7E-05					3.5E19*
	C_3	3.5E-05					

5 Concluding Remarks

We have presented a method of numerical conformal mapping of multiply-connected domains with closed boundary Jordan curves onto the unit disk with concentric circular slits. The advantages of the method are:

- High accuracy by simple computation for domains with curved boundaries.
- An explicit form of approximate mapping function continuous with the principal value of logarithmic function.

Conventional methods of numerical conformal mapping do not necessarily give an approximate mapping function which is continuous in the problem domain though case-by-case correction is possible.

See Amano [2], and Amano and Sugihara [4] for the numerical conformal mapping of unbounded multiply-connected domains onto parallel, circular and radial slit domains.

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