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HYDRODYNAMIC LIMITS OF SOME KINETIC EQUATIONS *

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1. INTRODUCTION. One of the most important problems in applied mathematics is

understanding relationships between the models

(I) *on the microscopic level (dynamics of the particles),*

(II) *on the statistical level (dynamics of one or more test–particles, in the framework of kinetic theory),*

(III) *on the macroscopic level (dynamics of the continuum).*

A huge number of papers on this topic exists (see [CIP], [La2]), but the relationships between the different models are still not fully understood.

The aim of this paper is to discuss the relationships between **(II)** and **(III)**.

In kinetic theory **(II)** a statistical description is used for systems composed of a large number of particles. The evolution of such a system is described by the probability density function of one particle (one–particle distribution function) $f = f(t, \mathbf{x}, \mathbf{v})$ satisfying a kinetic equation; $t \geq 0$, $\mathbf{x} = (x_1, x_2, x_3) \in \Omega \subseteq \mathbb{R}^3$, and $\mathbf{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$ are time, space and velocity variables, respectively. Throughout this paper Ω is either $\mathbb{R}^3$ or the three–dimensional torus $\mathbb{T}^3$.

The following moments of function f correspond with fluid dynamics **(III)**:

$$\varrho(t, \mathbf{x}) = \int_{\mathbb{R}^3} f(t, \mathbf{x}, \mathbf{v}) \, \mathrm{d}\mathbf{v} \,, \tag{1.1a}$$

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$$\mathbf{u}(t,\mathbf{x}) = \frac{1}{\varrho(t,\mathbf{x})} \int_{\mathbb{R}^3} \mathbf{v}\, f(t,\mathbf{x},\mathbf{v})\, \mathrm{d}\mathbf{v}\,, \tag{1.1b}$$

$$\mathfrak{e}(t,\mathbf{x}) = \frac{1}{2\varrho(t,\mathbf{x})} \left(\int_{\mathbb{R}^3} |\mathbf{v}|^2 f(t,\mathbf{x},\mathbf{v})\, \mathrm{d}\mathbf{v} - \varrho(t,\mathbf{x})|\mathbf{u}(t,\mathbf{x})|^2 \right) \tag{1.1c}$$

and represent the (dimensionless) mass density, macroscopic velocity vector and internal energy, respectively; The (dimensionless) temperature is

$$T(t,\mathbf{x}) = \frac{1}{3\varrho(t,\mathbf{x})} \left(\int_{\mathbb{R}^3} |\mathbf{v}|^2 f(t,\mathbf{x},\mathbf{v})\, \mathrm{d}\mathbf{v} - \varrho(t,\mathbf{x})|\mathbf{u}(t,\mathbf{x})|^2 \right)\,. \tag{1.1d}$$

The most widely known and used is the Boltzmann kinetic equation ([CC], [FK], [CIP], [La2], [Uk])

$$\mathfrak{M}\frac{\partial f}{\partial t} + \mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{x}} = \frac{1}{\varepsilon} J(f)\,, \tag{1.2}$$

where

$$J(f)(\mathbf{x},\mathbf{v}) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} \Big(f(\mathbf{x},\mathbf{w}')f(\mathbf{x},\mathbf{v}') - f(\mathbf{x},\mathbf{w})f(\mathbf{x},\mathbf{v}) \Big) B\big(\mathbf{n},\mathbf{w}-\mathbf{v}\big) \mathrm{d}\mathbf{n}\mathrm{d}\mathbf{w}\,,$$

$$\mathbb{S}^2 = \left\{ \mathbf{n} \in \mathbb{R}^3 : \quad |\mathbf{n}| = 1 \right\},$$

$\mathbf{v}'$ and $\mathbf{w}'$ are the velocities of a pair of particles after a collision, which are related to the velocities $\mathbf{v}$ and $\mathbf{w}$, before the collision, by

$$\mathbf{v}' = \mathbf{v} + ((\mathbf{w}-\mathbf{v})\cdot\mathbf{n})\mathbf{n}\,, \qquad \mathbf{w}' = \mathbf{w} - ((\mathbf{w}-\mathbf{v})\cdot\mathbf{n})\mathbf{n}\,;$$

ε and $\mathfrak{M}$ are the Knudsen and Mach numbers, respectively; and B is the collision kernel. Throughout the paper, the collision kernels corresponding to Grad's cutt-off hard potentials ([Gr]) are considered. The important example of a such collision kernel is the one corresponding to the hard sphere potential

$$B\big(\mathbf{n},\mathbf{w}-\mathbf{v}\big) = \mathbf{n}\cdot(\mathbf{w}-\mathbf{v}) \vee 0\,, \tag{1.3}$$

where $a_1 \vee a_2 = \max\{a_1, a_2\}$ (and $a_1 \wedge a_2 = \min\{a_1, a_2\}$).

It is well known that the concept of continuum theory is valid only in the so-called hydrodynamic limit

$$\varepsilon \downarrow 0\,. \tag{1.4}$$

This limit (for bounded $\mathfrak{M}$) formally leads to the equation

$$J(f) = 0\,, \tag{1.5}$$

which is the degenerate equation corresponding to the singularly perturbed equation (1.2). The unique class of positive solutions of Eq. (1.5) is that of Maxwellians

$$M[\varrho, \mathbf{u}, T](t, \mathbf{x}, \mathbf{v}) = \varrho(t, \mathbf{x}) \left(2\pi T(t, \mathbf{x})\right)^{-3/2} \exp\left(-\frac{|\mathbf{v} - \mathbf{u}(t, \mathbf{x})|^2}{2T(t, \mathbf{x})}\right),$$

where ϱ (local density), $\mathbf{u}$ (macroscopic velocity vector), and T (macroscopic temperature) are the fluid-dynamic parameters of the Maxwellian $M = M[\varrho, \mathbf{u}, T]$.

One can expect that in the limit (1.4) the solutions of Eq. (1.2) approach the Maxwellian $M[\varrho, \mathbf{u}, T]$, whose fluid-dynamic parameters solve a system of equations of continuum theory.

There are two classical and natural cases (cf. [La2] and references therein) for studying the hydrodynamic limit (1.4). The first one is for $\mathfrak{M}$ — fixed, e.g.

$$\varepsilon \downarrow 0 \quad \text{and} \quad \mathfrak{M} = 1\,, \tag{1.6}$$

which leads to the systems of equations of continuum for compressible fluids; In this case the Reynolds number $\mathfrak{R}$ is $\mathcal{O}\left(\frac{1}{\varepsilon}\right)$. The second one — for $\varepsilon \sim \mathfrak{M}$, e.g.

$$\varepsilon = \mathfrak{M} \downarrow 0\,, \tag{1.7}$$

which corresponds to incompressible fluids; In this case $\mathfrak{R}$ is $\mathcal{O}(1)$.

By the Hilbert or modified Hilbert expansion procedures ([Ca2], [La2]), one concludes that, in Limit (1.6), the Boltzmann equation (1.2) results in the Euler system and then in the Navier-Stokes system for which the viscosity and heat conduction terms are $\mathcal{O}(\varepsilon)$.

For the Hilbert expansion the existence of smooth solutions to the resulting continuum system is essential. It is well known that one cannot expect the global existence of smooth solutions to the Euler system due to the fact ([Si]) that singularities can appear in a finite time. On the other hand, as far as the Navier-Stokes approximation is concerned, the elliptic properties of the resulting Navier-Stokes operator, in Limit (1.6) cannot be preserved ($\mathfrak{R} = \mathcal{O}\left(\frac{1}{\varepsilon}\right)$).

In Limit (1.7), Eq. (1.2) results in the Navier-Stokes equation for incompressible fluids (see [BGL1-3], [DEL], [BU], [Go]). In this case one has to assume (see [BGL2], [Go] and [La2]) that the initial data (and then the solutions) to Eq. (1.2) are $\mathcal{O}(\varepsilon)$–close to a global Maxwellian (i.e. the Maxwellian constant in the time and space variables). The ambitious program of finding the relationships between the global weak solutions of the Navier-Stokes equation (due to Leray [Le]) and the renormalized solutions of the Boltzmann equation (due to DiPerna and Lions [DL]) has been started by Bardos, Golse and Levermore in a series of papers [BGL1-3], [Go]. The main difficulty in completing this program comes from the fact that it is strictly related to the local conservation laws (of mass, of momentum, and of energy) properties. The question whether the renormalized

DiPerna-Lions solution satisfies the local momentum and energy conservation laws still remains open (cf. discussion in [BGL3] and [Go]). Bardos, Golse and Levermore [BGL3] proved that if the DiPerna – Lions renormalized solutions satisfy the local momentum conservation and if some additional properties are fulfilled, then the weak convergence of the renormalized solutions to the Leray solutions holds.

Summarizing: the Boltzmann equation (1.2) is consistent with the compressible Navier-Stokes system, with the viscosity and heat conduction terms of the order of ε ($\mathfrak{R} = \mathcal{O}\left(\frac{1}{\varepsilon}\right)$) in Limit (1.6), and with the incompressible Navier-Stokes equation (with $\mathfrak{R} = \mathcal{O}(1)$) for $\mathcal{O}(\varepsilon)$–perturbation of a global equilibrium in Limit (1.7).

Contrarily, an asymptotic relationship between the Boltzmann equation and the compressible Navier-Stokes system with viscosity and heat conduction terms which are $\mathcal{O}(1)$ in the hydrodynamic limit (1.6) cannot hold true.

On the other hand, in Limit (1.6) some kinetic equations ([La3,4]) result in a compressible Navier-Stokes-type system, with viscosity and heat conduction terms independent of all small parameters of the kinetic models — under a particular assumption about the relation between the small parameters (cf. (4.2)).

In this paper we consider only Limit (1.6) (Limit (1.7), for other kinetic equations than the Boltzmann one, has been launched in the paper [JL]).

As far as the smooth solutions are concerned, in the mathematical literature, the two following approaches can be distinguished:

A↑) **MACRO (III)** $\Longrightarrow$ **MICRO (II)** ([Ca1,2], [La2]);

A↓) **MICRO (II)** $\Longrightarrow$ **MACRO (III)** ([Ni], [UA], [AU], [Uk]).

In Approach A↑ one shows that the existence of smooth solution to the system describing dynamics of continuum implies both the existence of a smooth solution to the kinetic equation and the validity of the Hilbert procedure.

In Approach A↓ first the existence of an analytic solution to the kinetic equation and then its convergence to the local Maxwellian (whose fluid–dynamic parameters solve the macroscopic system) are proved. Both approaches have a local (in time) nature, but the time interval does not depend on the small parameter(s) (is "*macroscopic*").

2. ENSKOG EQUATION. In the case of the Boltzmann equation, the overall dimensions of the particles are neglected (the collision operator J has a local nature with respect to the space variable $\mathbf{x}$). This physical idealization leads to serious mathematical difficulties: the operator J does not act in the space $L_1(\Omega \times \mathbb{R}^3)$, in which natural a *priori* bounds hold.

However, in the case of dense gases, one should replace the mass–point Boltzmann model by a model which can take into account the overall dimensions of particles.

One such attempt leads to the Enskog equation — a quite successful model of the kinetic theory of moderately dense gases (cf. [CC], [FK], [BLPT] and [Ar2], [AC], [EsP]), in which each particle is assumed to be a hard sphere with nonzero diameter. Such an assumption leads to nonlocal (with respect to $\mathbf{x}$) nature of the equation.

In the present paper only the simplified case of the Enskog equation (referred to as the Boltzmann–Enskog equation), for which the pair correlation function is equal to 1, is considered. Further, the analysis is valid for a general case, under a suitable assumption about the behaviour of the pair correlation function (cf. [La2]).

The Boltzmann–Enskog is

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\frac{\partial}{\mathbf{x}}\right) f = \frac{1}{\varepsilon} E_a(f) \,, \tag{2.1}$$

$$E_a(f)(\mathbf{x},\mathbf{v}) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} \Big(f(\mathbf{x}+a\mathbf{n},\mathbf{w}') f(\mathbf{x},\mathbf{v}') - f(\mathbf{x}-a\mathbf{n},\mathbf{w}) f(\mathbf{x},\mathbf{v}) \Big) \big(\mathbf{n}\cdot(\mathbf{w}-\mathbf{v}) \vee 0\big) d\mathbf{n} d\mathbf{w} \,,$$

where a is the (dimensionless) diameter of the (hard sphere) particles. The standard notation is used — a particle with the center at $\mathbf{x}$ and the velocity $\mathbf{v}$ collides with a particle with the center at $\mathbf{x} - a\mathbf{n}$ and the velocity $\mathbf{w}$. The collision kernel corresponds to the hard sphere potential (1.3).

The mathematical theory for the Enskog equation can be found in [Ar1,2], [AC1,2], [BLPT], [EsP] (see also references therein). Note that in the $\mathbf{x}$–one–dimensional case the operator E_a, for $a > 0$, acts in the $L_1(\Omega)$–space setting ([Ce2]).

A degenerate equation corresponding to the Boltzmann–Enskog equation (for fixed $a > 0$) is

$$E_a(f) = 0 \,. \tag{2.2}$$

The class of solutions of Eq. (2.2), for fixed $a > 0$, is too small to describe a reasonable hydrodynamic (cf. [At]). Therefore the following hydrodynamic limit

$$\varepsilon \downarrow 0 \,, \qquad a \downarrow 0 \tag{2.3}$$

should be considered.

One can distinguish the following important cases:

(i) $a \ll \varepsilon$, (e.g. $a = b\varepsilon^p$ for $p > 1$);

(ii) $a \gg \varepsilon$, (e.g. $a = b\varepsilon^p$ for $0 < p < 1$);

(iii) $a = b\varepsilon$, for $b = \text{const}$.

Case (i) leads to the hydrodynamic limits which are exactly the same as those resulting from the Boltzmann equation (cf. [La2]).

In Case (ii) one should expect similar difficulties as for the case when $a > 0$ is fixed in the limit (1.4).

In Case (iii), the Boltzmann-Enskog equation (formally) results in the following Enskog–Euler system

$$\frac{\partial \varrho}{\partial t} + \sum_{i=1}^{3} \frac{\partial}{\partial x_i}(\varrho u_i) = 0\,, \tag{2.4a}$$

$$\varrho \frac{\partial u_j}{\partial t} + \varrho \sum_{i=1}^{3} u_i \frac{\partial u_j}{\partial x_i} + \frac{\partial}{\partial x_j}\Big(\varrho T(1 + \frac{2\pi}{3}\, b\, \varrho)\Big) = 0\,, \qquad j = 1,2,3\,, \tag{2.4b}$$

$$\frac{\partial T}{\partial t} + \sum_{i=1}^{3} u_i \frac{\partial T}{\partial x_i} + \frac{2}{3} T \left(1 + \frac{2\pi}{3}\, b\varrho\right) \sum_{i=1}^{3} \frac{\partial u_i}{\partial x_i} = 0\,. \tag{2.4c}$$

The mathematical theory on the relationships between the Boltzmann–Enskog equation (2.1) and the Enskog–Euler system (2.4) was developed in the paper [La5] by applying the idea of Ukai and Asano [UA] (Approach A↓).

Let $a = b\,\varepsilon$, $b =$ const, and ω be a global Maxwellian, i.e. $\omega(\mathbf{v}) = M[\varrho_\star, \mathbf{u}_\star, T_\star](\mathbf{v})$, where $\varrho_\star > 0$, $\mathbf{u}_\star \in \mathbb{R}^3$, $T_\star > 0$ are given constants.

The initial datum F is assumed in the form

$$F = \omega + \omega^{\frac{1}{2}} G\,, \tag{2.5a}$$

where G is independent of ε, and the solution is looked for in the form

$$f = \omega + \omega^{\frac{1}{2}} g\,. \tag{2.5b}$$

In terms of g the Cauchy problem for the Boltzmann–Enskog equation reads

$$\frac{\partial g}{\partial t} = -\mathbf{v} \cdot \frac{\partial g}{\partial \mathbf{x}} + \frac{1}{\varepsilon} L g + \frac{1}{\varepsilon} Q_\varepsilon g + \frac{1}{\varepsilon}\Gamma_\varepsilon(g,g)\,, \qquad g\Big|_{t=0} = G\,, \tag{2.6}$$

where L is the Boltzmann linearized collision operator $Lg = 2\omega^{-\frac{1}{2}} J\left(\omega, \omega^{\frac{1}{2}} g\right)$, and

$$Q_\varepsilon g = 2\omega^{-\frac{1}{2}} E_{b\varepsilon}\left(\omega, \omega^{\frac{1}{2}} g\right) - Lg\,, \qquad \Gamma_\varepsilon(g_1, g_2) = \omega^{-\frac{1}{2}} E_{b\varepsilon}\left(\omega^{\frac{1}{2}} g_1, \omega^{\frac{1}{2}} g_2\right)\,.$$

Consider the following integral version of Problem (2.6)

$$g(t) = e^{tB_\varepsilon} G + \frac{1}{\varepsilon} \int_0^t e^{(t-t_1)B_\varepsilon} \left(Q_\varepsilon g(t_1) + \Gamma_\varepsilon(g,g)(t_1)\right) \mathrm{d}t_1\,, \tag{2.7}$$

$$(t, \mathbf{x}, \mathbf{v}) \in]0, \infty[\times \mathbb{R}^3 \times \mathbb{R}^3$$

(the proofs are, however, also applicable to the case of $\mathbf{x} \in \mathbb{T}^3$). Here e^{tB_ε} denotes the semigroup generated by the linear operator

$$B_\varepsilon = -\mathbf{v} \cdot \frac{\partial}{\partial \mathbf{x}} + \frac{1}{\varepsilon} L\,.$$

Let $\mathbb{B}^\alpha$ denote the space equipped with the norm

$$\|g\|^{(\alpha)} = \sup_{\mathbf{v}\in\mathbb{R}^3} |\langle\mathbf{v}\rangle^\alpha g(\mathbf{v})| \,, \qquad \alpha \in \mathbb{R}^1 ,$$

where $\langle\mathbf{v}\rangle^\alpha = (1+|\mathbf{v}|)^\alpha$ and $\alpha \in \mathbb{R}^1$. Let $\hat{g} = \mathcal{F}g$ denote the Fourier transform of a function $g \in \mathcal{S}'(\mathbb{R}^3 \times \mathbb{R}^3)$ with respect to the position variable $\mathbf{x}$,

$$\hat{g}(\mathbf{k},\mathbf{v}) = \mathcal{F}g(\mathbf{k},\mathbf{v}) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{-i\mathbf{k}\cdot\mathbf{x}} g(\mathbf{x},\mathbf{v}) d\mathbf{x} \,, \qquad \mathbf{k} \in \mathbb{R}^3 .$$

Let $\mathbb{X}^{(\alpha)}_{\beta,\gamma}$ be the space equipped with the norm

$$\|g\|^{(\alpha)}_{\beta,\gamma} = \sup_{\mathbf{k},\mathbf{v}\in\mathbb{R}^3} \left|\langle\mathbf{v}\rangle^\alpha \langle\mathbf{k}\rangle^\beta \exp(\gamma|\mathbf{k}|)\hat{g}(\mathbf{k},\mathbf{v})\right| \,, \quad \alpha,\beta,\gamma \in \mathbb{R}^1_+ ,$$

α, β, γ are positive constants.

The space $\dot{\mathbb{X}}^{(\alpha)}_{\beta,\gamma}$ is a closed subspace of $\mathbb{X}^{(\alpha)}_{\beta,\gamma}$, such that

$$g \in \dot{\mathbb{X}}^{(\alpha)}_{\beta,\gamma} \quad \Leftrightarrow \quad g \in \mathbb{X}^{(\alpha)}_{\beta,\gamma} \,\&\, \left\|\mathcal{F}^{-1}(\chi(|\mathbf{k}|+|\mathbf{v}|>\xi)\hat{g}(\mathbf{k},\mathbf{v}))\right\|^{(\alpha)}_{\beta,\gamma} \longrightarrow 0 \,, \text{ as } \xi \to \infty ,$$

where $\chi(\text{truth}) = 1$ and $\chi(\text{false}) = 0$.

For $\eta \geq 0$ and an interval $I \subset \mathbb{R}^1$, let

$$\mathbb{Y}^{\alpha,\eta}_{\beta,\gamma}(I) = \left\{ g = g(t) : \quad \tilde{g}_\eta \in C^0_B(I; \dot{\mathbb{X}}^{(\alpha)}_{\beta,\gamma}) \,;\; \tilde{g}_\eta(t,\mathbf{x},\mathbf{v}) \equiv \mathcal{F}^{-1} \exp(-\eta t|\mathbf{k}|)\hat{g}(t,\mathbf{k},\mathbf{v}) \right\}$$

be the space equipped with the norm

$$\|g\|^{\alpha,\eta}_{\beta,\gamma,I} = \sup_{t\in I} \|g(t)\|^{(\alpha)}_{\beta\,,\gamma-\eta t} \,.$$

Let

$$\mathbb{Z}^{\alpha,\eta,1}_{\beta,\gamma,t_0} = C^0_B(]0,1]; \mathbb{Y}^{\alpha,\eta}_{\beta,\gamma}([0,t_0]))$$

be the Banach space of ε–dependent functions, equipped with the norm

$$\|g\|^{\alpha,\eta,1}_{\beta,\gamma,t_0} = \sup_{\substack{0<\varepsilon\leq 1\\ 0\leq t\leq t_0}} \|g(t)\|^{(\alpha)}_{\beta\,,\gamma-\eta t} \,.$$

Let

$$\hat{B}_\varepsilon(\mathbf{k})\hat{g}(\mathbf{k},\mathbf{v}) \equiv \mathcal{F}B_\varepsilon g(\mathbf{k},\mathbf{v}) \equiv \left(-i\mathbf{k}\cdot\mathbf{v} + \frac{1}{\varepsilon}L\right)\hat{g}(\mathbf{k},\mathbf{v}) \,.$$

The semigroup $\frac{1}{\varepsilon}e^{t\hat{B}_\varepsilon(\mathbf{k})}$ is such that ([ElP], [Uk], [An1,3])

$$e^{t\hat{B}_\varepsilon(\mathbf{k})} = \chi(\varepsilon|\mathbf{k}| \leq \kappa) \sum_{j=0}^{4} e^{\lambda^\varepsilon_j(\mathbf{k})t} P^\varepsilon_j(\mathbf{k}) + U_\varepsilon(t,\mathbf{k}) \,,$$

where κ is a positive constant; $\lambda^\varepsilon_j \in C^\infty([-\kappa,\kappa])$ (for $j = 0,...,4$) are such that

$\Re\lambda_j^\varepsilon(\mathbf{k}) \le 0$ and have the asymptotic expansion

$$\lambda_j^\varepsilon(\mathbf{k}) = i\lambda_j^{(1)}|\mathbf{k}| - \varepsilon\lambda_j^{(2)}|\mathbf{k}|^2 + \mathcal{O}\left(\varepsilon^2|\mathbf{k}|^3\right), \qquad |\mathbf{k}| \to 0,$$

with coefficients $\lambda_j^{(1)} \in \mathbb{R}^1$ and $\lambda_j^{(2)} > 0$;

$$P_j^\varepsilon(\mathbf{k}) = P_j^{(0)}\left(\frac{\mathbf{k}}{|\mathbf{k}|}\right) + \varepsilon|\mathbf{k}|P_j^{(1)}\left(\frac{\mathbf{k}}{|\mathbf{k}|}\right) + \varepsilon^2|\mathbf{k}|^2P_j^{(2)}(\mathbf{k}), \ |\mathbf{k}| \to 0, \ \mathbf{k} \neq 0.$$

For each fixed $\mathbf{k}$, the operators $P_j^{(0)}(\mathbf{k})$ are orthogonal projections on $L_2(\mathbb{R}^3)$; $P^{(0)} = \sum_{j=0}^4 P_j^{(0)}(\frac{\mathbf{k}}{|\mathbf{k}|})$ is the orthogonal projection onto $\mathcal{N}^{(0)} = \mathrm{lin}\{\omega^{\frac{1}{2}}\psi_i : i = 0,...,4\}$ in $L_2(\mathbb{R}^3)$, where $\psi_0, \ldots, \psi_4$ are the collision invariants

$$\psi_0 \equiv 1, \quad \psi_i(\mathbf{v}) = v_i \quad (i = 1,2,3), \quad \psi_4(\mathbf{v}) = |\mathbf{v}|^2; \tag{2.8}$$

$P^{(0)}$ is independent of $\frac{\mathbf{k}}{|\mathbf{k}|}$;

For each fixed α, the operator $P_j^{(l)}$ satisfies (for $l = 0,1,2$ and $j = 0,...,4$)

$$\|P_j^{(l)}g\|^{(\alpha)} \le c\,\|g\|^{(\alpha')}, \qquad \forall\,\alpha' > \frac{3}{2};$$

The operator U_ε can be decomposed

$$U_\varepsilon(t,\mathbf{k}) = e^{t\hat{A}_\varepsilon(\mathbf{k})} + \tilde{U}_\varepsilon(t,\mathbf{k}),$$

where $\hat{A}_\varepsilon(\mathbf{k}) = -\left(i\mathbf{k}\cdot\mathbf{v} + \frac{1}{\varepsilon}\nu(\mathbf{v})\right)$, and for each fixed $\alpha > \frac{5}{2}$ the operator $\tilde{U}_\varepsilon$ satisfies

$$\|\tilde{U}_\varepsilon(t,\mathbf{k})\,g\|^{(\alpha)} \le c\,\exp(-\sigma\frac{t}{\varepsilon})\|g\|^{(\alpha-1)}, \quad \sigma > 0.$$

Therefore

$$\frac{1}{\varepsilon}e^{t\hat{B}_\varepsilon(\mathbf{k})}\big(I - P_0\big) = |\mathbf{k}|\mathfrak{A}(\mathbf{k}) + \frac{1}{\varepsilon}e^{-\sigma_0\frac{t}{\varepsilon}}\mathfrak{B}(\mathbf{k}),$$

where $\mathfrak{A}$ and $\mathfrak{B}$ are uniformly bounded (for $\varepsilon > 0$) operators and $\sigma_0 = \mathrm{const} > 0$ and the singular factor $\frac{1}{\varepsilon}$ "is replaced" ([Uk]) by a unbounded operator (a pseudodifferential operator with the symbol $|\mathbf{k}|$).

On the other hand, the operator $\frac{1}{\varepsilon}e^{t\hat{B}_\varepsilon(\mathbf{k})}Q_\varepsilon$, again can be treated as a pseudodifferential operator with the symbol $|\mathbf{k}|$, instead of being singular with respect to ε.

In the proof the following elementary inequalities are needed

(i) *For any $\beta_1 > 3$ and any $\beta_2 > 3$*

$$\textbf{(IQ1)} \qquad \int_{\mathbb{R}^3} \frac{1}{(1+|\mathbf{k}_1|)^{\beta_1}(1+|\mathbf{k}-\mathbf{k}_1|)^{\beta_2}}\mathrm{d}\mathbf{k}_1 \le \frac{c}{(1+|\mathbf{k}|)^{\beta_1\wedge\beta_2}};$$

(ii) *For ξ, γ, $\eta > 0$*

$$\textbf{(IQ2)} \quad \int_0^t \exp\big(-\xi\,(t-t_1) - (\gamma - \eta\,t_1)\,|\mathbf{k}|\big)\,\mathrm{d}t_1 \le \frac{1}{\xi + \eta\,|\mathbf{k}|}\exp\big(-(\gamma - \eta\,t)|\mathbf{k}|\big).$$

The main result of the paper [La5] is

THEOREM 2.1. *Let α, β, γ, b, η and t_0 be properly chosen (independent of ε). If the initial data*

$$G \in \dot{\mathbb{X}}^{(\alpha)}_{\beta,\gamma} \tag{2.9a}$$

satisfies the smallness condition

$$\|G\|^{(\alpha)}_{\beta,\gamma} < \vartheta_0 \,, \tag{2.9b}$$

where ϑ_0 is a given constant.

Then

(i.) there exists a unique classical solution g of the Cauchy problem for the Boltzmann–Enskog equation (2.1) on the time interval $[0, t_0]$, such that

$$g \in \mathbb{Z}^{\alpha,\eta,1}_{\beta,\gamma,t_0} \,, \tag{2.10}$$

$$\frac{\partial}{\partial t} g \in C^0(]0,1]; \mathbb{Y}^{\alpha-1,\eta}_{\beta-1,\gamma}([0,t_0])) \,. \tag{2.11}$$

(ii) *$g(t) \to g_0(t)$, as $\varepsilon \downarrow 0$, strongly in $\mathbb{Y}^{\alpha,\eta}_{\beta,\gamma}([\delta,t_0])$, for any $\delta \in]0,t_0[$;*

(iii) *$f_0(t) = \omega + \omega^{\frac{1}{2}} g_0(t)$ is the Maxwellian such that $(\varrho, \mathbf{u}, T)$ is a classical solution of the Cauchy problem for the Enskog–Euler system (2.4) with the initial data*

$$\varrho\Big|_{t=0} = (\psi_0, F)_{L_2(\mathbb{R}^3)} \,, \qquad T\Big|_{t=0} = (\psi_4, F)_{L_2(\mathbb{R}^3)} \,, \tag{2.12a}$$

$$u_j\Big|_{t=0} = (\psi_j, F)_{L_2(\mathbb{R}^3)} \,, \qquad j = 1,2,3 \,. \tag{2.12b}$$

As a by-product, Theorem 2.1 delivers an existence result for the Enskog–Euler system (2.4). For the Euler system (i.e. for $b = 0$), which is a symmetric hyperbolic system provided that $\varrho > 0$, the (local) existence and uniqueness theorem is available for the Cauchy problem with analytical initial data $(\varrho, \mathbf{u}, T)\Big|_{t=0}$ such that

$$\varrho\Big|_{t=0} > 0 \,. \tag{2.13}$$

This assumption was essential in the proof by Nishida [Ni] of the convergence of solution of the Boltzmann equation to the Maxwellian, which fluid–dynamic parameters solve the Euler system. This type of assumption was also essential in the methods reviewed in the lecture [La2]. On the other hand, Assumption (2.13) was removed in methods by Ukai and Asano [UA]. It is not necessary either in the paper [La5]. To author's knowledge, the existence result, which follows from Theorem 2.1, is the first for the Enskog–Euler system.

Like for the Boltzmann equation ([UA]), when the initial layer vanishes, i.e. for

$$g_0 = P^{(0)} g_0 \,, \tag{2.14}$$

Theorem 2.1 (ii) holds with $\delta = 0$.

3. STOCHASTIC KINETIC EQUATION. A large number of particle limit for a system of stochastic particles ([LP], [La1], and [Ce1], [Sk], [An2,3], [Vo]) leads to the stochastic kinetic equation

$$(\frac{\partial}{\partial t} + \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{x}}) f = \frac{1}{\varepsilon} S(f), \tag{3.1}$$

where

$$S(f)(\mathbf{x}, \mathbf{v}) = \int_{\mathbb{R}^3} \int_{\Omega} \Big(f(\mathbf{y}, \mathbf{w}^*) f(\mathbf{x}, \mathbf{v}^*) \mathcal{B}(\mathbf{y} - \mathbf{x}, \mathbf{w} - \mathbf{v}) - f(\mathbf{y}, \mathbf{w}) f(\mathbf{x}, \mathbf{v}) \mathcal{B}(\mathbf{x} - \mathbf{y}, \mathbf{w} - \mathbf{v}) \Big) d\mathbf{y} d\mathbf{w},$$

the velocities $\mathbf{v}^*$ and $\mathbf{w}^*$ are functions of $\mathbf{v}$ and $\mathbf{w}$ as well as of the distance between the two particles $\mathbf{y} - \mathbf{x}$, according to

$$\mathbf{v}^* = \mathbf{v} + ((\mathbf{w} - \mathbf{v}) \cdot \mathbf{n})\mathbf{n}, \qquad \mathbf{w}^* = \mathbf{w} - ((\mathbf{w} - \mathbf{v}) \cdot \mathbf{n})\mathbf{n}, \tag{3.2}$$

where now

$$\mathbf{n} = \frac{\mathbf{y} - \mathbf{x}}{|\mathbf{y} - \mathbf{x}|}, \qquad \text{for} \quad \mathbf{y} \neq \mathbf{x}, \quad \mathbf{y}, \mathbf{x} \in \Omega,$$

$\mathcal{B}$ is such that (cf. [Ce1], [LP], [La1])

$$\mathcal{B}(\mathbf{x}, \mathbf{v}) = 0 \quad \text{for} \quad \mathbf{x} \cdot \mathbf{v} < 0, \qquad \forall \mathbf{x} \in \Omega, \ \mathbf{v} \in \mathbb{R}^3. \tag{3.3}$$

In this Section it is assumed that (cf. [La4])

$$\mathcal{B}(\mathbf{x}, \mathbf{v}) = \frac{3}{R^3 - r^3} \chi(r \leq |\mathbf{x}| \leq R) \chi(\mathbf{x} \cdot \mathbf{v} \geq 0) B(\frac{\mathbf{x}}{|\mathbf{x}|}, \mathbf{v}), \tag{3.4}$$

where $0 < r < R < \infty$ and B is the collision kernel corresponding to Grad's cut–off hard potentials.

Equation (3.1) can also be considered in the symmetrized form ([Mo], [Po], [BP]), i.e. when Assumption (3.3) is replaced by

$$\mathcal{B}(-\mathbf{x}, \mathbf{v}) = \mathcal{B}(\mathbf{x}, \mathbf{v}), \qquad \forall \mathbf{x} \in \Omega \quad \forall \mathbf{v} \in \mathbb{R}^3. \tag{3.5}$$

The two assumptions (3.3) and (3.5) lead to different hydrodynamic limits ([La3,4]). Case (3.5) is discussed in Section 4.

Note that Eq. (3.1) formally leads to the Boltzmann equation for

$$\mathcal{B}(\mathbf{x}, \mathbf{v}) = \delta(\mathbf{x}) \chi(\mathbf{x} \cdot \mathbf{v} \geq 0) B(\frac{\mathbf{x}}{|\mathbf{x}|}, \mathbf{v}). \tag{3.6}$$

and therefore Eq. (3.1) can be regarded as a modification (mollification) of the Boltzmann equation (1.2). A mollified kinetic equation was first proposed by Morgenstern [Mo] and, more generally, by Povzner [Po].

Equation (3.1) is referred to as the stochastic kinetic equation (or Povzner equation). Actually, Povzner considered Eq. (3.1) assuming (3.5). Under the same assumption a class of more general equations was investigated by Bellomo and Polewczak [BP].

One can expect that the solutions of Eq. (3.1) with (3.4), in the limit $r < R \downarrow 0$ (and for fixed $\varepsilon > 0$), approach the corresponding solution to the Boltzmann equation (1.2). In fact ([AC2]), in the limit $a \downarrow 0$, the solutions of the Boltzmann-Enskog equation (2.1) converge to the (DiPerna–Lions — [DL]) renormalized solutions of the Boltzmann equation. The same certainly should be true for the convergence of the solutions of Eq. (3.1).

On the other hand, in the limit $r \uparrow R$ (where both $R = a > 0$ and $\varepsilon > 0$ remain fixed) the solution of Eq. (3.1) with (3.4) should approach the corresponding solution of the Boltzmann–Enskog equation (2.1).

Equation (3.1) has nice mathematical properties, which are lacking for the Boltzmann equation (1.2): the collision operator S acts in the space $L_1(\Omega)$. Therefore Eq. (3.1) preserves some mathematical properties of the space homogeneous Boltzmann equation. In particular, the existence of unique smooth global solutions is known ([Mo], [Po], [An3], [La1], [BP]).

The theory of hydrodynamic limits for Eq. (3.1) was proposed in [La3,4].

Different relations between the small parameters ε and R can lead to different hydrodynamic equations analogously with the Boltzmann–Enskog case.

If

$$R = b\varepsilon\,, \qquad b = \text{const}\,, \tag{3.7}$$

then Eq. (3.1) results in the following Euler–type system

$$\frac{\partial \varrho}{\partial t} + \sum_{i=1}^{3} \frac{\partial}{\partial x_i}(\varrho u_i) = 0\,, \tag{3.8a}$$

$$\varrho\frac{\partial u_j}{\partial t} + \varrho\sum_{i=1}^{3} u_i \frac{\partial u_j}{\partial x_i} + \frac{\partial}{\partial x_j}(\varrho T) + b\sum_{i=1}^{3}\frac{\partial}{\partial x_i}\left(\varrho^2 \mathfrak{p}_{ij}(T)\right) = 0\,, \qquad j = 1,2,3\,, \tag{3.8b}$$

$$\frac{\partial T}{\partial t} + \sum_{i=1}^{3} u_i \frac{\partial T}{\partial x_i} + \frac{2}{3}T\sum_{i=1}^{3}\frac{\partial u_i}{\partial x_i} + \frac{2}{3}b\varrho\sum_{i,j=1}^{3}\mathfrak{p}_{ij}(T)\frac{\partial u_j}{\partial x_i} = 0 \tag{3.8c}$$

where

$$\mathfrak{p}_{ij}(T) = \frac{\sqrt{2T}}{2\pi^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} n_i n_j (\mathbf{n}\cdot(\mathbf{w}-\mathbf{v})\vee 0) B(\mathbf{n}, \sqrt{2T}(\mathbf{w}-\mathbf{v}))\exp(-|\mathbf{v}|^2 - |\mathbf{w}|^2)\,\mathrm{d}\mathbf{n}\,\mathrm{d}\mathbf{w}\,\mathrm{d}\mathbf{v}\,.$$

In the case of B corresponding to the hard sphere potential (1.3),

$$\mathfrak{p}_{ij}(T) = \frac{2}{3}\pi T \delta_{ij}, \tag{3.9}$$

where $\delta_{ii} = 1$, for $i = 1, 2, 3$, and $\delta_{ij} = 0$, for $i \neq j$, and System (3.8) with (3.9) becomes the Enskog–Euler system (2.4).

Note that setting $b = 0$ one recovers from System (3.8) — the classical system of the Euler equations for compressible fluids.

In much the same way as for the Boltzmann–Enskog equation (2.1), the existence and convergence results — analogous to those of Theorem (2.1) — may be proved for Eq. (3.1) and System (3.8). As a by product of this theory, the existence theorem for System (3.8) may be obtained.

The convergence rate, which could be found by the methods of [La5], was not sufficiently strong for a justification of the Hilbert procedure in Case (3.7). The situation is better for

$$R = b\varepsilon^p, \qquad b = \text{const}, \qquad p > 6. \tag{3.10}$$

In this case, however, Eq. (3.1) results in the classical Euler system ("$b = 0$") rather than in System (3.8) with $b > 0$ — see [La4].

4. SYMMETRIZED ENSKOG AND STOCHASTIC KINETIC EQUATIONS.

The symmetrized Boltzmann–Enskog equation is defined as follows

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\frac{\partial}{\partial \mathbf{x}}\right) f = \frac{1}{\varepsilon}\tilde{E}_a(f); \tag{4.1}$$

$$\tilde{E}_a(f)(\mathbf{x},\mathbf{v}) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{S}^2}\Big(f(\mathbf{x}+a\mathbf{n},\mathbf{w}')f(\mathbf{x},\mathbf{v}') - f(\mathbf{x}-a\mathbf{n},\mathbf{w})f(\mathbf{x},\mathbf{v})\Big)\big|\mathbf{n}\cdot(\mathbf{w}-\mathbf{v})\big|d\mathbf{n}d\mathbf{w}.$$

The symmetrization of the kernel in the Boltzmann–Enskog equation was introduced by Arkeryd [Ar1]. In the physical case there are collisions only if $\mathbf{n}\cdot(\mathbf{w}-\mathbf{v}) \geq 0$, while, for mathematical purposes, Arkeryd used the whole $\mathbb{S}^2$ as a range of integration, with the same velocities $\mathbf{v}'$, $\mathbf{w}'$, for both $\mathbf{n}$ and $-\mathbf{n}$.

Under the assumption

$$a = b\sqrt{\varepsilon}, \qquad b = \text{const}, \tag{4.2}$$

Equation (4.1) results ([La3,4]) in the following system (a Navier–Stokes–type system)

$$\frac{\partial \varrho}{\partial t} + \sum_{i=1}^{3}\frac{\partial}{\partial x_i}(\varrho u_i) = 0, \tag{4.3a}$$

$$\varrho\frac{\partial u_j}{\partial t}+\varrho\sum_{i=1}^{3}u_i\frac{\partial u_j}{\partial x_i}+\frac{\partial}{\partial x_j}(\varrho T)=\frac{8\sqrt{\pi}}{15}b\sum_{i=1}^{3}\frac{\partial}{\partial x_i}\Big(\varrho^2\sqrt{T}(\frac{\partial u_j}{\partial x_i}+\frac{\partial u_i}{\partial x_j})\Big)+ \tag{4.3b}$$
$$\frac{8\sqrt{\pi}}{15}b\frac{\partial}{\partial x_j}\Big(\varrho^2\sqrt{T}\sum_{k=1}^{3}\frac{\partial u_k}{\partial x_k}\Big), \qquad j=1,2,3,$$

$$\varrho\frac{\partial T}{\partial t}+\varrho\sum_{i=1}^{3}u_i\frac{\partial T}{\partial x_i}+\frac{2}{3}\varrho T\sum_{i=1}^{3}\frac{\partial u_i}{\partial x_i}=\frac{16\sqrt{\pi}}{45}b\varrho^2\sqrt{T}\sum_{i,j=1}^{3}\Big(\frac{\partial u_j}{\partial x_i}+\frac{\partial u_i}{\partial x_j}\Big)\frac{\partial u_j}{\partial x_i}+ \tag{4.3c}$$
$$\frac{16\sqrt{\pi}}{45}b\varrho^2\sqrt{T}\sum_{j=1}^{3}\Big(\sum_{i=1}^{3}\frac{\partial u_i}{\partial x_i}\Big)^2+\frac{8\sqrt{\pi}}{9}b\sum_{i=1}^{3}\frac{\partial}{\partial x_i}\Big(\varrho^2\sqrt{T}\frac{\partial T}{\partial x_i}\Big).$$

The viscosity and heat conduction terms of System (4.3) are independent of the small parameters.

The symmetrized stochastic kinetic equation (the Povzner equation) is defined by (3.1) with (3.5). Here it is assumed that

$$\mathcal{B}(\mathbf{x},\mathbf{v})=\frac{3}{R^3-r^3}\chi(r\le|\mathbf{x}|\le R)B(\frac{\mathbf{x}}{|\mathbf{x}|},\mathbf{v}), \qquad \text{for} \quad \mathbf{x}\cdot\mathbf{v}\ge 0, \tag{4.4a}$$

$$\mathcal{B}(\mathbf{x},\mathbf{v})=\mathcal{B}(-\mathbf{x},\mathbf{v}), \tag{4.4b}$$

where $0<r<R<+\infty$ and B corresponds to Grad's cut–off hard potentials.

Under Assumptions (4.4), (4.2), and for

$$R=b\sqrt{\varepsilon}, \qquad b=\text{const}, \tag{4.5}$$

Equation (3.1) results ([La3,4]) in the followig system

$$\frac{\partial\varrho}{\partial t}+\sum_{i=1}^{3}\frac{\partial}{\partial x_i}(\varrho u_i)=0, \tag{4.6a}$$

$$\varrho\frac{\partial u_j}{\partial t}+\varrho\sum_{i=1}^{3}u_i\frac{\partial u_j}{\partial x_i}+\frac{\partial}{\partial x_j}(\varrho T)=b\sum_{i,k,l=1}^{3}\frac{\partial}{\partial x_i}\Big(\varrho^2\mu_{ijkl}(T)\frac{\partial u_l}{\partial x_k}\Big), \qquad j=1,2,3, \tag{4.6b}$$

$$\varrho\frac{\partial T}{\partial t}+\varrho\sum_{i=1}^{3}u_i\frac{\partial T}{\partial x_i}+\frac{2}{3}\varrho T\sum_{i=1}^{3}\frac{\partial u_i}{\partial x_i}= \tag{4.6c}$$
$$\frac{2}{3}b\varrho^2\sum_{i,j,k,l=1}^{3}\mu_{ijkl}(T)\frac{\partial u_j}{\partial x_i}\frac{\partial u_l}{\partial x_k}+\frac{2}{3}b\sum_{i,j=1}^{3}\frac{\partial}{\partial x_i}\Big(\varrho^2\mu_{ij}^H(T)\frac{\partial T}{\partial x_j}\Big),$$

where

$$\mu_{ijkl}(T)=\frac{1}{\pi^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}n_in_jn_k\Big(\mathbf{n}\cdot(\mathbf{w}-\mathbf{v})\vee 0\Big)\times$$
$$B\big(\mathbf{n},\sqrt{2T}(\mathbf{w}-\mathbf{v})\big)(w_l-v_l)\exp\big(-|\mathbf{w}|^2-|\mathbf{v}|^2\big)\,\mathrm{d}\mathbf{n}\,\mathrm{d}\mathbf{w}\,\mathrm{d}\mathbf{v}$$

and

$$\mu_{ij}^H(T) = \frac{1}{\pi^3} \int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{S}^2} n_i n_j \Big(\mathbf{n}\cdot(\mathbf{w}-\mathbf{v}) \vee 0\Big) \times$$
$$\mathbf{n}\cdot(\mathbf{w}+\mathbf{v}) B\big(\mathbf{n}, \sqrt{2T}(\mathbf{w}-\mathbf{v})\big)(|\mathbf{w}|^2 - |\mathbf{v}|^2)\exp\big(-|\mathbf{w}|^2 - |\mathbf{v}|^2\big)\, d\mathbf{n}\, d\mathbf{w}\, d\mathbf{v}\,.$$

System (4.6) is a Navier-Stokes-type system of equations of compressible fluids ([VZ], [Li2]). Its viscosity and heat conduction terms are independent of the small parameters ε, r, R of Eq. (3.1). In the case of B corresponding to the hard sphere potential (1.3), System (4.6) becomes (4.3).

In papers [La3,4] it was proved that the existence of smooth solutions of System (4.3) (or System (4.6)) implies both existence of solution to Eq. (4.1) (or Eq. (3.1) with (4.4)) and the corresponding asymptotic relationship (Approach A$\uparrow$).

In order to state these theorems — some preliminaries are needed.

Let $\mathbb{B}_\infty(W_{\mathbb{R}^3})$ be the space of continuous, real-valued functions on $\mathbb{R}^3$ with the norm

$$\|f;\mathbb{B}_\infty(W_{\mathbb{R}^3})\| = \sup_{\mathbb{R}^3} |W_{\mathbb{R}^3} f|\,,$$

for a strictly positive, smooth function $W_{\mathbb{R}^3}$ on $\mathbb{R}^3$.

Let $\mathbb{B}_\infty^\alpha = \mathbb{B}_\infty(\langle.\rangle^\alpha)$ and $\mathbb{B}_\infty^\alpha(W_{\mathbb{R}^3}) = \mathbb{B}_\infty(\langle.\rangle^\alpha W_{\mathbb{R}^3})$.

$C^k(\mathbb{T}^3, W_{\mathbb{T}^3})$ is the space of the functions which are continuous, together with all their derivatives of orders $|\gamma| \le k$, and equipped with the norm

$$\|f; C^k(\mathbb{T}^3; W_{\mathbb{T}^3})\| = \sup_{\substack{0\le|\gamma|\le k\\ \mathbf{x}\in\mathbb{T}^3}} \left| W_{\mathbb{T}^3} \frac{\partial^{|\gamma|} f}{\partial \mathbf{x}^\gamma}\right|\,,$$

for a strictly positive, smooth function $W_{\mathbb{T}^3}$ on $\mathbb{T}^3$; Let $C^k(\mathbb{T}^3) = C^k(\mathbb{T}^3; 1)$.

The basic assumption is the following

Assumption 4.1. *Let ϱ, $\mathbf{u}$, T be smooth functions*

$$\varrho\,,\, u_i\,,\, T\,:\, [0,t_0]\times\mathbb{T}^3 \to \mathbb{R}^1\,, \qquad i = 1,2,3\,,$$

for some $t_0 \in]0,+\infty[$, such that

$$\varrho(t,\mathbf{x}) > c_1 > 0\,, \qquad T(t,\mathbf{x}) > c_2 > 0\,, \qquad \forall\, t\in[0,t_0]\,, \quad \forall\, \mathbf{x}\in\mathbb{T}^3\,,$$

where c_1 and c_2 are constants (independent of the small parameters).

Now let $M = M[\varrho, \mathbf{u}, T]$, where ϱ, $\mathbf{u}$, T are as in Assumption 4.1. Let

$$M_0 = M\big|_{t=0} = M\left[\varrho\big|_{t=0}, \mathbf{u}\big|_{t=0}, T\big|_{t=0}\right].$$

A simple consequence of Assumption 4.1 is the existence, for each $\alpha\in\mathbb{R}^1$, of positive constants c^-, c^+, T^- and T^+ — independent of ε — and such that

$$c^-\omega^-(\mathbf{v}) \le \langle\mathbf{v}\rangle^\alpha M(t,\mathbf{x},\mathbf{v}) \le c^+\omega^+(\mathbf{v})\,, \quad \forall\, t\in[0,t_0], \forall\,\mathbf{x}\in\mathbb{T}^3, \forall\,\mathbf{v}\in\mathbb{R}^3\,, \qquad (4.7)$$

where ω^- and ω^+ are the global Maxwellians $\omega^- = M[1,\mathbf{0},T^-]$, $\omega^+ = M[1,\mathbf{0},T^+]$.

Let $\mathbb{Y}_0^{\alpha,k}$ and $\mathbb{Y}^{\alpha,k}$ be the spaces equipped with the norms:

$$\|\cdot\|_0^{\alpha,k} = \|(\|\cdot; C^k(\mathbb{T}^3; M_0^{-\frac{1}{2}})\|); \mathbb{B}_\infty^\alpha\|,$$
$$\|\cdot\|^{\alpha,k} = \|(\|\cdot; C^k(\mathbb{T}^3)\|); \mathbb{B}_\infty^\alpha((\omega^+)^{-\frac{1}{2}})\|,$$

respectively.

Theorem 4.1. ([La3,4]) *Let* (4.2) (*or* (4.5)) *hold, and let* $t_0 \in]0,+\infty[$ *be such that on the time interval* $[0,t_0]$ *there exists a solution* $(\varrho,\mathbf{u},T)$ *of System* (4.3) (*or System* (4.6)) *satisfying* Assumption 4.1; *Let the initial data be such that*

$$f\big|_{t=0} = M_0 + G, \tag{4.8}$$

where $M_0 = M\left[\varrho\big|_{t=0},\mathbf{u}\big|_{t=0},T\big|_{t=0}\right]$ *and* G *satisfies the smallness condition*

$$\|G\|_0^{0,4} \le \kappa, \tag{4.9a}$$

and

$$\int G\mathrm{d}\mathbf{v} = \int v_i G\mathrm{d}\mathbf{v} = \int |\mathbf{v}|^2 G\mathrm{d}\mathbf{v} = 0, \qquad i = 1,2,3, \tag{4.9b}$$

$$G \in \mathbb{Y}_0^{\alpha,k}, \tag{4.9c}$$

with α *and* k *large enough and where* κ *is a given positive constant (independent of* ε*).*

If $0 < \varepsilon \le \varepsilon_0$, *where* ε_0 *is a given positive constant (depending on* t_0*), then a solution* f *of the Cauchy problem for Eq.* (4.1) (*or for Eq.* (3.1), (4.4)) *exists in* $L_\infty(0,t_0;\mathbb{Y}^{\alpha_0,k_0})$, *for some* $\alpha_0 > 0$, $k_0 > 0$, *and*

$$\sup_{0\le t\le t_0} \|f(t) - M[\varrho,\mathbf{u},T](t) - L\left(\frac{t}{\varepsilon}\right)\|^{\alpha_0,k_0} \le c\varepsilon, \tag{4.10}$$

where c *is a constant (depending on* t_0*) and* $L \in C^0\left([0,\infty[;\mathbb{Y}_0^{\alpha_1,k_1}\right)$ *is such that*

$$\|L(\tau)\|_0^{\alpha_1,k_1} \le c_G e^{-\delta\tau}, \tag{4.11}$$

for some $\delta > 0$, $\alpha_1 \ge \alpha_0$ *and* $k_1 \ge k_0$; c_G *is a constant depending on* G *and such that* $c_G = 0$ *for* $G = 0$. *Moreover,*

$$f \in C^0\left([0,t_0];\mathbb{Y}^{\alpha_0,k_0-1}\right) \cap C^1\left(]0,t_0[;\mathbb{Y}^{\alpha_0,k_0-2}\right) \tag{4.12}$$

(*provided that* $k_0 \ge 2$).

5. BOLTZMANN EQUATION WITH DISSIPATIVE COLLISIONS. The Boltzmann equation with dissipative collisions ([ElP]) is defined as follows

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\frac{\partial}{\partial \mathbf{x}}\right)f = \frac{1}{\varepsilon}J_\beta(f), \tag{5.1}$$

where

$$J_\beta(f)(\mathbf{x},\mathbf{v}) =$$

$$\int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{S}^2}\Big(\frac{1}{(2\beta-1)^2}f(\mathbf{x},\hat{\mathbf{w}})f(\mathbf{x},\hat{\mathbf{v}}) - f(\mathbf{x},\mathbf{v})f(t,\mathbf{x},\mathbf{w})\Big)\,(\mathbf{n}\cdot(\mathbf{w}-\mathbf{v})\vee 0)\,\mathrm{d}\mathbf{n}\,\mathrm{d}\mathbf{w}\,,$$

$$\hat{\mathbf{v}} = \mathbf{v} + \frac{1-\beta}{1-2\beta}\mathbf{n}\cdot(\mathbf{w}-\mathbf{v})\,, \qquad \hat{\mathbf{w}} = \mathbf{w} - \frac{1-\beta}{1-2\beta}\mathbf{n}\cdot(\mathbf{w}-\mathbf{v})\,, \tag{5.2}$$

are the velocities before a collision which produce the velocities $\mathbf{v}$, $\mathbf{w}$, respectively, after the collision, $\beta\in]0,\frac{1}{2}[$ is a dimensionless parameter characterizing energy dissipation.

The unique solution to the degenerate equation (for $\beta > 0$ – fixed)

$$J_\beta(f) = 0 \tag{5.3}$$

is the trivial one $f\equiv 0$ — see [BEL].

Under the assumption

$$\beta = b\varepsilon\,, \qquad b = \text{const}\,, \tag{5.4}$$

Equation (5.1) formally results (cf. [BEL]) in the following Euler system

$$\frac{\partial\varrho}{\partial t} + \sum_{i=1}^{3}\frac{\partial}{\partial x_i}(\varrho u_i) = 0\,, \tag{5.5a}$$

$$\varrho\frac{\partial u_j}{\partial t} + \varrho\sum_{i=1}^{3}u_i\frac{\partial u_j}{\partial x_i} + \frac{\partial}{\partial x_j}(\varrho T) = 0\,, \qquad j = 1,2,3\,, \tag{5.5b}$$

$$\frac{\partial T}{\partial t} + \sum_{i=1}^{3}u_i\frac{\partial T}{\partial x_i} + \frac{2}{3}T\sum_{i=1}^{3}\frac{\partial u_i}{\partial x_i} = -c_0\,b\varrho T^{3/2}\,, \tag{5.5c}$$

where c_0 is a given positive constant.

6. UEHLING–UHLENBECK QUANTUM EQUATION. The Uehling–Uhlenbeck equation ([UU], [CC], [KB] and [Su], [Do], [Li1]), describing the evolution of a gas of quantum particles, is defined as follows

$$(\frac{\partial}{\partial t} + \mathbf{v}\cdot\frac{\partial}{\partial \mathbf{x}})f = \frac{1}{\varepsilon}\Omega_{\frac{\lambda}{\varepsilon}}(f)\,, \tag{6.1}$$

$$\Omega_{\frac{\lambda}{\varepsilon}}(f)(\mathbf{x},\mathbf{v}) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{S}^2}\Big(f(\mathbf{x},\mathbf{v}')f(\mathbf{x},\mathbf{w}')\Big(1-\frac{\lambda}{\varepsilon}f(\mathbf{x},\mathbf{v})\Big)\Big(1-\frac{\lambda}{\varepsilon}f(\mathbf{x},\mathbf{w})\Big) -$$

$$f(\mathbf{x},\mathbf{v})f(\mathbf{x},\mathbf{w})\Big(1-\frac{\lambda}{\varepsilon}f(\mathbf{x},\mathbf{v}')\Big)\Big(1-\frac{\lambda}{\varepsilon}f(\mathbf{x},\mathbf{w}')\Big)\Big)B(\mathbf{w}-\mathbf{v},\mathbf{n})\mathrm{d}\mathbf{n}\mathrm{d}\mathbf{w}\,;$$

λ is a parameter (proportional to $\hbar^3$), that is

(+) positive for fermions and

(—) negative for bosons.

One can expect that the class of solutions of the degenerate equation corresponding to the singularly perturbed equation (6.1), for $\varepsilon \to 0$ and fixed $\lambda > 0$, is too small to describe a reasonable hydrodynamic. Therefore the following limit

$$\varepsilon \downarrow 0\,, \qquad \lambda \downarrow 0\,, \tag{6.2}$$

should be considered.

One can distinguish the following important cases

(i.) $|\lambda| \ll \varepsilon$, (e.g. $|\lambda| = \varepsilon^p$ for $p > 1$) — it can be treated in the same way as the corresponding problem for the classical Boltzmann equation;

(ii.) $|\lambda| \gg \varepsilon$, (e.g. $|\lambda| = \varepsilon^p$ for $p < 1$) — it is not physically consistent;

(iii.) $|\lambda| \sim \varepsilon$, (e.g. $|\lambda| = \varepsilon$).

Consider the two cases

$$\lambda = \varepsilon \tag{6.3a}$$

— fermions — the subscript $+$ is used, e.g. $\mathfrak{Q}_+ = \mathfrak{Q}_{+1}$, and

$$\lambda = -\varepsilon \tag{6.3b}$$

— bosons — the subscript $-$ is used, e.g. $\mathfrak{Q}_- = \mathfrak{Q}_{-1}$.

Solutions to the degenerate equations

$$\mathfrak{Q}_\pm(f) = 0 \tag{6.4}$$

are given by

$$\mathfrak{F}_\pm(t,x,v) = \frac{M(t,x,v)}{1 \pm M(t,x,v)}\,, \tag{6.5}$$

where $M = M_\pm[\gamma, \mathbf{u}, \tau]$ are Maxwellians with fluid–dynamic parameters $\gamma = \gamma_\pm$, $\mathbf{u} = \mathbf{u}_\pm$, $\tau = \tau_\pm$; $\mathfrak{F}_+$ is called the Fermi–Dirac distribution, whereas $\mathfrak{F}_-$ — the Bose–Einstein distribution.

The fluid dynamic parameters $\varrho_\pm$, $\mathbf{u}_\pm$, $\mathfrak{e}_\pm$ of $\mathfrak{F}_\pm$ are related to the fluid–dynamic parameters $\gamma_\pm$, $u_\pm$ and $\tau_\pm$ of the corresponding Maxwellians by some given functions (see (2.10) in [AL2]).

Formally, in Cases (6.3), Eqs. (5.1) results in the classical Euler system at the 0–th order of approximation ([AL1]) and in the following Navier–Stokes system at the 1–st order of approximation ([La2])

$$\frac{\partial \varrho}{\partial t} + \sum_{i=1}^{3} \frac{\partial}{\partial x_i}(\varrho u_i) = 0\,, \tag{6.6a}$$

$$\varrho\frac{\partial u_j}{\partial t}+\varrho\sum_{i=1}^{3}u_i\frac{\partial u_j}{\partial x_i}+\frac{2}{3}\frac{\partial}{\partial x_j}(\varrho\mathfrak{e})=\varepsilon\Bigg(\sum_{i=1}^{3}\frac{\partial}{\partial x_i}\left(\mu_\pm(\varrho,\mathfrak{e})\left(\frac{\partial u_j}{\partial x_i}+\frac{\partial u_i}{\partial x_j}\right)\right)-$$
$$\frac{2}{3}\frac{\partial}{\partial x_j}\left(\mu_\pm(\varrho,\mathfrak{e})\sum_{i=1}^{3}\frac{\partial u_i}{\partial x_i}\right)\Bigg)\,,\qquad j=1,2,3\,, \tag{6.6b}$$

$$\frac{\partial\mathfrak{e}}{\partial t}+\sum_{i=1}^{3}u_i\frac{\partial\mathfrak{e}}{\partial x_i}+\frac{2}{3}\mathfrak{e}\sum_{i=1}^{3}\frac{\partial u_i}{\partial x_i}=\frac{\varepsilon}{\mathcal{G}_\pm(\varrho,\mathfrak{e})}\Bigg(\sum_{i=1}^{3}\frac{\partial}{\partial x_i}\left(\mu_\pm^{(1)}(\varrho,\mathfrak{e})\frac{\partial\mathfrak{e}}{\partial x_i}+\mu_\pm^{(2)}(\varrho,\mathfrak{e})\frac{\partial\varrho}{\partial x_i}\right)+$$
$$\mu_\pm(\varrho,\mathfrak{e})\sum_{i,k=1}^{3}\frac{\partial u_k}{\partial x_i}\left(\frac{\partial u_k}{\partial x_i}+\frac{\partial u_i}{\partial x_k}\right)-\frac{2}{3}\mu_\pm(\varrho,\mathfrak{e})\Big(\sum_{i=1}^{3}\frac{\partial u_i}{\partial x_i}\Big)^2\Bigg)\,, \tag{6.6c}$$

where $\mu_\pm(\varrho,\mathfrak{e})$, $\mathcal{G}_\pm(\varrho,\mathfrak{e})$, $\mu_\pm^{(1)}(\varrho,\mathfrak{e})$, $\mu_\pm^{(2)}(\varrho,\mathfrak{e})$ are given functions (cf. [AL2]) depending on B.

In papers [AL1,2] it was proved that the existence of smooth solutions of the Euler system or of System (6.6) implies both existence of solution to the kinetic equations (6.1) and the corresponding asymptotic relationships (Approach A$\uparrow$).

The results were obtained under the assumption that the solution $(\varrho,\mathbf{u},\mathfrak{e})$ to the hydrodynamic system satisfies the following inequality

$$\mathfrak{e}>l_\pm\varrho^{\frac{2}{3}}\,, \tag{6.7}$$

for every $(t,\mathbf{x})\in[0,t_0]\times\mathbb{T}^3$, where l_+ and l_- are given positive constants, respectively, for fermions $(+)$ and for bosons $(-)$. Inequalities (6.7) are essential for the asymptotic relationships (see [AL2]). In fact, under Conditions (6.7) the correspondence between $(\gamma_\pm,\mathbf{u}_\pm,\tau_\pm)$ and the fluid–dynamic parameters $(\varrho_\pm,\mathbf{u}_\pm,\mathfrak{e}_\pm)$ of $\mathfrak{F}_\pm$ is one–to–one, respectively for $+$ and $-$.

In order to state the theorems from [AL2] — some preliminaries are needed.

Assumption 6.1. *Let $\gamma_\pm$, $\mathbf{u}_\pm$ and $\tau_\pm$ be smooth functions*

$$\gamma_\pm\,,\ u_{i\pm}\,,\ \tau_\pm\ :\ [0,t_0]\times\mathbb{T}^3\to\mathbb{R}^1\,,\qquad i=1,2,3\,,$$

for some $t_0\in]0,+\infty[$, such that

$$\gamma_\pm(t,\mathbf{x})>c_1>0\,,\qquad \tau_\pm(t,\mathbf{x})>c_2>0\,,\qquad \forall t\in[0,t_0]\,,\quad \forall\mathbf{x}\in\mathbb{T}^3\,, \tag{6.8}$$

where c_1 and c_2 are constants. Moreover let

$$\gamma_-(t,\mathbf{x})\le\delta<1\,,\qquad \forall t\in[0,t_0]\,,\quad \forall\mathbf{x}\in\mathbb{T}^3\,. \tag{6.9}$$

Consider Maxwellians with parameters $\gamma_\pm$, $\mathbf{u}_\pm$, $\tau_\pm$ satisfying Assumption 6.1 and let $\mathfrak{F}_\pm$ be defined by Eqs. (6.5).

A simple consequence of Assumption 6.1 is the existence, for each $\alpha \in \mathbb{R}^1$, of positive constants $c_\pm^-$, $c_\pm^+$, $\tau_\pm^-$, $\tau_\pm^+$ — independent of ε — and such that

$$c_\pm^- \omega_\pm^-(\mathbf{v}) \le \langle \mathbf{v} \rangle^\alpha \mathfrak{F}_\pm(t, \mathbf{x}, \mathbf{v}) \le c_\pm^+ \omega_\pm^+(\mathbf{v}) \qquad \forall t \in [0, t_0], \ \forall \mathbf{x} \in \mathbb{T}^3, \ \forall \mathbf{v} \in \mathbb{R}^3, \quad (6.10)$$

where $\omega_\pm^-$ and $\omega_\pm^+$ are the global Maxwellians $\omega_\pm^- = M[1, \mathbf{0}, \tau_\pm^-]$, $\omega_\pm^+ = M[1, \mathbf{0}, \tau_\pm^+]$.

Let $\mathbb{Y}_\pm^{\alpha,k}$ and $\| \cdot \|_\pm^{\alpha,k}$ be defined as $\mathbb{Y}^{\alpha,k}$ and $\| \cdot \|^{\alpha,k}$ in Section 4, but with $\omega_\pm^+$ instead of ω^+.

The results of [AL2] both for the classical Euler system and for the Navier–Stokes system (6.6) as well as both for fermions and for bosons can be summarized as follows

Theorem 6.1. *Let either of Conditions* (6.3) *hold and let* $t_0 \in]0, \infty[$ *be independent of* ε *such that on the time interval* $[0, t_0]$ *there exists a smooth solution* $(\varrho_\pm, \mathbf{u}_\pm, \mathfrak{e}_\pm)$ *of the Navier–Stokes system* (6.6) *(or the classical Euler system) that satisfies* (6.7) *and corresponds to parameters* $(\gamma_\pm, \mathbf{u}_\pm, \tau_\pm)$ *satisfying Assumption* 6.1 *together with the conditions of uniformly boundedness with respect to* ε *of the quantities*

$$\sup_{[0,t_0]\times\mathbb{T}^3} \gamma_\pm, \qquad \sup_{[0,t_0]\times\mathbb{T}^3} |\mathbf{u}_\pm|, \qquad \sup_{[0,t_0]\times\mathbb{T}^3} \tau_\pm;$$

Let the initial data be either of the functions (6.5) *with parameters* $\gamma_\pm\big|_{t=0}$, $\mathbf{u}_\pm\big|_{t=0}$, $\tau_\pm\big|_{t=0}$ *corresponding to* $\varrho_\pm\big|_{t=0}$, $\mathbf{u}_\pm\big|_{t=0}$, $\mathfrak{e}_\pm\big|_{t=0}$.

If $0 < \varepsilon \le \varepsilon_0$, *where* ε_0 *is a given positive constant (depending on* t_0*), then a solution* f *of the Cauchy problem for Eq.* (6.1) *exists in* $L_\infty(0, t_0; \mathbb{Y}_\pm^{\alpha,k})$, *for some* $\alpha > 0$, $k > 0$, *and*

$$f \in C^0\left([0, t_0]; \mathbb{Y}_\pm^{\alpha,k-1}\right) \cap C^1\left(]0, t_0[; \mathbb{Y}_\pm^{\alpha,k-2}\right) \quad (\text{if } \ k \ge 2), \qquad (6.10)$$

$$\sup_{[0,t_0]} \left\| f - \mathfrak{F}_\pm \right\|_\pm^{\alpha,k} \le c\,\varepsilon, \qquad (6.11)$$

where $\mathfrak{F}_\pm$ *is defined by parameters* $(\gamma_\pm,\ \mathbf{u}_\pm,\ \tau_\pm)$ *corresponding to* $(\varrho_\pm,\ \mathbf{u}_\pm,\ \mathfrak{e}_\pm)$ *and* c *is a positive constant (depending on* t_0*).*

An analogous theorem can be obtained for more general data if one includes initial layer as in Theorem (4.1).

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