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Delamination Models : link with Fracture Mechanics*

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Abstract

This paper proposed a delamination model permitting to recover the fracture mechanics approach as well as the interface damage approach. The mathematical properties of this model are studied. It is governed by a functional which is neither smooth nor convex. Existence theorem is proved for a regularized interface model. The convergence of the solutions of the regularized problem to a solution of the nonsmooth initial problem is established. Numerical tests has been performed in the case of a laminate composed as two plates.

1 Introduction

A great amount of structures are now made of laminated composites or of materials bonded together by a thin layer of adhesive. Under some repeated loadings or accidental chocks, some defects may occur in the adhesive layer on the interface. The growth in size and in number of these defects on the interface is called the delamination process. It can induce a loss of stiffness and strength of the composite and it can even lead to the complete debonding of two parts of a structure.

Due to the growing use of composites laminated and of adhesive in the industry, it is more and more important to understand this phenomenon and to have a relevant model in order to predict the evolution of a composite structure under specific loadings.

This is confirmed by the considerable literature dealing with the delamination problem .

Two different approaches are used.

1) The study of the delamination of composites may be carried out by adopting a fracture mechanics approach [1, 2, 3, 4] . In fact when there is separation of two bonded parts of a structure, the growth of the debonded area is equivalent to the propagation of a fracture in an *a priori* known direction.

2) But, when the progressive damage of the interface is taken into account, then the study may be carried out by introducing appropriated interface constitutive laws [5] .

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Formally, these two approaches are quite different. But, from a physical point of view, it seems reasonable to think that they can be related.

This article has three main purposes :

1) to present a delamination model which allows to recover either the classical fracture mechanics theory or, via regularization of the involved functionals, a wide variety of interface constitutive models,

2) to give the corresponding existences theorems,

3) to propose some numerical procedure and to use it in some specifics cases.

A model of delamination is first presented. The functional involved in this model is neither smooth nor convex. Nevertheless, it can be proved that this functional is subdifferentiable. The main mathematical properties of this nonsmooth model are studied and the link with fracture mechanics is shown.

Then a regularization of the non-differentiable functional is introduced,

To remove the difficulty due to the non-differentiability, a regularized problem is introduced, which, in fact, corresponds to a relevant mechanical modelization of the delamination and the corresponding constitutive law is described.

A theorem of existence for a solution of the regularized problem is established.

Then the convergence of solutions of the regularized problems to a solution of the initial non smooth problem is proved.

Since the three dimensional analysis needs strong computational efforts, a plate model is adopted to perform numerical tests. Finite element formulation and a numerical procedure based on a predictor-corrector scheme are developed. Finally, numerical results relative to simple structural problems are treated.

2 The delamination model

The object of this paragraph is the presentation, in the framework of the small displacements and deformations theory, of the delamination model proposed in [10] and based on the Frémond adhesive model [6, 7, 8, 9] .

Delamination means the debonding of two parts of a composite laminate, hence in order to describe the mechanical state of such a solid, a new variable is introduced on the interface between the two parts. In Frémond model, this variable, denoted β represent the adhesion of the two parts on the interface : β is equal to 1 if the adhesion is complete , β is equal to 0 if the debonding is complete, and eventually, β can take values between 0 and 1 and, in this case, it can be interpreted as the proportion of active adhesive link by unit of area on this interface. But in articles on delamination, it is more usual to speak of damage on the interface. In this case the variable is denoted by d or here, by γ . When the two parts are completely bonded together, there is no damage and $\gamma = 0$ and, on the contrary, if there is no more link, there is total damage and $\gamma = 1$. So, in fact, γ can be considered as $\gamma = 1 - \beta$.

The main difference between the two notions is that the damage can only increase in time.

To present this model, only one interface, denoted by S , of a composite laminate, V , is considered and, for sake of simplicity, this interface is assumed to be flat.

The Cartesian coordinate system (x_1, x_2, x_3) is introduced such that the plane spanned by the axes x_1, x_2 contains the interface S. The time is denoted by t. The whole body can be regarded as two superimposed connected plates V_1 and V_2 which can be separated.

The displacement vector $\mathbf{u}(x_1, x_2, x_3, t)$ is defined on $(V_1 \cup V_2) \times \mathbf{R}^+$, and its restriction to V_i is denoted by $\mathbf{u}^{(i)}$ with $i = 1, 2$. The relative displacement vector of two points on the surface S, belonging to the plates V_1 and V_2 and characterized by the same in-plane coordinates, is defined as

$$\mathbf{s}(x_1, x_2, t) = \mathbf{u}^{(2)}(x_1, x_2, 0, t) - \mathbf{u}^{(1)}(x_1, x_2, 0, t) \tag{1}$$

The relative displacement $\mathbf{s}$ can be decomposed in its normal s_n and tangential $\mathbf{s}_t$ components on the surface S, according to the formula $\mathbf{s} = s_n\mathbf{n} + \mathbf{s}_t$, where $s_n = \mathbf{s}.\mathbf{n}$ with $\mathbf{n}$ the unit vector oriented along the outward normal to V_1 on S.

The variable $\gamma(x_1, x_2, t)$ represents the damage state on the set $S \times \mathbf{R}^+$. This variable γ can vary between 0 and 1 .

The relation between the damage variable and $\mathbf{s} = \mathbf{u}^{(2)} - \mathbf{u}^{(1)}$ which measures the gap between the two parts of the composite laminate, must be now specified. The simplest assumption is made. If, on a point of the interface S , there is no more contact, $\mathbf{u}^{(2)} - \mathbf{u}^{(1)} \neq 0$, then the damage is complete and $\gamma = 1$. If the damage is not complete , $0 \leq \gamma < 1$, then there must be contact, $\mathbf{u}^{(2)} - \mathbf{u}^{(1)} = 0$ on this point. Besides, the non interpenetration of the two parts V_1 and V_2 imposes on S that the normal relative displacement s_n must be positive. Hence, the proposed simple interface model satisfy the following conditions on S :

$$(1 - \gamma)\mathbf{s} = \mathbf{0} \quad s_n \geq 0 \quad 0 \leq \gamma \leq 1 \tag{2}$$

The first of the conditions (2) imposes that the relative displacement is zero until the damage is total. The second relation of (2) represents the condition of unilateral contact between the two plates constituting the laminate.

The problem is completed by a consistent evolution law for the damage parameter. Herein, a very simple evolution law is considered, based on the physical evidence that delamination cannot recede. As a consequence, it is assumed that the damage parameter is an increasing function of time:

$$\dot{\gamma} \geq 0 \tag{3}$$

It can be proved that this simple and intuitive evolution law satisfies the Clausius-Duhem inequality on dissipation [10].

In the following it is supposed that the external forces do not dependent on the evolution variable t and inertial terms are neglected, hence a static problem is considered. The sets K and A are introduced as:

$$K = \{(\mathbf{s}, \gamma) : (1 - \gamma)\mathbf{s} = \mathbf{0} \quad s_n \geq 0\} \qquad A = \{\gamma : \gamma_i \leq \gamma \leq 1\} \tag{4}$$

where γ_i represents the initial damage state and is assumed to verify $0 \leq \gamma_i \leq 1$. Consequently $\gamma =$ on the part S_d which is initially delaminated, that is where $\gamma_i = 1$, the damage must remain equal to 1.

The admissible solutions must belong to the set K and to the set A in order to verify the assumed constitutive law and to prevent the decrease of the damage variable.

The problem of the delamination of two plates connected by an interface material modeled by equations (2) and (3), is formulated in variational form in the framework of the small displacements and strains theory.

If ϵ represents the strain tensor in the body V_i, it is defined in this framework as the symmetric part of the gradient operator on $\mathbf{u}^{(i)}$ such that

$$\epsilon(\mathbf{u}^{(i)}) = \frac{1}{2}(grad(\mathbf{u}^{(i)}) + grad(\mathbf{u}^{(i)})^T) \tag{5}$$

And if linear elasticity is assumed then $\sigma^{(i)}$ which represents the stress tensor in the body V_i, is defined by :

$$\sigma^{(i)}(\mathbf{u}) = \mathbf{A}^{(i)}(\epsilon(\mathbf{u}^{(i)})) \tag{6}$$

with $\mathbf{A}^{(i)}$ the fourth order elasticity tensor of the plate V_i . It is a linear symmetric operator depending on the mechanical properties of the solid. The associated elastic energy is the half of the quadratic form associated to the symmetric bilinear form $a^{(i)}$, defined on V_i by :

$$a^{(i)}(\mathbf{u}^{(i)}, \mathbf{v}^{(i)}) = \int_{V_i} \sigma^{(i)}(\mathbf{u}^{(i)})\ \epsilon(\mathbf{v}^{(i)}) d\mathbf{x} \tag{7}$$

The potential energy of the two loaded plates constituting the laminate is defined as

$$\Lambda(\mathbf{u}) = \frac{1}{2} \sum_{i=1}^{2} \left\{ a^{(i)}(\mathbf{u}^{(i)}, \mathbf{u}^{(i)}) - f^{(i)}(\mathbf{u}^{(i)}) \right\} \tag{8}$$

where $f^{(i)}$ is a linear operator which characterizes the potential of the external forces. These forces can be either surfacic forces on the external boundary of the complete structure V or volumic forces in V_i .

The total potential energy π of the system is the sum of the potential energy of the two loaded plates and of the potential energy of the interface:

$$\pi(\mathbf{u}, \gamma) = \Lambda(\mathbf{u}) + D(\mathbf{s}, \gamma) \tag{9}$$

In order to obtain only admissible solutions, I_K and I_A , the indicator functions of the sets K and A, are introduced in the potential energy of the interface :

$$D(\mathbf{s}, \gamma) = \int_S I_K(\mathbf{s}, \gamma) ds - \int_S (1 - \gamma)\omega ds + \int_S I_A(\gamma) ds \tag{10}$$

where ω is the Dupre's energy of adhesion [12] ; this quantity is the energy which must be furnish to debond a unit area of the interface S.

Since the set K is not convex, the functional $D(\mathbf{s}, \gamma)$ which contains the indicator function is neither smooth nor convex. Nevertheless, this functional is subdifferentiable as pointed out in [7] .For sake of self-consistence, the definition of the local subdifferential in x of a functional F defined on a reflexive Banach space X is recalled here [13] [14] :

$$x^* \in \partial F(x) \quad if and only if \quad \forall y \in V(x) \subset X \qquad F(y) \geq \langle x^*, \ y - x\rangle + F(x)$$

where $V(x)$ is a neighborhood of x. If the set $\partial F(x)$ is empty the functional F is not subdifferentiable in x .

It is clear that an indicator function of a set K is subdifferentiable in any point of K : since it takes only two values : 0 and $+\infty$, the subdifferential in a point of K contains at least 0 . Even if the set K is not a convex set., its indicator function I_K is subdifferentiable. Hence the functional $D(\mathbf{s}, \gamma)$ is subdifferentiable too.

Finally, the delamination problem consists in finding a stationary point for the potential π, hence the solution state $(\mathbf{u}^o, \gamma^o)$ is obtained by solving the problem:

$$find(\mathbf{u}^o, \gamma^o) such that 0 \in \partial\pi(\mathbf{u}^o, \gamma^o) \tag{11}$$

where $\partial\pi$ represents the subdifferential of the non-convex functional π.

In order to have a well-posed problem, it is assumed that the displacement is equal to zero on Γ_{0i} $(i = 1, 2)$, which is a subset of strictly positive measure of the external boundary of the open set V_i . Let $\mathbf{H}^1_{0i}$ denote the functions of $(H^1(V_i))^3$ whose traces on Γ_{0i} are $\mathbf{0}$. The space $\mathbf{H}^1_0$ is defined as:

$$\mathbf{H}^1_0 = \left\{\mathbf{u} defined on V_1 \cup V_2 \quad / \quad \mathbf{u}^{(i)} \in \mathbf{H}^1_{0i}\right\}$$

In this framework the quantity $\mathbf{s}$ depends linearly on $\mathbf{u}$, $\mathbf{s} = T\mathbf{u}$, where $T\mathbf{u}$ which represents the difference of the traces of the variables $\mathbf{u}^{(i)}$ on the interface S, is well defined and belongs to $(H^{1/2}(S))^3$.

The following hypotheses are made :

$$\begin{array}{ll} a^{(i)} & is a bounded bilinear form, coercive on \mathbf{H}^1_{0i} \qquad (H1) \\ f^{(i)} & is a bounded linear functional on \mathbf{H}^1_{0i} \qquad (H2) \\ \omega & is a positive constant \qquad (H3) \end{array}$$

The problem (11) can be written in the following variational form:

find $(\mathbf{u}^o, \gamma^o)$ in $\mathbf{H}^1_0 \times L^2(S)$ /

$$\begin{array}{ll} \forall \delta\mathbf{u} \in \mathbf{H}^1_0 & \sum_{i=1}^{2} \left\{a^{(i)}(\mathbf{u}^{(i)o}, \delta\mathbf{u}^{(i)}) - f^{(i)}(\delta\mathbf{u}^{(i)})\right\} + \int_S \mathbf{t}^o \bullet T\delta\mathbf{u} \ ds = 0 \\ \forall \delta\gamma \in L^2(S) & \int_S (\omega - Y^o + q^o)\delta\gamma \ ds = 0 \\ & q^o \in \partial I_A(\gamma^o) \qquad (\mathbf{t}^o, -Y^o) \in \partial I_K(\mathbf{s}^o, \gamma^o) \end{array} \tag{12}$$

The strong formulation of the problem can be written as :

$$\begin{array}{rcll} div(\sigma^{(i)}(\mathbf{u}^{(i)o})) + \mathbf{f}^{(i)}_{int} & = & \mathbf{0} & in V_i \\ \mathbf{u}^{(i)o} & = & \mathbf{0} & on \Gamma_{0i} \end{array} \tag{13}$$

$$
\begin{aligned}
\sigma^{(i)o}.\mathbf{n} &= \mathbf{T}_i \quad on \partial V_i \backslash (S \cup \Gamma_{0i}) && (14)\\
\sigma^{(1)o}.\mathbf{n} &= -\mathbf{t}^o \quad on \partial V_1 \cap S && (15)\\
\sigma^{(2)o}.\mathbf{n} &= \mathbf{t}^o \quad on \partial V_2 \cap S && (16)\\
\omega - Y^o + q^o &= 0 \quad on\ S && (17)\\
q^o &\in \partial I_A(\gamma^o) \quad and \quad (\mathbf{t}^o, -Y^o) \in \partial I_K(\mathbf{s}^o, \gamma^o) && (18)
\end{aligned}
$$

where $\mathbf{f}_{vol}^{(i)}$ is a volumic density of forces and $\mathbf{T}_i$ and $\mathbf{t}^o$ surfacic densities of forces. The external forces applied on the system are $\mathbf{f}_{vol}^{(i)}$ and $\mathbf{T}_i$. But $\mathbf{t}^o$ is an internal force of the system. The quantity $-Y^o$ is a dual variable of the damage, it will be shown that it is the opposite of an energy per unit of area on S.

3 Link with fracture mechanics

The careful examination of the precise properties of the subdifferentiable of I_K and I_A , will now permit to establish a clear link with fracture mechanics.

The set K is not convex but its particular structure as the union of two convex subsets, allows to explicit its subdifferential as the Cartesian product of its subdifferentials with respect to each variable $\mathbf{s}$ and γ. In general, when variables are not independent, the Cartesian product of the subdifferentials with respect to each variable does not furnish the subdifferential. Nevertheless it is possible in some special case as specified in the following Proposition.

Proposition 1: Let Q_1 and Q_2 be two convex sets of $\mathbf{R}^n$ and $\mathbf{R}^m$, respectively. The set Q is the nonconvex set of $\mathbf{R}^{n+m}$ containing (a, b), with $a \in Q_1$ and $b \in Q_2$, and defined by:

$$
Q = \left\{ (x, y) \in \mathbf{R}^n \times \mathbf{R}^m \ / \ \left[\begin{array}{l} (x, y) = (a, y) with y \in Q_2 \\ or \\ (x, y) = (x, b) with x \in Q_1 \end{array} \right. \right\} \tag{19}
$$

Then, the subdifferential of the function I_Q exists in any point of Q and

$$
\partial I_Q(x, y) = \partial_x I_Q(x, y) \times \partial_y I_Q(x, y) \tag{20}
$$

The Proposition 1 can be successfully applied to the particular case of the indicator function of the set K. It allows to say that $\mathbf{t}^o \in \partial_{\mathbf{s}} I_K(\mathbf{s}^o, \gamma^o)$ and $-Y^o \in \partial_\gamma I_K(\mathbf{s}^o, \gamma^o)$. Furthermore, it is easy to describe in detail, for each possible value taken by $(\mathbf{s}^o, \gamma^o)$, the structure of the subdifferential of I_K.

Proposition 2: In each point of the surface S the relationship $(\mathbf{t}, -Y) \in \partial I_K(\mathbf{s}, \gamma)$ leads to:

$$
\begin{array}{llllllll}
\gamma \neq 1 & \mathbf{s} = \mathbf{0} & & \Longrightarrow & t_n \in \mathbf{R} & \mathbf{t}_t \in \mathbf{R}^2 & Y = 0 \\
\gamma = 1 & s_n > 0 & & \Longrightarrow & t_n = 0 & \mathbf{t}_t = \mathbf{0} & Y \in \mathbf{R} \\
\gamma = 1 & s_n = 0 & \mathbf{s}_t \neq \mathbf{0} & \Longrightarrow & t_n \in \mathbf{R}^- & \mathbf{t}_t = \mathbf{0} & Y \in \mathbf{R} \\
\gamma = 1 & s_n = 0 & \mathbf{s}_t = \mathbf{0} & \Longrightarrow & t_n \in \mathbf{R}^- & \mathbf{t}_t = \mathbf{0} & Y = 0
\end{array} \tag{21}
$$

where the vector $\mathbf{t}$ is decomposed in its normal t_n and tangential $\mathbf{t}_t$ components on the surface S, such that $\mathbf{t} = t_n\mathbf{n} + \mathbf{t}_t$.

From a mechanical point of view, $\mathbf{t}$ represents the density of force per unit of area applied by the plate V_1 on V_2. The three last relations of (21) imply that when the delamination is complete, the stress on interface S is only due to a frictionless unilateral contact.

The following proposition is obtained by using the properties of $q^o \in \partial I_A(\gamma^o)$ solution of the problem (12) and the properties of $\partial I_K(\mathbf{s}^o, \gamma^o)$ and $\partial I_A(\gamma^o)$.

Proposition 3: In each point of the surface S only the following three cases are possible for the solution of the problem (12):

$$\begin{array}{ll} \gamma^o = \gamma_i & q^o = Y^o - \omega \leq 0 \\ \gamma_i < \gamma^o < 1 & q^o = Y^o - \omega = 0 \\ \gamma^o = 1 & q^o = Y^o - \omega \geq 0 \end{array} \tag{22}$$

The relations (22) are direct consequences of the second and the third equations of (12).

This proposition makes clear the relation of this model with Fracture Mechanics. The Griffith criterion says that a fracture can increase only when the rate of variation of the elastic energy of the solid with respect to the eventual variation of the surface of the fracture is greater or equal to a limit value called the limit energy release rate. Here Proposition 3 shows that there is no delamination when $Y^o - \omega < 0$ and there is complete delamination ($\gamma^o = 1$) when $Y^o - \omega > 0$. In fact, as it has been already shown [10], Y^o is the local value, in the point (x_1, x_2) of the surface S, of the energy release rate. The delamination occurs only when Y^o is greater than ω, which correspond to the Griffith limit energy release rate. Hence, delamination depends essentially on the sign of the quantity $Y^o - \omega$.

Furthermore, the Propositions 2 and 3 can be used to put in evidence some special features of the solution of equations (12).

Proposition 4: If $(\mathbf{u}^o, \gamma^o)$ is a solution of problem (12) then either $\gamma^o = \gamma_i$ or $\gamma^o = 1$. Furthermore, in a point of S where $\gamma^o = 1$, the relative displacement $\mathbf{s}^o$ is not equal to zero.

If $\gamma^o \neq 1$, from the Proposition 2 it results $Y^o = 0$, and so the second equation of (12) gives $q^o = Y^o - \omega \leq 0$. Then the Proposition 3 implies $\gamma^o = \gamma_i$. On the contrary, if $\gamma^o = 1$ then $q^o = Y^o - \omega \geq 0$ and so $Y^o > 0$, hence the Proposition 2 ensures $\mathbf{s}^o \neq \mathbf{0}$.

This shows that the solution of problem (12) has an on-off character similar to the problem of Fracture Mechanics: in each point of S , the two plates are either in complete adhesion or completely delaminated. It must be emphasized that herein the energy release rate Y is not introduced *a priori*, as in the Fracture Mechanics Theory, but $-Y$ is obtained as an element of the subdifferential of the indicator function I_K with respect to the damage variable γ (the sign is chosen so that $-Y$ represents the density of energy per unit area lost by the system when the delamination occurs).

4 The regularized model

The nondifferentiability of the functional π may represent a strong difficulty both from a mathematical and computational point of view in the treatment of the delamination problem. Hence, a regularization of the total potential energy appears suitable. The regularization is obtained by replacing, into the original functional π, the functions I_K and I_A with new differentiable ones, which tend to the original irregular functions when some regularization parameters approach to zero.

Let the scalars $\eta_n > 0$ and $\eta_t > 0$ be regularization parameters, the function I_K is replaced with:

$$I_K^r(\mathbf{s},\gamma) = \frac{1}{2\eta_n}\left[(1-\gamma)(s_n^+)^2 + (s_n^-)^2\right] + \frac{1}{2\eta_t}(1-\gamma)\left\|\mathbf{s}_t\right\|^2 \tag{23}$$

where $(.)^+$ and $(.)^-$ indicate the positive and negative parts of $(.)$, respectively, such that $(.) = (.)^+ - (.)^-$, and $\|\|$ denotes the norm of a vector.

An internal regularization of the function I_A is used. It is based on an homographic regularization of its subdifferential ∂I_A. Hence, the indicator function I_A is replaced on S with:

$$I_A^r(\gamma) = r\frac{1-\gamma_i}{2}\int_0^{\nu(\gamma)} f(y)dy \tag{24}$$

where the scalar $r > 0$ is a regularization parameter and

$$f(\nu) = \frac{\nu}{1-|\nu|} \tag{25}$$

with

$$\nu(\gamma) = \frac{2}{1-\gamma_i}(\gamma - \frac{1+\gamma_i}{2}) \tag{26}$$

The derivative of the function I_A^r is:

$$\forall\gamma \in]\gamma_i, 1[\qquad \frac{dI_A^r}{d\gamma}(\gamma) = r\ f(\nu(\gamma)) \tag{27}$$

It is a simple matter to verify that the graph of I_A^r tends to the graph of I_A , and the derivative of I_A^r tends to an element of the subdifferential of I_A for $r \to 0$.

Finally, the new regularized total potential energy π^r is defined as:

$$\pi^r(\mathbf{u},\gamma) = \Lambda(\mathbf{u}) + D^r(T\mathbf{u},\gamma) \tag{28}$$

where the regularized adhesion energy D^r can be written as:

$$\begin{aligned} D^r(T\mathbf{u},\gamma) = D^r(\mathbf{s},\gamma) &= \textstyle\int_S \left\{\frac{1}{2\eta_n}[(1-\gamma)(s_n^+)^2 + (s_n^-)^2] + \frac{1}{2\eta_t}(1-\gamma)\left\|\mathbf{s}_t\right\|^2\right\} ds \\ &+ \textstyle\int_S I_A^r(\gamma)ds - \int_S (1-\gamma)\omega ds \end{aligned} \tag{29}$$

The partial derivatives of $I_K^r(\mathbf{s},\gamma)$ with respect to $\mathbf{s}$ and to γ, are:

$$\frac{\partial I_K^r}{\partial \mathbf{s}}(\mathbf{s},\gamma) = \frac{1}{\eta_n}\left[(1-\gamma)s_n^+ + s_n^-\right]\mathbf{n} + \frac{1}{\eta_t}(1-\gamma)\mathbf{s}_t \tag{30}$$

$$\frac{\partial I_K^r}{\partial \gamma}(\mathbf{s}) = -\frac{1}{2\eta_n}(s_n^+)^2 - \frac{1}{2\eta_t}\|\mathbf{s}_t\|^2 \tag{31}$$

The corresponding regularized problem consists to
find $(\mathbf{u}^r, \gamma^r)$ in $\mathbf{H}_0^1 \times L^2(S)$ with $\gamma^r = 1$ on S_d such that :

$$\forall \mathbf{v} \in \mathbf{H}_0^1 \qquad \sum_{i=1}^{2}\left\{a^{(i)}(\mathbf{u}^{(i)r}, \mathbf{v}^{(i)}) - f^{(i)}(\mathbf{v}^{(i)})\right\} + \int_S \mathbf{t}^r \bullet T\mathbf{v}\, ds = 0$$

$$\forall \delta\gamma \in L^2(S_a) \qquad \int_S (\omega - Y^r + q^r)\delta\gamma\, ds = 0 \tag{32}$$

$$q^r = \frac{dI_A^r}{d\gamma}(\gamma^r) \qquad \mathbf{t}^r = \frac{\partial I_K^r}{\partial \mathbf{s}}(\mathbf{s}^r, \gamma^r) \qquad -Y^r = \frac{\partial I_K^r}{\partial \gamma}(\mathbf{s}^r)$$

It can be noted that equations (32) are similar to equations (12) with q^o, $\mathbf{t}^o$ and Y^o replaced by q^r, $\mathbf{t}^r$ and Y^r, respectively.

The local form of the second equation of (32), the continuity and the strict monotonicity of the composed function $f \circ \nu$ from $]\gamma_i, 1[$ to $\mathbf{R}$, allow to write on the part S_a of S where $\gamma_i \neq 1$:

$$\gamma^r = \nu^{-1} \circ f^{-1}\left(\frac{Y^r - \omega}{r}\right) \tag{33}$$

The damage parameter is an explicit function of the energy release rate Y^r and hence, using equation (31) and the last one of (32), γ^r can be written as function of the relative displacement $\mathbf{s}^r$:

$$\gamma^r = \hat{\gamma}(\mathbf{s}^r) = \nu^{-1} \circ f^{-1}\left[\frac{1}{r}\left(\frac{1}{2\eta_n}(s_n^{r+})^2 + \frac{1}{2\eta_t}\|\mathbf{s}_t^r\|^2 - \omega\right)\right] \tag{34}$$

5 Mechanical meaning of the regularized model

According to equation (30) and the last of (32) the interface is modelled by an distribution of nonlinear springs oriented in the normal and tangential directions to the surface S . The stiffness of the springs depends on the damage of the interface and of the direction. It must be equal to zero if the delamination is complete except in compression in the normal direction since, in this case, the corresponding term is only due to the penalization of the unilateral contact condition, namely $s_n^- = 0$. In the tangential direction, the stiffness of the springs is $\frac{1}{\eta_t}(1-\gamma)$. In traction in the normal direction, the stiffness is $\frac{1}{\eta_n}(1-\gamma)$. If $(1-\gamma)$ represent the density of unbroken links between the two part of V, it means that the stiffness is proportional to this quantity which seems quite reasonable.

The equation (31) shows that the quantity Y^r is a positive quantity, it represents in fact the density of the adhesion energy on the interface S .

The equation (34) gives the explicit relationship between the displacement on S and the damage. In many articles dealing with delamination and using a special constitutive law on the interface, the relation is rather stated between the damage and the stress on the interface, but of course the stress depends on the displacement so the two points of view are related.

Of course other regularizations may be proposed and they will leads to different interface constitutive laws

6 Existence of solution of the regularized problem

A constructive method is used to prove the existence of the solution of the regularized problem. The proof is carried out in two steps. Initially, the existence of the solution for the elastostatic problem corresponding to a fixed value of the damage function is given and the boundness of this solution is proved. Then an iterative procedure is introduced and the convergence of a subsequence to a solution of the regularized problem is proved.

Proposition 5: For any γ such that $0 \leq \gamma \leq 1$ almost everywhere on S, there exists a unique $\mathbf{u}^\gamma$ solution of the following variational problem:

find $\mathbf{u}^\gamma \in \mathbf{H}_0^1$ /

$$\begin{aligned} \forall \mathbf{v} \in \mathbf{H}_0^1 \qquad 0 \;=\; & \sum_{i=1}^{2} \left\{ a^{(i)}(\mathbf{u}^{(i)\gamma}, \mathbf{v}^{(i)}) - f^{(i)}(\mathbf{v}^{(i)}) \right\} \\ + \; & \int_S \frac{1}{\eta_n} \left[(1-\gamma) s_n^{\gamma+} + s_n^{\gamma-} \right] \mathbf{n} \bullet T\mathbf{v} \, ds \\ + \; & \int_S \frac{1}{\eta_t} (1-\gamma) \mathbf{s}_t^\gamma \bullet T\mathbf{v} \, ds \end{aligned} \tag{35}$$

where $\mathbf{s}^\gamma = T\mathbf{u}^\gamma$. Moreover this solution verifies the following estimates:

$$\left\| \mathbf{u}^{(i)\gamma} \right\|_{(H^1(V_i))^3} \leq C \qquad (i = 1, 2) \tag{36}$$

$$\int_S \left\{ \frac{1}{\eta_n} \left[(1-\gamma)(s_n^{\gamma+})^2 + (s_n^{\gamma-})^2 \right] + \frac{1}{\eta_t} (1-\gamma) \left\| \mathbf{s}_t^\gamma \right\|^2 \right\} ds \leq C \tag{37}$$

where the constant C is independent of the parameters η_n, η_t and of the function γ.
The resolution of the equation (35) is equivalent to the research of the stationary points for the functional $J^\gamma(\mathbf{u}) = \pi^r(\mathbf{u}, \gamma) = \Lambda(\mathbf{u}) + D^r(T\mathbf{u}, \gamma)$. It can be emphasized that $\Lambda(\mathbf{u})$ is a continuous strictly convex functional on $\mathbf{H}_0^1$ and $D^r(T\mathbf{u}, \gamma)$ is a convex functional of $\mathbf{s} = T\mathbf{u}$ for fixed γ with $0 \leq \gamma \leq 1$; since $\mathbf{s}$ depends linearly on $\mathbf{u}$, then $D^r(T\mathbf{u}, \gamma)$ is also convex with respect to $\mathbf{u}$. Hence, $J^\gamma(\mathbf{u})$ is strictly convex on $\mathbf{H}_0^1$ and then it has a unique stationary point which is the minimum on $\mathbf{H}_0^1$.
Setting $\mathbf{v}^{(i)} = \mathbf{u}^{(i)\gamma}$ $(i = 1, 2)$ the equation (35) leads to:

$$\begin{aligned} & \sum_{i=1}^{2} \left\{ a^{(i)}(\mathbf{u}^{(i)\gamma}, \mathbf{u}^{(i)\gamma}) - f^{(i)}(\mathbf{u}^{(i)\gamma}) \right\} + \\ & \int_S \left\{ \frac{1}{\eta_n} \left[(1-\gamma)(s_n^{\gamma+})^2 + (s_n^{\gamma-})^2 \right] + \frac{1}{\eta_t} (1-\gamma) \left\| \mathbf{s}_t^\gamma \right\|^2 \right\} ds = 0 \end{aligned} \tag{38}$$

Since the integral on S in the left hand side of equation (38) is always nonnegative for $0 \leq \gamma \leq 1$, the estimates (36) and (37) can be deduced from the coercivity of $a^{(i)}$ on $\mathbf{H}_{0i}^1$ and from the linearity of $f^{(i)}$.

A sequence $\left\{ (\mathbf{u}^{[n]}, \gamma^{[n]}) \right\}$ is defined in the following way:

1. let $\gamma^{[0]} = \gamma_i$;
2. for a given $\gamma = \gamma^{[n-1]}$, let $\mathbf{u}^{[n]}$ be the unique solution of equation (35);
3. let $\gamma^{[n]} = 1$ on S_d and $\gamma^{[n]} = \widehat{\gamma}(\mathbf{s}^{[n]})$ on S_a with $\widehat{\gamma}$ given by equation (34);
4. with the new value $\gamma = \gamma^{[n]}$ go back to step 2.

The sequence $\left\{(\mathbf{u}^{[n]}, \gamma^{[n]})\right\}$, obtained using the described procedure, verifies the following theorem:

Theorem 6: There exists a subsequence of $\left\{(\mathbf{u}^{[n]}, \gamma^{[n]})\right\}$ still denoted by $\left\{(\mathbf{u}^{[n]}, \gamma^{[n]})\right\}$ such that
- $\left\{\mathbf{u}^{[n]}\right\}$ converges weakly to $\mathbf{u}^\infty$ in $\mathbf{H}_0^1$
- $\left\{\mathbf{s}^{[n]}\right\}$ converges to $\mathbf{s}^\infty = T\mathbf{u}^\infty$ strongly in $(L^2(S))^3$ and almost everywhere on S
- $\left\{\gamma^{[n]}\right\}$ converges almost everywhere to γ^∞ with $\gamma_i \leq \gamma^\infty \leq 1$ and γ^∞ is such that $\gamma^\infty = 1$ on S_d and $\gamma^\infty = \widehat{\gamma}(\mathbf{s}^\infty)$ on S_a.

The limit $(\mathbf{u}^\infty, \gamma^\infty)$ is solution of the regularized problem (32), hence $\gamma^\infty = \gamma^r$ and $\mathbf{u}^\infty = \mathbf{u}^r$. Furthermore, $(\mathbf{u}^r, \gamma^r)$ verifies the estimates (36) and (37) for $\gamma = \gamma^r$ and $\mathbf{u}^\gamma = \mathbf{u}^r$.

The Proposition 5 implies that the sequences $\left\{\mathbf{u}^{[n](i)}\right\}$ are bounded in $(H^1(V_i))^3$. Since bounded sequences in Hilbert spaces are weakly sequentially compact, there exists a subsequence $\left\{\mathbf{u}^{[n](i)}\right\}$ that converges to $\mathbf{u}^{\infty(i)}$ weakly in $(H^1(V_i))^3$ and the corresponding traces on S are bounded in $(H^{1/2}(S))^3$. The classical Rellich-Kondrachov theorem [15] is used to obtain the strong convergence of $\left\{\mathbf{s}^{[n]}\right\}$. Since the dimension of S is 2, the injection of the space $(H^{1/2}(S))^3$ is continuous in any $(L^p(S))^3$ for $p \leq 4$, and compact for $p < 4$. Thus, there exists a subsequence of $\left\{\mathbf{s}^{[n]}\right\}$ that converges strongly in $(L^2(S))^3$ and even a subsubsequence, still denoted $\left\{\mathbf{s}^{[n]}\right\}$ which converges almost everywhere on S to a limit $\mathbf{s}^\infty$.

On the other hand, the continuity of $\widehat{\gamma}$ implies that $\left\{\gamma^{[n]}\right\}$ converges almost everywhere to $\gamma^\infty = \widehat{\gamma}(\mathbf{s}^\infty)$ on S_a.

In order to prove that $(\mathbf{u}^\infty, \gamma^\infty)$ is solution of the regularized problem (32), and hence $\gamma^\infty = \gamma^r$ and $\mathbf{u}^\infty = \mathbf{u}^r$, it must be verified that $\mathbf{u}^\infty$ is the solution of equation (35) with $\gamma = \gamma^\infty$. By construction $\mathbf{u}^{[n]}$ is the solution of equation (35) with $\gamma = \gamma^{[n-1]}$, and $\mathbf{s}^{[n]} = T\mathbf{u}^{[n]}$. Since $\widehat{\gamma}$ is an application on $]\gamma_i, 1[$, the sequence $\left\{\gamma^{[n]}\right\}$ remains bounded and converges to γ^∞ in $L^\infty(S)$. Finally, the weak convergence of $\left\{\mathbf{u}^{[n]}\right\}$ to $\mathbf{u}^\infty$ in $\mathbf{H}_0^1$, the strong convergence of the sequence $\left\{\mathbf{s}^{[n]}\right\}$ to $\mathbf{s}^\infty$ in $(L^2(S))^3$ and the convergence of $\left\{\gamma^{[n]}\right\}$ in $L^\infty(S)$ weak* allow to pass to the limit in the equation (35). Hence $(\mathbf{u}^\infty, \gamma^\infty)$ verifies (35) and the estimates (36) and (37).

7 Convergence of the regularized solution

It is expected that when the regularization parameters r, η_n and η_t tend to 0, the sequence of solutions $\{(\mathbf{u}^r, \gamma^r)\}$ of the regularized problems (32) tends to a solution $(\mathbf{u}^o, \gamma^o)$ of the initial nonsmooth problem (12). In order to prove this convergence the weak compactness of bounded sequences in Hilbert spaces is used again.

Theorem 7: Let $(\mathbf{u}^r, \gamma^r)$ be a solution of problem (32). Under the hypotheses $(H1)-(H3)$, assuming that $\eta_n > 0$ and $\eta_t > 0$ tend to 0 for r tending to 0, there exists a subsequence of $\{(\mathbf{u}^r, \gamma^r)\}$ still denoted by $\{(\mathbf{u}^r, \gamma^r)\}$ such that

- $\{\mathbf{u}^r\}$ converges weakly to $\mathbf{u}^o$ in $\mathbf{H}_0^1$
- $\{\mathbf{s}^r\}$ converges to $\mathbf{s}^o = T\mathbf{u}^o$ strongly in $(L^2(S))^3$ and almost everywhere on S
- $\{\gamma^r\}$ converges to γ^o in $L^\infty(S)$ weak* with $\gamma_i \leq \gamma^o \leq 1$ and $\gamma^o = 1$ on S_d
- $\{\mathbf{t}^r\}$ converges to $\mathbf{t}^o$ weakly in the dual space of $(H^{1/2}(S))^3$.

These limits verify:

$$\forall \mathbf{v} \in \mathbf{H}_0^1 \qquad \sum_{i=1}^{2} \left\{ a^{(i)}(\mathbf{u}^{(i)o}, \mathbf{v}^{(i)}) - f^{(i)}(\mathbf{v}^{(i)}) \right\} + \int_S \mathbf{t}^o \bullet T\mathbf{v} \; ds = 0 \tag{39}$$

$$(1-\gamma^o)\mathbf{s}^o = \mathbf{0} \quad \gamma_i \leq \gamma^o \leq 1 \quad s_n^o \geq 0 \tag{40}$$

Using the estimate (36), and following the same method as in Theorem 6 it can be proved that there exists a subsequence of $\{(\mathbf{u}^r, \gamma^r)\}$ such that $\{\mathbf{u}^r\}$ converges weakly to $\mathbf{u}^o$ in $\mathbf{H}_0^1$ and $\{\mathbf{s}^r\}$ converges to $\mathbf{s}^o = T\mathbf{u}^o$ strongly in $(L^2(S))^3$ and almost everywhere on S. The condition $\gamma_i \leq \gamma^r \leq 1$ ensures that the sequence $\{\gamma^r\}$ converges in $L^\infty(S)$ weak*. Thus, when the regularization parameters tend to 0, the estimate (37) leads to the relations (40). As a consequence, $(\mathbf{u}^o, \gamma^o)$ is an admissible solution, i.e. it satisfies the relations (2). Next, the convergence of the sequence $\{\mathbf{t}^r\}$ is investigated. Because of the first equation of the (32) the sequence $\{\mathbf{t}^r\}$ is bounded in the dual space of $(H^{1/2}(S))^3$. Hence there exists a subsequence which converges weakly to a limit denoted $\mathbf{t}^o$. Thus, when r tends to 0, the equation (32) leads to the first of the relations (12).

In order to complete the proof of the convergence of the sequence $\{(\mathbf{u}^r, \gamma^r)\}$ to a solution of equations (12), it remains to prove the last relations of (12), or equivalently that the properties described in the Propositions 2-4 are verified.

The Proposition 4 shows that the damage parameter in the initial nonsmooth problem (12) has an on-off character. In fact, on S_a, the initially adherent area of S, γ can only take two values: γ_i (where no damage occurs) and 1 (where complete delamination occurs, and in this case the relative displacement must be different from 0). It is expected that the limit of the sequence $\{\gamma^r\}$ exhibits the same feature. Recalling the equation (33), the function $\nu^{-1} \circ f^{-1}(y/r)$ defining the damage state on S_a, when $r \to 0$, tends to γ_i for $y < 0$, to 1 for $y > 0$, and its limit has a discontinuity in $y = 0$. In the following proposition a partial result concerning the case when the limit $\mathbf{s}^o$ of $\{\mathbf{s}^r\}$ is different from zero, is obtained.

Proposition 8: On the subset $S^o = \{(x_1, x_2) \in S_a \;/\; \mathbf{s}^o(x_1, x_2) \neq \mathbf{0}\}$ the sequence $\{Y^r\}$ tends almost everywhere to $+\infty$ and the sequence $\{\gamma^r\}$ tends almost everywhere to 1. The convergence of $\{Y^r\}$ to $+\infty$ is a direct consequence of the definition (31) and of the last relation in (32). Further, the equation (33) implies the convergence of $\{\gamma^r\}$ to 1.

More generally, it is necessary to define the delaminated zone to known where the sequence $\{\gamma^r\}$ converges to 1 or to γ_i. As pointed out in Proposition 3, the sign of $Y^o - \omega$, limit of the sequence $\{Y^r - \omega\}$, defines the zone where delamination occurs. The next proposition shows in which sense this limit exists.

Proposition 9: The sequence $\{(1-\gamma^r)Y^r\}$ is a bounded sequence of positive functions in $L^1(S)$ hence it converges to a positive measure on S. On any subset of S where $(1-\gamma^r)$ is bounded from below., the sequence $\{Y^r\}$ tends to a bounded positive measure.
These are direct consequence of the estimate (37).

Now, the rate of convergence of γ^r to γ_i or to 1 is studied. To this end, some subdomains of S_a are initially defined as follows. For any fixed positive real numbers ρ and r let $S^r_{a\rho} \subset S_a$ and $S^r_{d\rho} \subset S_a$ be:

$$S^r_{a\rho} = \{(x_1, x_2) \in S_a \;/\; Y^r(x_1, x_2) - \omega \leq -\rho\} \tag{41}$$

$$S^r_{d\rho} = \{(x_1, x_2) \in S_a \;/\; Y^r(x_1, x_2) - \omega \geq \rho\} \tag{42}$$

It is a simple matter to verify that the sets introduced satisfy the properties:

- for $r \to 0$ the sequence of sets $\left\{\cap_{r \leq r^o} S^r_{a\rho}\right\}$, with $r^o > 0$, is increasing and converges to a set denoted $S_{a\rho}$ included in the adherent part of S_a for the solution $(\mathbf{u}^o, \gamma^o)$;
- for $r \to 0$ the sequence of sets $\left\{\cap_{r \leq r^o} S^r_{d\rho}\right\}$, with $r^o > 0$, is increasing and converges to a set denoted $S_{d\rho}$ included in the delaminated part of S_a for the solution $(\mathbf{u}^o, \gamma^o)$.

Proposition 10: On $S^r_{a\rho}$ and on $S^r_{d\rho}$, the following estimates hold:

$$\begin{aligned} 0 < 1 - \gamma^r < \frac{r}{2\rho} & \qquad on S^r_{a\rho} \\ 0 < \gamma^r - \gamma_i < \frac{r}{2\rho} & \qquad on S^r_{d\rho} \end{aligned} \tag{43}$$

Using relations (25), (26), (27) and (33), the following relation between the regularized parameter and the quantity $(Y^r - \omega)$ is established:

$$\begin{aligned} 0 < 1 - \gamma^r &= \frac{1-\gamma_i}{2} \frac{r}{r + |Y^r - \omega|} \\ &< \frac{1}{2} \frac{r}{r + |Y^r - \omega|} \\ &< \frac{r}{2\rho} \qquad on S^r_{a\rho} \end{aligned} \tag{44}$$

Analogously, it can be obtained:

$$0 < \gamma^r - \gamma_i = \frac{1-\gamma_i}{2} \frac{r}{r + (Y^r - \omega)} \qquad on S^r_{d\rho} \tag{45}$$

and hence the relations (43) are proved.

The Proposition 7 implies the uniform convergence of $\{\gamma^r\}$ to 1 on any closed subset strictly included in $S_{d\rho}$, and to γ_i on any closed subset strictly included in $S_{a\rho}$.

8 Convergences of $\mathbf{t}^r$ and Y^r

The convergence of the gradient of a regularized functional to an element of the subdifferential of the initial nonsmooth one is a well-known result of convex analysis [16, 17]. The functional governing the delamination problem is not convex, and so this classical result does not apply. To overcome this difficulty, it is possible to prove that, at least locally, the relations $\mathbf{t}^o \in \partial_{\mathbf{s}} I_K(\mathbf{s}^o, \gamma^o)$ and $-Y^o \in \partial_\gamma I_K(\mathbf{s}^o, \gamma^o)$ are verified. To this end, it is necessary to give a local meaning to $\mathbf{t}^o$, regarded as the limit of $\mathbf{t}^r$.

The local regularity of $\mathbf{t}^r$, depends on the regularity of the displacement field $\mathbf{u}^r$. In fact, $\mathbf{t}^r$ is the tension at the interface S, which is related to the stress tensor by the relation $\mathbf{t}^r = \sigma^{(1)}\mathbf{n} = -\sigma^{(2)}\mathbf{n}$ on S. Then, because of relation (6), $\mathbf{t}^r$ is a linear function of the symmetric part of the gradient of $\mathbf{u}^r$. The L^2 regularity of $\mathbf{t}^r$ is ensured when $\mathbf{u}^{r(i)} \in (H^{1+1/2})^3$, at least near the interface S. In fact, no global regularity results better than $(H^1(V_i))^3$ can be expected for $\mathbf{u}^{r(i)}$. The lack of regularity results is due to two reasons:

- equations (32) define an elliptic nonlinear problem,
- the boundary conditions are of mixed type.

It is well known that even when very smooth data for the problem are considered, the solution can be very irregular in the neighborhood of the points where the boundary conditions change from the Dirichlet type to the Neumann type [18, 19].

Nevertheless, it is possible to establish a local regularity result for $\mathbf{u}^{r(i)}$ in any open set of V_i containing a part of S and away from all points presenting some difficulty (corners, change in the type of the boundary condition and on the delamination front defined for the initial nonsmooth problem).

Theorem 11: Let Θ be an open set included in V and containing the closure of the open set ϑ. Let ϑ be such that the subset $\vartheta \cap S$ is not empty and contained in the interior of S and either in $S^r_{a\rho}$ or in $S^r_{d\rho}$ for some fixed positive constant ρ. Under the hypotheses $(H1) - (H3)$ and assuming that:
- the external forces are applied only on the boundary $\partial V \backslash \Gamma_0$,
- the elasticity coefficients $E_{ijhk} \in C^{0,1}(\overline{V}_i)$ with $i = 1, 2$,
- $\gamma^i \in C^{0,1}(\overline{V}_i)$ with $i = 1, 2$,

the solution $\mathbf{u}^r$ of the regularized problem (32) verifies the following inequality:

$$\left\|\mathbf{u}^{r(i)}\right\|_{(H^2(\vartheta\cap V_i))^3} \leq C \left\|\mathbf{u}^{r(i)}\right\|_{(H^1(\Theta\cap V_i))^3} \qquad with i = 1, 2 \tag{46}$$

for r small enough, e.g. $r < 2\alpha\rho \min(\eta_n, \eta_t)$ where α is the coercivity constant of the elasticity operator and C is a constant independent of the regularization parameters.
(Sketch): The fully developed proof of this regularity result, omitted herein, is long and fairly technical. However, the main steps of the proof are presented.
The nonlinearity of the problem is handled using the convexity properties of problem (35) and the uniform convergence of the sequence $\{\gamma^r\}$ to 1 in $S^r_{d\rho} \cap S$ or to γ_i in $S^r_{a\rho} \cap S$, given in Proposition 10. The proof is divided into three steps. The first step consists in

assuming that γ is a constant function with $0 \leq \gamma \leq 1$. Under these hypotheses, the local regularity result searched can be proved using the translation method proposed in [20]. Unfortunately, the damage parameter γ could not be constant on S. So, in the second step, the same regularity result is obtained assuming enough regularity on the function γ, e.g. $\gamma \in C^{0,1}(\Theta \cap V_i)$. Actually, the damage parameter involved in the solution $(\mathbf{u}^r, \gamma^r)$ of the regularized delamination problem (32) can be expressed, by means of relation (34), as a continuously derivable function of $\mathbf{s}^r = T\mathbf{u}^r$. Since $\mathbf{s}^r \in (H^{1/2}(S))^3$ the function γ^r cannot be regular enough to apply the result obtained in the second step for problem (35). On the other hand, by substituting the equation (34) into (30), and recalling the last of equations (32), it results that the tension at the interface $\mathbf{t}^r$ is not a monotone function of the relative displacement $\mathbf{s}^r$. Consequently, it is cannot be the derivative of a convex functional of $\mathbf{s}^r$ and then any convexity property cannot be applied. Nevertheless, the uniform convergence of the sequence $\{\gamma^r\}$ to 1 on $S^r_{d\rho} \cap S$ or to γ_i on $S^r_{a\rho} \cap S$ has been proved in Proposition 10. Hence, the third step is devoted to the derivation of the local regularity result for $\mathbf{u}^r$ under the assumptions that the data γ_i is regular enough, e.g. $\gamma_i \in C^{0,1}(\Theta \cap V_i)$, and that $|\gamma^r - 1| \leq \alpha$ on $S^r_{d\rho} \cap S$ and $|\gamma_i - \gamma^r| \leq \alpha$ on $S^r_{a\rho} \cap S$, where α is the coercivity constant. In order to satisfy these last two conditions it is sufficient to choose r such that $r < 2\alpha\rho \min(\eta_n, \eta_t)$, i.e. r must be small enough with respect to the coercivity constant α and to the regularization parameters η_n and η_t.

It is important to note that for a problem with fixed values for η_n and η_t all the regularity results obtained hold for any r small enough, even $r \to 0$. The case with η_n and η_t fixed has a clear mechanical meaning. In fact, the quantities $1/\eta_n$ and $1/\eta_t$ may represent the stiffness of the interface material.

Corollary 12: Under the assumptions given in Theorem 7, it results that the relative displacement $\mathbf{s}^r$ is bounded in $(C^{0,1/2}(\vartheta \cap S))^3$ and the interface stress $\mathbf{t}^r$ is bounded in $(H^{1/2}(\vartheta \cap S))^3$ by constants independent from the regularization parameters.

Under more regular assumptions on the elasticity coefficients E_{ijhk} and on the initial damage function γ_i, better regularity results can be obtained.

Corollary 13: Under the hypotheses of Theorem 7, and assuming $E_{ijhk} \in C^{1,1}(\overline{V}_i)$ and $\gamma_i \in C^{1,1}(\overline{V}_i)$ it results that $\mathbf{s}^r$ is bounded in $(C^{1,1/2}(\vartheta \cap S))^3$ and the tension $\mathbf{t}^r$ is bounded in $(C^{0,1/2}(\vartheta \cap S))^3$ by constants independent of the regularization parameters.

Now, it is possible to prove that the limit of the sequence $\{(\mathbf{t}^r, -Y^r)\}$ evaluated at any point $(x_1, x_2) \in S_{a\rho}$ or $(x_1, x_2) \in S_{d\rho}$, for some strictly positive real number ρ, belongs to the subdifferential of $I_K(\mathbf{s}^o(x_1, x_2), \gamma^o(x_1, x_2))$. In other words, the limit of $\{(\mathbf{t}^r, -Y^r)\}$ belongs to the subdifferential of $I_K(\mathbf{s}^o, \gamma^o)$ at least away from the delamination front. According to the Proposition 4 and Proposition 2 , the damage variable can assume only the values 0 or γ_i, and the limit of the sequence $\{(\mathbf{t}^r, -Y^r)\}$ must verify one of the first three relations given in (21).

The delaminated case is considered first and the density of force $\mathbf{t}^o$, is proved to be due to unilateral contact.

Proposition 14: Let $(\mathbf{u}^r, \gamma^r)$ be the solution of problem (32) under the hypotheses of Theorem 7.1. In any closed subset strictly included in $S_{a\rho} \subset S_a$, for some strictly positive real number ρ, the sequence $\{\mathbf{t}^r\}$ converges to $\mathbf{t}^o$. When the regularization parameter r tends to 0 more rapidly than η_n and η_t, then the limit $\mathbf{t}^o$ verifies:

$$\mathbf{t}_n^o \leq 0 \qquad \mathbf{t}_t^o = (0,0) \tag{47}$$

$$s_n^o \leq 0 \implies \begin{array}{c} \mathbf{t}_n^o = 0 \\ \mathbf{t}_t^o = (0,0) \end{array} \tag{48}$$

The tangential component of $\mathbf{t}^r$, can be written using the relation (30) as:

$$\mathbf{t}_t^r = (1-\gamma^r)\frac{1}{\eta_t}\mathbf{s}_t^r \tag{49}$$

It can be noted that on the delaminated part $S_{d\rho}$, the relative displacement $\mathbf{s}^o$, regarded as the limit of $\mathbf{s}^r$, is bounded, as emphasized in the Corollary 12, and may be different from 0. Hence in order to prove the relation (47), it is necessary to prove that the quantities $(1-\gamma^r)/\eta_t$ and $(1-\gamma^r)/\eta_n$ go to zero. To this end, the estimate (43) leads to:

$$\frac{1-\gamma^r}{\eta_t} \leq \frac{r}{2\rho\eta_t} \tag{50}$$

If r is assumed to tend to 0 more rapidly than η_t, the right side of equation (50) tends to 0. Consequently, $\mathbf{t}_t^r$ goes to $(0,0)$. In the same way it can be proved that the quantity $(1-\gamma^r)/\eta_n$ tends to 0. Hence the relation (47) is proved.
Furthermore, if $s_n^o > 0$ on a close subset of $S_{d\rho}$ then, for any r small enough, $s_n^r \geq 0$, i.e. $(s_n^r)^- = 0$, and thus $(t_n^r)^- = 0$. Consequently, $(t_n^o)^- = 0$, which ends the proof.

Now it remains to verify that on the adherent zone the energy release rate is zero, as obtained in the first of relations (21). This results will be established under stronger local regularity hypotheses.

Proposition 15: Under the hypotheses of Corollary 13 and on any closed subset $\Xi \subset S_{a\rho} \subseteq S_a$ for some strictly positive real number ρ, the sequence $\{Y^r\}$ converges uniformly to 0.
The Corollary 13 ensures that $\mathbf{t}^r$ is uniformly bounded in $(C^{0,1/2}(\Xi))^3$. In particular, the relation (30) implies

$$\left|\frac{1}{\eta_n}(1-\gamma^r)(s_n^r)^+\right| \leq C \qquad \left\|\frac{1}{\eta_t}(1-\gamma^r)\mathbf{s}_t^r\right\| \leq C \tag{51}$$

Since $1-\gamma_i > 0$ on S_a, the assumed regularity on γ_i ensures the existence of a strictly positive lower bound m_i for the function $1-\gamma_i$ on the closed set Ξ. On the other hand, on $S_{a\rho}$ the sequence $\{\gamma^r\}$ tends uniformly to $\gamma^o = \gamma_i$ and on $S_{a\rho}$ the estimate (43) holds true. For r small enough the following inequality is then verified:

$$1-\gamma^r \geq |1-\gamma_i| - \frac{r}{2\rho} \geq \frac{1}{2}m_i > 0 \tag{52}$$

By taking into account the inequality (51) this yields:

$$|(s_n^r)^+| \leq \frac{2\eta_n}{m_i} C \qquad \|\mathbf{s}_t^r\| \leq \frac{2\eta_t}{m_i} C \tag{53}$$

From the relation (31) the following inequality can be derived:

$$0 \leq Y^r = \frac{1}{2\eta_n}\left[(s_n^r)^+\right]^2 + \frac{1}{2\eta_t}\|\mathbf{s}_t^r\|^2 \leq \frac{4C^2}{m_i^2}\max(\eta_n, \eta_t) \tag{54}$$

Finally, when the regularization parameters tend to 0, the sequence $\{Y^r\}$ tends to $Y^o = 0$ on Ξ.

9 The numerical procedure and the plate model

The numerical integration algorithm adopted is a predictor-corrector scheme based on the constructive method used for the proof of existence of a solution.

The problem of two beams in adhesion has been used as a test. The numerical results has been compared to the analytic solution obtained using the regularization of ∂I_K only ([?]). The results obtained using the finite element method show a very good agreement with the analytical solution even when a rare mesh is adopted.

Then the problem of the delamination of composite laminates during a drilling process is investigated.

If the solid V is assumed to be composed of two plates V_1 and V_2 bonded together along the surface S, then it is more convenient from a numerical point of view to use plate theory in order to work with finite elements in two dimensions rather than in three dimensions.

The first order shear deformation plate theory, that is the Mindlin-Reissner plate theory, is considered to model the behavior of each plate. The displacement vector for each plate is represented in the following manner :

$$\mathbf{u}^{(i)}(x_1, x_2, x_3) = \mathbf{v}^{(i)}(x_1, x_2) + (x_3 - d^{(i)})\, \varphi^{(i)}(x_1, x_2) \tag{55}$$

where $d^{(i)}$ is the coordinate along the x_3−axis of the mid-plane of the plate V_i at rest, the vector $\mathbf{v}^{(i)}$ is the displacement of the mid-plane and $\varphi^{(i)}$ represents the rotation of the normal to the mid- plane.

This Mindlin-Reissner plate model is used to treat the problem of a composite laminate made of three orthotropic laminae. In the center of the cross-ply (90°/0°/90°) laminate a circular hole is present only in the two upper laminae of the plate. The central point of the third lamina is subject to a transversal imposed displacement δ oriented downwards. The delamination is supposed to appear between the second and the third lamina. Because of the double symmetry the structure only a quarter of the laminate is considered as shown in figure 1. Computations are developed for several values of the imposed displacement δ. The figure 2 shows the delaminated zone for different values of δ. It can be noted that the delamination has not a circular shape, the delamination is more important in the direction of the major stiffness of the third layer. The precise values of each parameters can be found in ([10]).

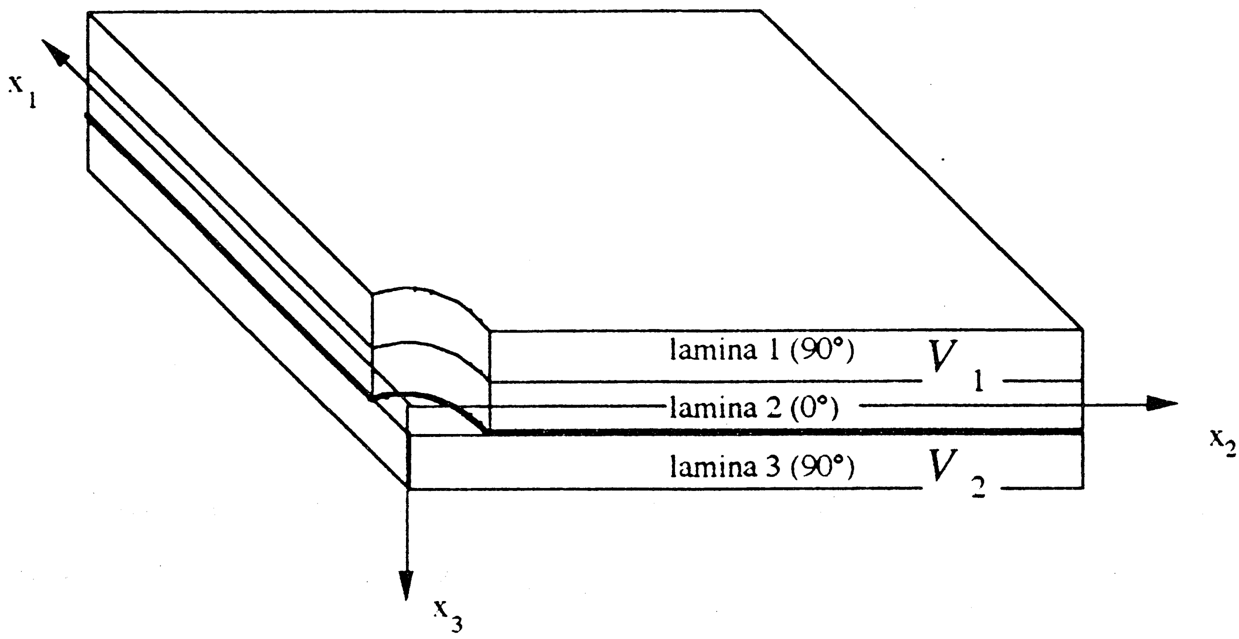

Figure 1 - A quarter of a cross-ply laminate with a hole in the first two laminae

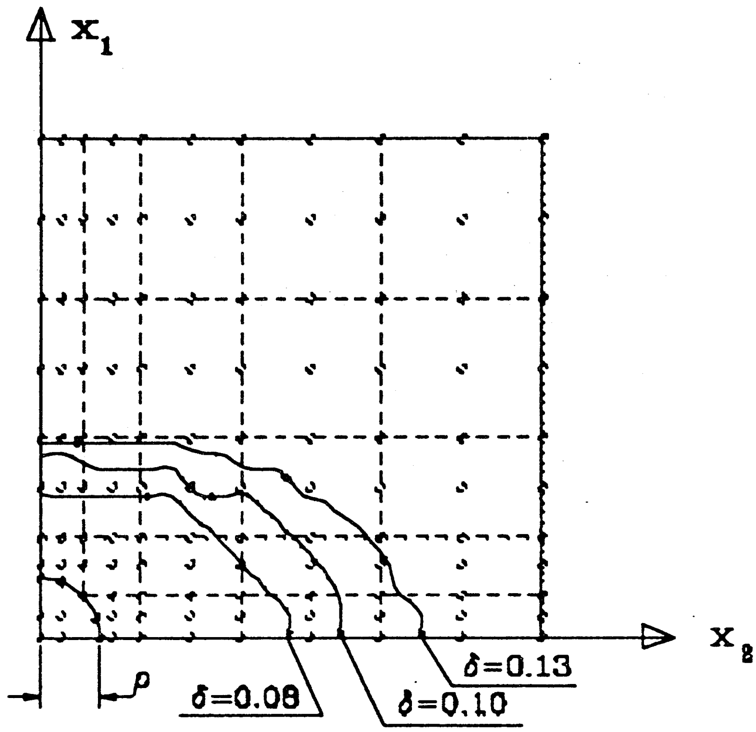

Figure 2 - Zones in adhesion and delaminated for δ=*0.08, 0.10 and 0.13cm*

10 Conclusions

A delamination model has been carried out starting from a very simple assumption concerning the behavior of the interface : two point of the interface are in adhesion only if their relative displacement is zero. The corresponding model is governed by a nonsmooth functional involving two variables: the relative displacement of the joined parts and the damage parameter which is equal to 1 when there is no more adhesion. A regularized model has been given which corresponds to a relevant mechanical mode. But in fact, the initial non-smooth model may generate a lot of various regularized models with different mechanical meaning. Some fundamental mathematical properties of the initial nonsmooth and the regularized delamination problems have been obtained. In particular, a constructive proof of the existence of a solution for the regularized problem has been given. The convergence of the regularized solution to a solution of the initial nonsmooth problem has been proved. A numerical procedure has been used for beam and plate problems. The delaminated area round a hole during a drilling process has been calculated.

Finally, the present approach is quite general, it permit to recover naturally the fracture mechanics approach and it suggests that the others approaches, based on special constitutive laws involving the notion of the damage of the interface, are in some sense regularization of the proposed non-smooth model.

More precise evolution law concerning the damage variable can be choosen [9] . It is worth noting that this model

References

[1] W.J. Bottega and A. Maewal, Delamination Buckling and Growth in Laminates. *J. Appl. Mech.* 50 , 184-189 (1983).

[2] H. Chai, and C.D. Babcock, Two-Dimensional Modelling of Compressive Failure in Delaminated Laminates. *J. Comp. Mater.* 19 , pp. 67-98 (1985).

[3] A.C. Garg, Delamination - A damage model in composite structures. *Engng. Fract. Mech.* 29 , 557-584 (1988).

[4] L.M. Kachanov, *Delamination Buckling of Composite Materials.* Kluwer Academic, (1988).

[5] O. Allix, P. Ladeveze, Interlaminar interface modelling for the prediction of the delamination. *Comp. Struct.* 22 ,pp 235-242 (1992).

[6] M. Frémond, Contact Unilatéral avec Adhérence. *Unilateral Problems in Structural Analysis*, G. Del Piero and F. Maceri, Eds., Springer-Verlag (1985).

[7] M. Frémond, Adhérence des Solides, *J. Méc. Théor. Appl.* 6, 383-407 (1987).

[8] M.Frémond, Contact with Adhesion. *Topics in Nonsmooth Mechanics*, J.J. Moreau, P.D. Panagiotopoulos and G. Strang, Eds., Birkhäuser (1988).

[9] N. Point, Approche Mathématique de Problèmes à Frontières Libres. Application à des Exemples Physiques. Thèse de Doctorat d'Etat es-Sciences Mathématiques de l'Université Paris XIII (1989).

[10] N. Point, and E. Sacco, A delamination model for laminated composites. *Int. J. Solids Structures* 33, 483-509 (1995).

[11] N. Point, and E. Sacco, Delamination of beams: A method for the evaluation of the strain energy release rates of DBC specimen., *Int. J. Fracture* 79 225-247 (1996).

[12] D. Maugis, and M. Barquins, Fracture mechanics and the adherence of viscoelastic bodies. *J. Phys. D., Appl. Phys.* 11 , 1989-2023 (1978).

[13] R.T. Rockafellar, Integrals which are convex functionals, *Pacif. J. Math.* 24, 525-539 (1968)

[14] R.T. Rockafellar, Integrals which are convex functionals II, *Pacif. J. Math.* 39, 439-469 (1971)

[15] J.L. Lions, *Quelques methodes de resolution des problemes aux limites non lineaires.* Dunod, Gauthier-Villars, (1969).

[16] V. Barbu, *Nonlinear-semigroups and differential equations on Banach spaces.* Noordhoff, Leiden, (1976).

[17] V. Mosco, Convergence of convex sets and of solutions of variational inequalities, *Advances in Math.* 3, 510-585 (1969).

[18] E. Shamir, Regularization of mixed second-order elliptic problems, *Israël J. Math.* 6 , 150- 168 (1968).

[19] J-F. Rodrigues, *Obstacle Problems in Mathematical Physics.* North-Holland, (1991).

[20] S. Agmon, A.Douglis, and L. Nirenberg, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions, *Comm. Pure Appl. Math.* 12, 623-727 (1959).