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Solving Linear Differential Equation through Companion Matrix

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1. Introduction

The Newton equation of motion gives Hamilton equations. The Hamilton equations are equivalently represented as Lagrange equations which yield an Euler-Lagrange equation. In this statement, it is worthwhile to note that there exists a certain equivalence between the homogeneous linear differential equation of a variable of rank n and a linear system of n differential equations $\dot{\mathbf{x}} = A\mathbf{x}$ with a coefficient matrix A of rank n. Another example of this kind is found in the theory of relaxation as the relation of differential general linear equation of a pair of macroscopic conjugate variables to the linear system of differential equations on n pairs of microscopic conjugate variables. The macroscopic variables are usually observable physical quantities, while the microscopic variables difficult to observe, consist of n pairs of conjugate variables corresponding to n different relaxation times. Conjugate variables are, for instance, strain vs. stress, temperature vs. entropy, electric displacement vs. electric field, magnetic flux density vs. magnetic field, chemical potential vs. concentration and so on.

The above relation is summarized to an equivalent relation between a one-variable linear differential equation of rank n and a system $\dot{\mathbf{x}} = A\mathbf{x}$ with $A \in \mathrm{GL}(n;K)$ (K: Field), where $\mathrm{GL}(n;K)$ is the group of all general linear transformations of n-dimensional vector space over K or all nonsingular matrices of order n with K components. Let $f(z)$ be a polynomial of degree n and $d_t \overset{\mathrm{def}}{=} \dfrac{d}{dt}$. For a given $f(d_t)x = 0$, a companion matrix of f gives a linear system $\dot{\mathbf{x}} = A\mathbf{x}$ with $\mathbf{x} = (x_i)$, $x_1 = x$ and $x_i = \dot{x}_{i-1}$ $(i = 2,3,\cdots,n)$. The converse does not always hold. For instance, a symmetric matrix with a 2-folded eigenvalue is not similar to any companion matrix. This paper deals with such converse problem.

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2. Companion Matrix

Let $f(z)$ be a polynomial in a K coefficient polynomial ring $K[z]$ with a field K:

$$f(z)=\sum_{i=0}^{n} a_i z^{n-i} \quad (a_i \in K, a_0=1)\,. \tag{1}$$

Here, K is the real or complex number field. The companion matrix of f is defined as a square matrix A of order n whose characteristic polynomial is f; that is, $\Phi_A(z)\overset{\text{def}}{=}|zE-A| = f(z)$, where $\Phi_A(z)$ is the characteristic polynomial of A, and E the unit matrix. Matrices

$$A_1=\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & 1 \\ -a_n & -a_{n-1} & \cdots & -a_2 & -a_1 \end{pmatrix}, \quad A_2=\begin{pmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & 1 & 0 & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}$$

are often cited as companion matrices of the Frobenius form[1,2]. Hereafter, A_1 is denoted by A_f. Let P be a non-singular matrix, i.e., $P \in \mathrm{GL}(n,K)$. Since $\Phi_{P^{-1}AP}(z)=\Phi_A(z)$, $P^{-1}AP$ with a companion matrix A of f is also a companion matrix of f. The converse does not hold; in fact, for $A=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $B=\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, A and B are companion matrices of $(z-1)^2$, although A is not similar to B. By Hamilton-Cayley's theorem, $\Phi_A(A)=\mathrm{O}$. Then,

Proposition 1

$$\exists P \in \mathrm{GL}(n;K);\ P^{-1}AP=A_f \ \Rightarrow\ f(A)=\mathrm{O}\,.$$

Let $N=\{1,2,\cdots,n\}$ and $\Omega \subset N$ with $|\Omega|=i$ where $|\Omega|$ stands for the order of Ω. A submatrix of A associated with Ω, denoted by A_Ω, is defined as a matrix whose jth rows and columns are deleted from A for all $j \in N-\Omega$.

Theorem 2 If A is a companion matrix of f, then $a_i=(-1)^i \sum_{|\Omega|=i} |A_\Omega|$, where the summation ranges over all Ω with $|\Omega|=i$ and $|A_\Omega|$ exhibits the determinant of A_Ω.

Proof. Since A is a companion matrix of f,

$$|zE-A|=\sum_{i=0}^{n} a_i z^{n-i} \quad (a_0=1)\,. \tag{2}$$

By definition of the determinant, $|zE-A|=\sum_{\sigma\in\mathfrak{S}_n} \operatorname{sgn}\sigma \prod_{j=1}^{n}\left(\delta_{j\sigma(j)}z-a_{j\sigma(j)}\right)$. Here δ_{ij} is Kronecker's delta and $\operatorname{sgn}\sigma$ the signature of a permutation σ in the symmetric group $\mathfrak{S}_n$ of order n. By comparing the coefficients of degree $n-i$ in (2),

$$a_i=\sum_{|\Omega|=i}\ \sum_{\sigma\in\mathfrak{S}_i(\Omega)} \operatorname{sgn}\sigma \prod_{j\in\Omega}\left(-a_{j\sigma(j)}\right)\,.$$

Here, $\mathfrak{S}_i(\Omega)$ is the set of all bijective transformations of Ω, and the first summation is carried out over all subsets of N of order i. Then, $a_i = (-1)^i \sum_{|\Omega|=i} |A_\Omega|$. □

Especially for $i=1$ and n, it follows directly from $\mathrm{Tr}(P^{-1}AP) = \mathrm{Tr}A$ and $|P^{-1}AP| = |A|$ that $a_1 = -\mathrm{Tr}A$ and $a_n = (-1)^n |A|$. The following is readily deduced from Theorem 2.

Corollary 3

$$P^{-1}AP = A_f \ \ (P \in \mathrm{GL}(n;K)) \ \ \Rightarrow \ \ a_i = (-1)^i \sum_{|\Omega|=i} |A_\Omega| \ .$$

3. Homogeneous Linear Differential Equation (HLDE)

Let K be a topological field, $C^\infty(K)$ the set of all infinitely differentiable functions. Substitution of d_t for z in (1) yields a differential operator $f(d_t)$ of $C^\infty(K)$ to $C^\infty(K)$. Let x be a function of $t \ \ (\in K)$ and consider the homogeneous linear differential equation:

$$f(d_t)x = 0 \ . \tag{3}$$

Here, $d_t^{\,0} \overset{\text{def}}{=} I$ with the identity operator I. Equation (3) is written in the form of $\dot{\mathbf{x}} = A_f \mathbf{x}$ with $\mathbf{x} = (x_i)$, $x_1 = x$, $x_2 = \dot{x}$, $\cdots$, $x_n = x^{(n-1)} = d_t^{\,n-1}x$ as mensioned in Introduction.

Now, consider the converse problem to find a representation of (3) equivalent to a given system of linear differential equations $\dot{\mathbf{x}} = A\mathbf{x} \ \ (A \in \mathrm{GL}(n;K))$.

Let $P = (\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_n)$ with column vectors $\mathbf{p}_i$. $AP = PA_f$ gives $A\mathbf{p}_1 = -a_n\mathbf{p}_n$, $A\mathbf{p}_2 = \mathbf{p}_1 - a_{n-1}\mathbf{p}_n$, $\cdots$, $A\mathbf{p}_n = \mathbf{p}_{n-1} - a_1\mathbf{p}_n$. Then, $\mathbf{p}_i = (A^{n-i} + a_1A^{n-i-1} + \cdots + a_{n-i}E)\mathbf{p}_n$. Thus, the following proposition holds.

Proposition 4

$$AP = PA_f \ \ \Rightarrow \ \ P = (\mathbf{p}_i) \ , \ \mathbf{p}_i = (A^{n-i} + a_1A^{n-i-1} + \cdots + a_{n-i}E)\mathbf{p}_n \ .$$

The converse of Prop. 4 holds for $P \in \mathrm{GL}(n;K)$.

Proposition 5

$$P = (\mathbf{p}_i) \ , \ \mathbf{p}_i = (A^{n-i} + a_1A^{n-i-1} + \cdots + a_{n-i}E)\mathbf{p}_n \text{ and } P \in \mathrm{GL}(n;K) \ \ \Rightarrow \ \ AP = PA_f$$

Proof. It suffices to show $A\mathbf{p}_1 = -a_n\mathbf{p}_n$. By Prop. 1, $A^n + a_1A^{n-1} + \cdots + a_{n-1}A = -a_nE$. Since $\mathbf{p}_1 = (A^{n-1} + a_1A^{n-2} + \cdots + a_{n-1}E)\mathbf{p}_n$, $A\mathbf{p}_1 = -a_n\mathbf{p}_n$. □

Let $\mathbf{x}_0$ be a given vector and $\dot{\mathbf{x}} = A\mathbf{x}$ with $\mathbf{x}(0) = \mathbf{x}_0$. Then, $\mathbf{x} = (\exp tA)\mathbf{x}_0$ is the unique solution of the initial value propblem of $\dot{\mathbf{x}} = A\mathbf{x}$ with $\mathbf{x}(0) = \mathbf{x}_0$. In the case of $AP = PA_f$ with $P \in \mathrm{GL}(n;K)$ and by setting $\mathbf{x} = P\mathbf{y}$, $\mathbf{y} = (\exp tA_f)P^{-1}\mathbf{x}_0$ is the solution of $\dot{\mathbf{y}} = A_f\mathbf{y}$, $\mathbf{y}(0) = P^{-1}\mathbf{x}_0$.

4. Jordan Canonical Form

Let J be a matrix of the Jordan canonical form similar to A, i.e.,$\exists U \in \mathrm{GL}(n,K)$; $J=U^{-1}AU$. Suppose $\exists Q \in \mathrm{GL}(n,K)$; $A_fQ=QJ$. Then, $P \overset{\text{def}}{=} UQ \in \mathrm{GL}(n,K)$; $AP=PA_f$. Now, let $P=(\mathbf{p}_1\mathbf{p}_2\cdots\mathbf{p}_n)$ with $\mathbf{p}_j \neq \mathbf{0}$ $(j=1,2,\cdots,n)$ and $\mathbf{p}_j={}^t(p_{1j}p_{2j}\cdots p_{nj})$ satisfying $A_fP=PJ$. Two cases are considered according to diagonal and nondiagonal J.

Case 1 J: Diagonal. Let λ_i $(i=1,2,\cdots,n)$ be eigenvalues of A_f. Since $A\in\mathrm{GL}(n,K)$ or $A_f\in\mathrm{GL}(n,K)$, all λ_i's are nonzero. P is assumed to be a matrix related to A_f as

$$A_fP=P\begin{pmatrix}\lambda_1 & & & \mathrm{O}\\ & \lambda_2 & & \\ & & \ddots & \\ \mathrm{O} & & & \lambda_n\end{pmatrix}.$$

Hence, $A_f\mathbf{p}_j=\lambda_j\mathbf{p}_j$ $(j=1,2,\cdots,n)$. Thus, $p_{i+1j}=\lambda_j p_{ij}=\lambda_j{}^i p_{1j}$ $(i=0,1,\cdots,n-1)$.
Then, $\mathbf{p}_j=p_{1j}{}^t(1\lambda_j\cdots\lambda_j{}^{n-1})\neq\mathbf{0}$. Therefore,

$$|P|=p_{11}p_{12}\cdots p_{1n}\begin{vmatrix}1 & 1 & \cdots & 1\\ \lambda_1 & \lambda_2 & \cdots & \lambda_n\\ \vdots & & & \vdots\\ \lambda_1{}^{n-1} & \lambda_2{}^{n-1} & \cdots & \lambda_n{}^{n-1}\end{vmatrix}=\prod_{j=1}^{n}p_{1j}\cdot\prod_{i>j}(\lambda_i-\lambda_j).$$

Hence it follows that

Proposition 6

(1) $\lambda_i\neq\lambda_j\ (i\neq j)\ \Rightarrow\ |P|\neq 0$. (2) $\exists i,j\ (i\neq j);\lambda_i=\lambda_j\ \Rightarrow\ |P|=0$.

Corollary 7

$$\exists P\in\mathrm{GL}(n;K); AP=PA_f \quad\Leftrightarrow\quad \lambda_i\neq\lambda_j\ (i\neq j).$$

Case 2 J: Nondiagonal, i.e., there exists a Jordan block of J of order larger than 1 (case of 2) in Prop. 8).

$$J=\begin{pmatrix}J_1 & & & \mathrm{O}\\ & J_2 & & \\ & & \ddots & \\ \mathrm{O} & & & J_r\end{pmatrix},\ J_i\text{: } n_i \text{ Jordan block such that }\begin{pmatrix}\lambda_i & 1 & & \mathrm{O}\\ & \lambda_i & \ddots & \\ & & \ddots & 1\\ \mathrm{O} & & & \lambda_i\end{pmatrix}.$$

J is denoted by $\overset{r}{\underset{i=1}{\oplus}}J_i$ or $J_1\oplus J_2\oplus\cdots\oplus J_n$. The characteristic polynomial of J is $f(z)=\prod_{i=1}^{r}(z-\lambda_i)^{n_i}$, $\sum_{i=1}^{r}n_i=n$. Let $A_f^{(i)}$ denote the companion matrix of $(z-\lambda_i)^{n_i}$. P_i such that $A_f^{(i)}P_i=P_iJ_i$ $(i=1,2,\cdots,r)$ resulted in $\left(\overset{r}{\underset{i=1}{\oplus}}A_f^{(i)}\right)\left(\overset{r}{\underset{i=1}{\oplus}}P_i\right)=\left(\overset{r}{\underset{i=1}{\oplus}}P_i\right)\left(\overset{r}{\underset{i=1}{\oplus}}J_i\right)$. Then, it suffices to discuss the case of $r=1$; that is, $J=\begin{pmatrix}\lambda & 1 & & \mathrm{O}\\ & \lambda & \ddots & \\ & & \ddots & 1\\ \mathrm{O} & & & \lambda\end{pmatrix}$.

5. Krylov Sequence of Vectors

Since $A \in \mathrm{GL}(n,K)$ is assumed, $\lambda \neq 0$. With $\hat{A}_f \overset{\text{def}}{=} A_f - \lambda E$, $A_f P = PJ$ yields $\hat{A}_f \mathbf{p}_1 = \mathbf{0}$, $\hat{A}_f \mathbf{p}_{i+1} = \mathbf{p}_i \quad (i = 1,2,\cdots,n-1)$, or $\hat{A}_f P = PE_1$, $E_1 = \begin{pmatrix} 0 & 1 & & \mathrm{O} \\ & 0 & \ddots & \\ & & \ddots & 1 \\ \mathrm{O} & & & 0 \end{pmatrix}$.

Such a sequence of vectors as $\{\mathbf{p}_i\}$ defined by $\mathbf{p}_i = \hat{A}_f \mathbf{p}_{i+1}$ with $\mathbf{p}_n \neq \mathbf{0}$ is called the Krylov sequence associated with $\hat{A}_f$ (Housholder 64). By setting $\mathbf{p}_n = \mathbf{p}$,

$$P = (\mathbf{p}_1 \mathbf{p}_2 \cdots \mathbf{p}_n) = \left(\hat{A}_f \mathbf{p}_2 \; \hat{A}_f \mathbf{p}_3 \cdots \hat{A}_f \mathbf{p}_n \; \mathbf{p}_n\right) = \left(\hat{A}_f^{\,n-1} \mathbf{p} \; \hat{A}_f^{\,n-2} \mathbf{p} \cdots \hat{A}_f \mathbf{p} \; \mathbf{p}\right).$$

Now, consider the determinant of *P*. Since $\mathbf{p}_i = \hat{A}_f^{\,n-i} \mathbf{p} = \left(A_f - \lambda E\right)^{n-i} \mathbf{p}$, $\mathbf{p}_i$ is a linear combination of $A_f^{\,n-i}\mathbf{p}$, $A_f^{\,n-i-1}\mathbf{p}, \cdots, \mathbf{p}$. Hence, $|P| = \left|A_f^{\,n-1}\mathbf{p} \; A_f^{\,n-2}\mathbf{p} \cdots A_f \mathbf{p} \; \mathbf{p}\right|$.

6. Construction of Nonsingular Matrix *P* for Nondiagonal *J*

The problem to solve is to show the existence of $P \in \mathrm{GL}(n;K);\, A_f P = PJ$ and construct such *P*. Let A_{ij} and E_i be defined as

$$A_{ij} \overset{\text{def}}{=} \left(a_{\mu\nu}^{(ij)}\right),\; a_{\mu\nu}^{(ij)} = \begin{cases} -a_{n-(\nu-j)+1} & (\mu = n-i) \\ 0 & (\text{otherwise}) \end{cases},\quad E_i = \left(e_{\mu\nu}^{(i)}\right),\; e_{\mu\nu}^{(i)} = \begin{cases} 1 \; (\nu - \mu = i) \\ 0 \; (\text{otherwise}) \end{cases},$$

which are further explicitly represented as

$$A_{ij} = \begin{array}{c} \\ \\ \\ \\ i\left\{\begin{array}{c} \\ \\ \\ \end{array}\right. \end{array}\!\!\begin{pmatrix} \overbrace{0 \;\cdots\; 0}^{j} & & \mathrm{O} & & \\ \vdots \quad\; \vdots & & & & \\ 0 \;\cdots\; 0 & -a_n & \cdots & -a_{j+1} \\ 0 \;\cdots\; 0 & \cdots & \cdots & 0 \\ \vdots \quad\; \vdots & & & \vdots \\ 0 \;\cdots\; 0 & \cdots & \cdots & 0 \end{pmatrix},\quad E_i = \begin{array}{c} \\ \\ \\ \\ i\left\{\begin{array}{c} \\ \\ \\ \end{array}\right. \end{array}\!\!\begin{pmatrix} \overbrace{0 \;\cdots\; 0}^{i} & 1 & & \mathrm{O} \\ \vdots \quad\; \vdots & & \ddots & \\ 0 \;\cdots\; 0 & \mathrm{O} & & 1 \\ 0 \;\cdots\; 0 & \cdots & \cdots & 0 \\ \vdots \quad\; \vdots & & & \vdots \\ 0 \;\cdots\; 0 & \cdots & \cdots & 0 \end{pmatrix}.$$

Then, $A_f = A_{00} + E_1$.

Proposition 8

(1) $A_{ij} A_{kl} = -a_{j+k+1} A_{il}$, (2) $E_i A_{jk} = A_{i+jk}$, $A_{jk} E_i = A_{jk+i}$,

(3) $E_i E_j = E_{i+j}$, $E_i^{\,m} = E_{mi}$.

Proof. (1)

$$A_{ij} A_{kl} = i\left\{\begin{pmatrix} & & & & \mathrm{O} & & & & \\ \overbrace{0 \;\cdots\; 0}^{j} & -a_n & \underbrace{-a_{n-1} \;\cdots}_{n-k} & -a_{j+k+1} & \cdots & -a_{j+1} \\ & & & & \mathrm{O} & & & & \end{pmatrix}\right. \begin{pmatrix} & & \mathrm{O} & & \\ \overbrace{0 \;\cdots\; 0}^{l} & -a_n & \cdots & -a_{l+1} \\ & & \mathrm{O} & & \end{pmatrix} (n-k$$

$$= -a_{j+k+1}\begin{array}{r c l} & \overbrace{\hphantom{0 \cdots 0}}^{l} & \\ & \begin{pmatrix} 0 & \cdots & 0 & & O & \\ \vdots & & \vdots & & & \\ 0 & \cdots & 0 & -a_n & \cdots & -a_{l+1} \\ \vdots & & \vdots & & & \\ 0 & \cdots & 0 & & O & \end{pmatrix} & \begin{array}{l} \\ \\ (n-i \\ \\ \\ \end{array} \\ i\{ & & \end{array} = -a_{j+k+1}A_{il}\,.$$

(2) For $i+j<n$,

$$E_iA_{jk} = \begin{array}{r c} & \overbrace{\hphantom{0 \cdots 0}}^{i} \\ & \begin{pmatrix} 0 & \cdots & 0 & 1 & & O \\ \vdots & & \vdots & & \ddots & \\ 0 & \cdots & 0 & O & & 1 \\ 0 & \cdots & 0 & \cdots & \cdots & 0 \\ \vdots & & \vdots & & & \vdots \\ 0 & \cdots & 0 & \cdots & \cdots & 0 \end{pmatrix} \\ i\{ & \end{array} \begin{array}{c l} \overbrace{\hphantom{0 \cdots 0}}^{k} & \\ \begin{pmatrix} 0 & \cdots & 0 & & O & \\ \vdots & & \vdots & & & \\ 0 & \cdots & 0 & -a_n & \cdots & -a_{k+1} \\ \vdots & & \vdots & & & \\ 0 & \cdots & 0 & & O & \end{pmatrix} & \begin{array}{l} \\ \\ (n-j \\ \\ \\ \end{array} \end{array}$$

$$= \begin{array}{r c l} & \overbrace{\hphantom{0 \cdots 0}}^{k} & \\ & \begin{pmatrix} 0 & \cdots & 0 & & O & \\ \vdots & & \vdots & & & \\ 0 & \cdots & 0 & -a_n & \cdots & -a_{k+1} \\ \vdots & & \vdots & & & \\ 0 & \cdots & 0 & & O & \end{pmatrix} & \begin{array}{l} \\ \\ (n-i-j \\ \\ \\ \end{array} \\ i+j\{ & & \end{array} = A_{i+jk}$$

Similarly, $A_{jk}E = A_{jk+i}$. (3) Readily proved. $\square$

N.B. (2) holds for $i+j \le n-1$. If $i+j \ge n$, then $E_iA_{jk} = A_{jk}E_i = O$.

By Prop. 8, only A_{ij}'s $(i+j \le m-1)$ appear in $A_f{}^m$. Then, $A_f{}^m$ is written as

$$A_f{}^m = \sum_{i=0}^{m-1}\sum_{j=0}^{m-1-i} c_{ij}^{(m)}A_{ij} + E_m \ (m \ge 1) \text{ and } A_f{}^0 \overset{\text{def}}{=} E_0\,.$$

Proposition 9

(1) $c_{0i+1}^{(m+1)} = c_{0i}^{(m)} \ (m \ge 1)$, (2) $c_{ij}^{(m)} = c_{0i+j}^{(m)} \ (i+j \le m-1)$,

(3) $c_{ij}^{(m)} = c_{00}^{(m-i-j)} \ (i+j \le m-1)$.

Proof. (3) is easily derived by applying (2) and then (1) to $c_{ij}^{(m)}$.

(1) $A_f{}^{m+1} = A_f{}^m(A_{00}+E_1)$. Since only A_{0i} in the right can join A_{0i+1} as $A_{0i+1} = A_{0i}E_1$, $c_{0i+1}^{(m+1)} = c_{0i}^{(m)}$.

(2) $(A_{00}+E_1)^m = \sum_{(e_1e_2\cdots e_m)\in\{0,1\}^m}\left(A_{00}{}^{1-e_1}E_1{}^{e_1}\right)\left(A_{00}{}^{1-e_2}E_1{}^{e_2}\right)\cdots\left(A_{00}{}^{1-e_m}E_1{}^{e_m}\right)$,

where $\{0,1\}^m$ is the product set of $\{0,1\}$. In the right, $c_{ij}^{(m)}$ is related to such terms as $\overbrace{E_1E_1\cdots E_1}^{i}A_{00}\hat{A}A_{00}\overbrace{E_1E_1\cdots E_1}^{j}$ with $\hat{A} = \left(A_{00}{}^{1-e_{i+2}}E_1{}^{e_{i+2}}\right)\left(A_{00}{}^{1-e_{i+3}}E_1{}^{e_{i+3}}\right)\cdots\left(A_{00}{}^{1-e_{m-j-1}}E_1{}^{e_{m-j-1}}\right)$. The value $c_{ij}^{(m)}$ is determined only by $A_{00}\hat{A}A_{00}$ and independent of i and j. Then, the proof is completed. □

From Prop. 9, it suffices to derive $c_{00}^{(m)}$. For simplicity, $c_{00}^{(m)}$ is, hereafter, denoted by $c^{(m)}$.

Proposition 10

(1) $c^{(1)}=1$, (2) $c^{(m)}=\sum_{i=1}^{m-1}(-a_i)c^{(m-i)} \quad (m\geq 2)$.

Proof. (1) $A_f=A_{00}+E_1$. $\therefore\ c^{(1)}=1$. (2) $c^{(m)}A_{00}=A_{00}\sum_{i=0}^{m-2}c_{i0}^{(m-1)}A_{i0}=\sum_{i=0}^{m-2}(-a_{i+1})c_{i0}^{(m-1)}A_{00}$

$=\sum_{i=0}^{m-2}(-a_{i+1})c_{00}^{(m-1-i)}A_{00}=\sum_{i=1}^{m-1}(-a_i)c_{00}^{(m-i)}A_{00}$ □

Let $Q=(q_{ij})=(A_f^{n-1}\mathbf{p}\ A_f^{n-2}\mathbf{p}\cdots A_f\mathbf{p}\ \mathbf{p})$, $\mathbf{p}={}^t(p_1p_2\cdots p_n)$, and

$$\mathbf{a}_j'=\left(\overbrace{0\cdots0}^{j}-a_n-a_{n-1}\cdots-a_{j+1}\right)=(i\text{th row of } A_{n-ij}) .$$

Proposition 11

$$q_{ij}=\begin{cases}p_{n+i-j} & (i\leq j)\\ \sum_{k=1}^{i-j}c^{(k)}\mathbf{a}_{i-j-k}'\mathbf{p} & (i>j)\end{cases} .$$

Proof.

$$A_f^{n-j}=\sum_{k=0}^{n-j-1}\sum_{l=0}^{n-j-1-k}c_{kl}^{(n-j)}A_{kl}+E_{n-j}=\sum_{k=j+1}^{n}\sum_{l=0}^{k-j-1}c_{n-kl}^{(n-j)}A_{n-kl}+E_{n-j} .$$

For $i\leq j$, $q_{ij}=(E_{n-j}\mathbf{p})_i=p_{n+i-j}$. For $i>j$, $(A_{n-kl}\mathbf{p})_i=0\ (k\neq i)$, $c_{n-kl}^{(n-j)}=c^{(k-j-l)}$. Hence,

$q_{ij}=\left(\sum_{l=0}^{i-j-1}c^{(i-j-l)}A_{n-il}\mathbf{p}\right)_i$. Let $k=i-j-l$. Then, $q_{ij}=\sum_{k=1}^{i-j}c^{(k)}\mathbf{a}_{i-j-k}'\mathbf{p}$. □

From Prop. 11 follows

Corollary 12 For $k\leq n-\max\{i,j\}$,

$$q_{i+k\,j+k}=q_{ij} .$$

Definition 13

$$q_{ij}^{(k)}\overset{\text{def}}{=}\begin{cases}q_{ij}^{(k-1)} & (i\leq k)\\ q_{ij}^{(k-1)}-\lambda q_{i-1j}^{(k-1)} & (i>k)\end{cases}\quad \text{with } q_{ik}^{(0)}=q_{ik} ,$$

$$r_{ij}\overset{\text{def}}{=}\sum_{k=j}^{n}\binom{n-j}{k-j}(-\lambda)^{k-j}q_{ik}^{(j)}\quad (i=1,2,\cdots,n) .$$

Proposition 14

(1) $q_{i+lj+l}^{(k)}=q_{ij}^{(k)}\ (i>k)$, (2) $r_{i+lj+l}=r_{ij}$, (3) $r_{ij}=0\ (i>j)$.

Proof. (1) holds for $k=1$ by Cor. 12. $q_{i+lj+l}^{(k)}=\begin{cases}q_{i+lj+l}^{(k-1)} & (i+l\leq k)\\ q_{i+lj+l}^{(k-1)}-\lambda q_{i+l-1j+l}^{(k-1)} & (i+l>k)\end{cases}$.

For $i>k$,

$$q^{(k)}_{i+l\,j+l}=q^{(k-1)}_{i+l\,j+l}-\lambda q^{(k-1)}_{i+l-1\,j+l}=q^{(k-1)}_{ij}-\lambda q^{(k-1)}_{i-1\,j}=q^{(k)}_{ij},$$

where the second equality is asserted by the supposition of mathematical induction on k.

(2) It suffices to show $r_{i+l\,j+l}=r_{i+l-1\,j+l-1}$. For $i+l\le j+l$, i.e., $i\le j$, Def. 13 yields

$$q^{(j+l)}_{i+l\,k}=q^{(i+l-1)}_{i+l\,k}=q^{(i+l-2)}_{i+l\,k}-\lambda q^{(i+l-2)}_{i+l-1\,k}.$$

Hence,

$$r_{i+l\,j+l}=\sum_{k=j+l}^{n}\binom{n-j-l}{k-j-l}(-\lambda)^{k-j-l}\left(q^{(i+l-2)}_{i+l\,k}-\lambda q^{(i+l-2)}_{i+l-1\,k}\right)$$

$$=\sum_{k=j+l}^{n}\binom{n-j-l}{k-j-l}(-\lambda)^{k-j-l}q^{(i+l-2)}_{i+l\,k}+\sum_{k=j+l+1}^{n}\binom{n-j-l}{k-j-l-1}(-\lambda)^{k-j-l}q^{(i+l-2)}_{i+l-1\,k-1}+(-\lambda)^{n-j-l+1}q^{(i+l-2)}_{i+l-1\,n}$$

$$=q^{(i+l-2)}_{i+l\,j+l}+\sum_{k=j+l+1}^{n}\binom{n-j-l+1}{k-j-l}(-\lambda)^{k-j-l}q^{(i+l-2)}_{i+l\,k}+(-\lambda)^{n-j-l+1}q^{(i+l-2)}_{i+l-1\,n}\quad(\because q^{(i+l-2)}_{i+l-1\,k-1}=q^{(i+l-2)}_{i+l\,k}\text{ by (1)})$$

$$=\sum_{k=j+l-1}^{n}\binom{n-(j+l-1)}{k-(j+l-1)}(-\lambda)^{k-(j+l-1)}q^{(i+l-2)}_{i+l-1\,k}=r_{i+l-1\,j+l-1}\quad(\because q^{(j+l-2)}_{i+l-1\,k}=q^{(i+l-1)}_{i+l-1\,k}\text{ by Def. 13}).$$

For $i>j$,

$$r_{i+l\,j+l}=\sum_{k=j+l}^{n}\binom{n-j-l}{k-j-l}(-\lambda)^{k-j-l}q^{(j+l)}_{i+l\,k}=\sum_{k=j+l}^{n}\binom{n-j-l}{k-j-l}(-\lambda)^{k-j-l}\left(q^{(j+l-1)}_{i+l\,k}-\lambda q^{(j+l-1)}_{i+l-1\,k}\right)$$

$$=\sum_{k=j+l}^{n}\binom{n-j-l}{k-j-l}(-\lambda)^{k-j-l}q^{(j+l-1)}_{i+l\,k}+\sum_{k=j+l+1}^{n}\binom{n-j-l}{k-j-l-1}(-\lambda)^{k-j-l}q^{(j+l-1)}_{i+l-1\,k-1}+(-\lambda)^{n-j-l+1}q^{(j+l-1)}_{i+l-1\,n}$$

$$=q^{(j+l-1)}_{i+l\,j+l}+\sum_{k=j+l+1}^{n}\binom{n-j-l+1}{k-j-l}(-\lambda)^{k-j-l}q^{(j+l-1)}_{i+l\,k}+(-\lambda)^{n-j-l+1}q^{(j+l-1)}_{i+l-1\,n}\quad(\because q^{(j+l-1)}_{i+l-1\,k-1}=q^{(j+l-1)}_{i+l\,k})$$

$$=\sum_{k=j+l-1}^{n}\binom{n-(j+l-1)}{k-(j+l-1)}(-\lambda)^{k-(j+l-1)}q^{(j+l-1)}_{i+l-1\,k}=r_{i+l-1\,j+l-1}.$$

(3) The proof is too long and then omitted. □

Definition 15

$$R_j\overset{\text{def}}{=}\begin{pmatrix} r_{11} & \cdots & r_{1j} & q^{(j)}_{1\,j+1} & \cdots & q^{(j)}_{1n}\\ & \ddots & \vdots & \vdots & & \vdots\\ \mathrm{O} & & r_{jj} & q^{(j)}_{j\,j+1} & \cdots & q^{(j)}_{jn}\\ & & & \vdots & & \vdots\\ & \mathrm{O} & & q^{(j)}_{n\,j+1} & \cdots & q^{(j)}_{nn}\end{pmatrix}\qquad (j=1,2,\cdots,n).$$

By Prop. 14,

$$R_n=\begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1n}\\ & \ddots & \ddots & \vdots\\ & & \ddots & r_{12}\\ \mathrm{O} & & & r_{11}\end{pmatrix}.$$

For $j\ge 1$,

$$r_{1j} = \sum_{k=j}^{n} \binom{n-j}{k-j} (-\lambda)^{k-j} q_{1k}^{(j-1)} = \sum_{k=j}^{n} \binom{n-j}{k-j} (-\lambda)^{k-j} q_{1k} = \sum_{k=j}^{n} \binom{n-j}{k-j} (-\lambda)^{k-j} p_{n+1-k} \quad .$$

Since $|P| = |R_i|$ $(i = 1,2,\cdots,n)$, the following theorem is obtained.

Theorem 16

$$|P| = \left\{ \sum_{k=1}^{n} \binom{n-1}{k-1} (-\lambda)^{n-k} p_k \right\}^n \quad .$$

Proof.

$$|P| = \left\{ \sum_{k=1}^{n} \binom{n-1}{k-1} (-\lambda)^{k-1} p_{n+1-k} \right\}^n = \left\{ \sum_{k=0}^{n-1} \binom{n-1}{k} (-\lambda)^{k} p_{n-k} \right\}^n = \left\{ \sum_{k=0}^{n-1} \binom{n-1}{n-k-1} (-\lambda)^{k} p_{n-k} \right\}^n$$

$$= \left\{ \sum_{k=1}^{n} \binom{n-1}{k-1} (-\lambda)^{n-k} p_k \right\}^n$$

□

Corollary 17

(1) $\mathbf{p} = {}^t\left(1\,(1+\lambda)\cdots(1+\lambda)^{n-1}\right) \quad \Rightarrow \quad |P| = 1.$

(2) $\mathbf{p} = {}^t\left(1\lambda\cdots\lambda^{n-1}\right) \quad \Rightarrow \quad |P| = 0.$

Proof. By Theorem 16,

$$|P| = \left\{ \sum_{k=1}^{n} \binom{n-1}{k-1} (-\lambda)^{(n-1)-(k-1)} (1+\lambda)^{k-1} \right\}^n = \left[\left\{ (1+\lambda) - \lambda \right\}^{n-1} \right]^n = 1 \, .$$

Similarly, (2) is shown. □

To construct a nonsingular P satisfying $A_f P = PJ$ or $\hat{A}_f P = PE_1$, it suffices to say $\hat{A}_f{}^n \mathbf{p} = \mathbf{0}$. Let $\mathbf{p} = {}^t\left(1\,(1+\lambda)\cdots(1+\lambda)^{n-1}\right)$, and $\mathbf{q}_i = {}^t\left(q_1^{(i)}\, q_2^{(i)} \cdots q_n^{(i)}\right)$ $(i = 1,2,\cdots n)$ be defined as

$$q_j^{(i)} \stackrel{\text{def}}{=} \begin{cases} 0 & (j \le n-i) \\ \displaystyle\sum_{k=0}^{i+(j-1)-n} \binom{j-1}{k} \lambda^k & (j > n-i) \end{cases} \quad .$$

Then, $\mathbf{q}_i = {}^t\left(0\cdots 0\, q_{n-i+1}^{(i)} \cdots q_n^{(i)}\right)$ and $\mathbf{q}_n = \mathbf{p}$.

Proposition 18

(1) $\hat{A}_f \mathbf{p} = \mathbf{p} - \mathbf{q}_1$,

(2) $\hat{A}_f \mathbf{q}_i = \mathbf{q}_{i+1} - \mathbf{q}_1 \quad (i \le n-1)$; especially for $i = n-1$, $\hat{A}_f \mathbf{q}_{n-1} = \mathbf{p} - \mathbf{q}_1$,

(3) $\hat{A}_f{}^i \mathbf{p} = \mathbf{p} - \mathbf{q}_i \quad (i \le n)$; especially for $i = n$, $\hat{A}_f{}^n \mathbf{p} = \mathbf{0}$.

Proof. (3) follows from (1), (2). In fact,

$$\hat{A}_f{}^2 \mathbf{p} = \hat{A}_f(\mathbf{p} - \mathbf{q}_1) = \hat{A}_f \mathbf{p} - \hat{A}_f \mathbf{q}_1 = (\mathbf{p} - \mathbf{q}_1) - (\mathbf{q}_2 - \mathbf{q}_1) = \mathbf{p} - \mathbf{q}_2 \ .$$

Recursive operation of $\hat{A}_f$ on $\mathbf{p}$ gives $\hat{A}_f^{\,i}\mathbf{p} = \mathbf{p} - \mathbf{q}_i$. For $i = n$, $\hat{A}_f^{\,n}\mathbf{p} = \mathbf{p} - \mathbf{q}_n = \mathbf{0}$.

(1) $f(z) = (z-\lambda)^n = \sum_{i=0}^{n} a_i z^{n-i} \quad a_i = \binom{n}{i}(-\lambda)^i$

$$\hat{A}_f = A_f - \lambda E = \begin{pmatrix} -\lambda & 1 & & & \text{O} \\ & -\lambda & 1 & & \\ & & \ddots & \ddots & \\ \text{O} & & & -\lambda & 1 \\ -a_n & -a_{n-1} & \cdots & -a_2 & -a_1 - \lambda \end{pmatrix} .$$

For $j \le n-1$,

$$\left(\hat{A}_f\mathbf{p}\right)_j = -\lambda(1+\lambda)^{j-1} + (1+\lambda)^j = (1+\lambda)^{j-1} .$$

For $j = n$,

$$\begin{aligned}\left(\hat{A}_f\mathbf{p}\right)_n &= \sum_{j=1}^{n}\left(-a_{n-j+1} - \delta_{jn}\lambda\right)(1+\lambda)^{j-1} \\ &= -\sum_{j=1}^{n}\binom{n}{n-(j-1)}(-\lambda)^{n-(j-1)}(1+\lambda)^{j-1} - \lambda(1+\lambda)^{j-1} \\ &= -\left\{(1+\lambda)-\lambda\right\}^n + (1+\lambda)^n - \lambda(1+\lambda)^{n-1} = (1+\lambda)^{n-1} - 1 .\end{aligned}$$

Thus,

$$\hat{A}_f\mathbf{p} = \mathbf{p} - \mathbf{q}_1 .$$

(2) For $j \le n-i$,

$$\left(\hat{A}_f\mathbf{q}_i\right)_j = -\lambda q_j^{(i)} + q_{j+1}^{(i)} = \begin{cases} 0 & (j \le n-i-1) \\ 1 & (j = n-i) \end{cases} .$$

For $n-i < j \le n-1$,

$$\begin{aligned}\left(\hat{A}_f\mathbf{q}_i\right)_j &= -\lambda\sum_{k=0}^{i+j-n-1}\binom{j-1}{k}\lambda^k + \sum_{k=0}^{i+j-n}\binom{j}{k}\lambda^k = -\sum_{k=1}^{i+j-n}\binom{j-1}{k-1}\lambda^k + \sum_{k=0}^{i+j-n}\binom{j}{k}\lambda^k \\ &= 1 + \sum_{k=1}^{i+j-n}\left\{\binom{j}{k} - \binom{j-1}{k-1}\right\}\lambda^k = 1 + \sum_{k=1}^{i+j-n}\binom{j-1}{k}\lambda^k = \sum_{k=0}^{(i+1)+(j-1)-n}\binom{j-1}{k}\lambda^k = q_j^{(i+1)} .\end{aligned}$$

The following lemma is shown before completing the proof for $j = n$.

Lemma 19

$$\sum_{k=0}^{p}(-1)^k\binom{n}{p-k}\binom{n-p+k}{k} = (-1)^p\sum_{k=0}^{p}(-1)^k\binom{n-k}{p-k}\binom{n}{k} = 0 \quad (p \le n) .$$

Proof. Replacing $p-k$ by k yields the first equality.

$$\binom{n-k}{p-k}\binom{n}{k} = \frac{(n-k)!}{(p-k)!(n-p)!}\cdot\frac{n!}{k!(n-k)!} = \frac{p!}{k!(p-k)!}\cdot\frac{n!}{p!(n-p)!} = \binom{p}{k}\binom{n}{p} ,$$

whence

$$\sum_{k=0}^{p}(-1)^k\binom{n-k}{p-k}\binom{n}{k} = \binom{n}{p}\sum_{k=0}^{p}(-1)^k\binom{p}{k} = \binom{n}{p}(1-1)^p = 0 .$$

□

Return to the proof of the proposition.

For $j=n$,

$$
\begin{aligned}
\left(\hat{A}_f\mathbf{q}_i\right)_n &= \sum_{l=1}^{n}(-a_{n-l+1})q_l^{(i)} - \lambda q_n^{(i)} \\
&= \sum_{l=n-i+1}^{n}\left\{(-1)^{n-l}\binom{n}{n-l+1}\lambda^{n-l+1}\cdot\sum_{k=0}^{i+(l-1)-n}\binom{l-1}{k}\lambda^k\right\} - \lambda\sum_{k=0}^{i-1}\binom{n-1}{k}\lambda^k \\
&= \sum_{l=n-i+1}^{n}\sum_{k=0}^{i+(l-1)-n}(-1)^{n-l}\binom{n}{n-l+1}\binom{l-1}{k}\lambda^{n-l+k+1} - \sum_{k=0}^{i-1}\binom{n-1}{k}\lambda^{k+1} \\
&= \sum_{p=1}^{i}\left\{(-1)^{p+1}\sum_{k=0}^{p-1}(-1)^k\binom{n}{p-k}\binom{n-p+k}{k} - \binom{n-1}{p-1}\right\}\lambda^p \\
&= \sum_{p=1}^{i}\left\{(-1)^{p+1}\sum_{k=0}^{p}(-1)^k\binom{n}{p-k}\binom{n-p+k}{k} + \binom{n}{p} - \binom{n-1}{p-1}\right\}\lambda^p .
\end{aligned}
$$

From

$$\binom{n}{p}=\binom{n-1}{p}+\binom{n-1}{p-1}$$

and Lemma 19 follows

$$\left(\hat{A}_f\mathbf{q}_i\right)_n = \sum_{p=1}^{(i+1)-1}\binom{n-1}{p}\lambda^p = q_n^{(i+1)}-1 .$$

Thus,

$$\hat{A}_f\mathbf{q}_i = \mathbf{q}_{i+1}-\mathbf{q}_1 . \qquad \square$$

It is, accordingly, verified that

Theorem 20

(1) $\forall J$: Nondiagonal Jordan block with eigenvalue λ, $\exists P\in \mathrm{GL}(n;K)$; $A_fP=PJ$.
P is expressed as $P=\left(\hat{A}_f^{\,n-1}\mathbf{p}\ \hat{A}_f^{\,n-2}\mathbf{p}\cdots\hat{A}_f\mathbf{p}\ \mathbf{p}\right)$ with $\mathbf{p}={}^t\left(1(1+\lambda)\cdots(1+\lambda)^{n-1}\right)$.

(2) $\forall J=\bigoplus_{i=1}^{r}J_i$, $\exists P_i(i=1,2,\cdots,r)$;

$$\left(\bigoplus_{i=1}^{r}A_f^{(i)}\right)\left(\bigoplus_{i=1}^{r}P_i\right)=\left(\bigoplus_{i=1}^{r}P_i\right)\left(\bigoplus_{i=1}^{r}J_i\right),\quad P_i=\left(\hat{A}_f^{(i)n_i-1}\mathbf{p}_i\ \hat{A}_f^{(i)n_i-2}\mathbf{p}_i\cdots\hat{A}_f^{(i)}\mathbf{p}_i\ \mathbf{p}_i\right)$$

with $\mathbf{p}_i={}^t\left(1(1+\lambda_i)\cdots(1+\lambda_i)^{n_i-1}\right)$.

7. Solution of HLDE through Companion Matrix

Let $f(z)$ be a polynomial of (1) and $f(z)=\prod_{i=1}^{r}(z-\lambda_i)^{n_i}$, $\lambda_i\neq\lambda_j\ (i\neq j)$, $\sum_{i=1}^{r}n_i=n$.
The general solution of $f(d_t)x=0$ is given by (Iwasaki 2000,Takahashi 96)

$$x=\sum_{i=1}^{r}\sum_{j=0}^{n_i-1}c_{i,j}e_{\lambda_i,j}\quad (c_{i,j}\in K)\ ,\quad c_{ij}=\frac{1}{(n_i-j)!}\left(\frac{\partial}{\partial\lambda_i}\right)^{n_i-j}\frac{1}{\prod_{k=1,k\neq i}^{n_i-j}(\lambda_i-\lambda_k)^{n_k}}\ , \tag{3}$$

where $e_{\lambda_i,j}=\frac{t^j}{j!}e^{\lambda_i t}$ $(j\geq 0)$. The general solution of $\dot{\mathbf{x}}=A\mathbf{x}$ is, then, expressed as $\mathbf{x}=P\mathbf{y}$, $\mathbf{y}={}^t\left(x\ \dot{x}\cdots x^{(n-1)}\right)$. Here, $x^{(i)}$ requires calculation of $d_t^{\,i}e_{\lambda,j}$. The following lemma is readily proved.

Lemma 21

$$d_t e_{\lambda,j}=\begin{cases}\lambda e_\lambda & (j=0)\\ e_{\lambda,j-1}+\lambda e_{\lambda,j} & (j\geq 1)\end{cases}\ .$$

Proposition 22

$$d_t^m e_{\lambda,j}=\sum_{k=\max\{j-m,0\}}^{j}\binom{m}{m-j+k}\lambda^{m-j+k}e_{\lambda,k}\ .$$

Proof. Since $d_t^0 e_{\lambda,j}=e_{\lambda,j}$, the proposition holds for $m=0$. In the case of $0\leq m\leq j$,

$$d_t^m e_{\lambda,j}=d_t\left(d_t^{m-1}e_{\lambda,j}\right)=\sum_{i=0}^{m-1}\binom{m-1}{i}\lambda^i d_t e_{\lambda,j-(m-1)+i}=\sum_{i=0}^{m-1}\binom{m-1}{i}\lambda^i\left(e_{\lambda,j-m+i}+\lambda e_{\lambda,j-(m-1)+i}\right)$$

$$=e_{\lambda,j-m}+\sum_{i=1}^{m-1}\left(\binom{m-1}{i}+\binom{m-1}{i-1}\right)\lambda^i e_{\lambda,j-m+i}+\lambda^m e_{\lambda,j}$$

$$=\sum_{i=0}^{m}\binom{m}{i}\lambda^i e_{\lambda,j-m+i}=\sum_{i=j-m}^{j}\binom{m}{m-j+i}\lambda^{m-j+i}e_{\lambda,i}\ .$$

Similarly to the case of $0\leq m\leq j$, the mathematical induction on m is adapted to $m>j$.

$$d_t^m e_{\lambda,j}=d_t\left(d_t^{m-1}e_{\lambda,j}\right)=\binom{m-1}{m-1-j}\lambda^{m-j}e_{\lambda,0}+\sum_{i=1}^{j}\binom{m-1}{m-1-j+i}\lambda^{m-1-j+i}\left(e_{\lambda,i-1}+\lambda e_{\lambda,i}\right)$$

$$=\left(\binom{m-1}{m-1-j}+\binom{m-1}{m-j}\right)\lambda^{m-j}e_{\lambda,0}+\sum_{i=2}^{j}\left(\binom{m-1}{m-1-j+i}+\binom{m-1}{m-1-j+i-1}\right)\lambda^{m-1-j+i}e_{\lambda,i-1}+\lambda^m e_{\lambda,j}$$

$$=\binom{m}{m-j}\lambda^{m-j}e_{\lambda,0}+\sum_{i=2}^{j}\binom{m}{m-1-j+i}\lambda^{m-1-j+i}e_{\lambda,i-1}+\lambda^m e_{\lambda,j}$$

$$=\binom{m}{m-j}\lambda^{m-j}e_{\lambda,0}+\sum_{i=1}^{j-1}\binom{m}{m-j+i}\lambda^{m-j+i}e_{\lambda,i}+\lambda^m e_{\lambda,j}=\sum_{i=0}^{j}\binom{m}{m-j+i}\lambda^{m-j+i}e_{\lambda,i}\ .\qquad\square$$

By Theorem 20 (2), it is, therefore, sufficient only to apply the above general solution (3) to the case of $f(z)=(z-\lambda_i)^{n_i}$, in order to solve the homogeneous linear differential equation.

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