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Complexity of Gröbner Bases for Toric Ideals of Acyclic Tournament Graphs

石関 隆幸 (Takayuki Ishizeki)†. 今井 浩 (Hiroshi Imai)†

†Department of Information Science, Faculty of Science, University of Tokyo
7-3-1 Hongo, Bunkyo-ku, Tokyo 113-0033, Japan
{ishizeki,imai}@is.s.u-tokyo.ac.jp

Abstract

Applications of Gröbner bases to some computationally hard problems in combinatorics using the discreteness of toric ideals have been studied in recent years. On the other hand, the properties of graphs may give insight into Gröbner bases. In this paper, we analyze toric ideals of acyclic tournament graphs, which are the most fundamental directed graphs. We focus especially on the number of elements of its reduced Gröbner bases. We show that there exist term orders for which reduced Gröbner bases remain in polynomial order by characterizing the bases in terms of circuits. We next analyze the number of elements of reduced Gröbner bases with respect to various term orders. We finally discuss applications to the minimum cost flow problem.

1 Introduction

Recently, some algebraic approaches to many computationally hard problems in combinatorics have been studied. The main tool is the *Gröbner basis*, which is an important tool in computational algebra and algebraic geometry. Gröbner bases have provided new insight into some combinatorial problems such as integer programming [2, 5, 6, 12] and computational statistics [6].

Related to some combinatorial problems in graph theory, toric ideals of graphs have been studied. De Loera, Sturmfels and Thomas [5] studied the toric ideals of undirected complete graphs, and applied them to the triangulation of second hypersimplex and perfect f-matching problem. Diaconis and Sturmfels [6] studied the toric ideals of bipartite graphs, and applied them for sampling from conditional distributions and transportation problem. From the viewpoint of in commutative algebra, Ohsugi and Hibi [10] studied the toric ideals of general undirected graphs, and showed the conditions when the toric ideals are generated by quadratic binomials. Conversely, the properties of graphs may give insight into Gröbner bases.

Gröbner bases of directed graphs are not well understood. In this paper, we study the toric ideals of acyclic tournament graphs, which are the most fundamental directed graphs. Any elements in the reduced Gröbner bases for toric ideals of these graphs correspond to the circuits in the graphs. So we can characterize the reduced Gröbner bases of toric ideals in terms of circuits.

We focus especially on the number of elements in reduced Gröbner bases. Analysis of the Gröbner bases of acyclic tournament graphs are very important. Acyclic tournament graphs contains any acyclic directed graphs as subgraphs, and undirected bipartite graphs can be regarded as the subgraphs of acyclic tournament graphs by directing each edge from one set of vertices in bipartite graphs to the other. By the elimination theorem(see [3]), reduced Gröbner bases of any subgraphs of acyclic tournament graphs can be obtained automatically if that of acyclic tournament graphs can be calculated. Thus the number of elements in reduced Gröbner bases of any subgraphs are less than those of acyclic tournament graphs. On the other hand, the number of elements in reduced Gröbner bases of graphs are related to the complexity of integer programming problem arising from the graphs.

In this paper, we show that the number of elements in reduced Gröbner bases remain in polynomial order by characterizing the bases in terms of circuits. We next analyze the number of elements of reduced Gröbner bases with respect to various term orders using TiGERS [8]. We finally discuss applications to the minimum cost flow problem on acyclic tournament graphs.

2 Preliminaries

In this section, we give basic definitions of Gröbner bases and toric ideals. We refer to [3, 4] for the introductions of Gröbner bases, and [11] for the introductions of toric ideals and their applications.

2.1 Gröbner Bases

Let k be a field and $k[x_1,\dots,x_n]$ be the ring of polynomials in n variables. For a non-negative integer vector $\alpha=(\alpha_1,\dots,\alpha_n)\in\mathbb{N}^n$, we write $x^\alpha := x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}$. We call α the *exponent vector* of monomial x^α.

Definition 2.1 *Let $\succ$ be a total order on $\mathbb{N}^n$. We call $\succ$ a* term order *on $\mathbb{N}^n$ if it satisfies the following:*

1. $^\forall\alpha,\beta,\gamma \in \mathbb{N}^n$, $\alpha \succ \beta \Longrightarrow \alpha+\gamma \succ \beta+\gamma$.
2. $^\forall\alpha \in \mathbb{N}^n \setminus \{0\}, \alpha \succ 0$

For a polynomial f and a term order $\succ$, we call the largest term in f with respect to $\succ$ initial term *of f and write $in_\succ(f)$, or short, $in(f)$.*

Remark 2.2 *In this paper, we line under the initial term of each polynomial.*

We give some examples of term orders.

Definition 2.3 *Fix a variable ordering $x_{i_1} \succ x_{i_2} \succ \cdots \succ x_{i_n}$. We say $\succ$ is a* purely lexicographic order *induced by this variable ordering if, for any α and β, $\alpha \succ \beta$ if and only if there exists $1 \le m \le n$ such that $\alpha_{i_k} = \beta_{i_k}$ for $k < m$ and $\alpha_{i_m} > \beta_{i_m}$.*

Definition 2.4 *Fix a variable ordering $x_{i_1} \succ x_{i_2} \succ \cdots \succ x_{i_n}$. We say $\succ$ is a* degree lexicographic order *induced by this variable ordering if, for any α and β, $\alpha \succ \beta$ if and only if $\sum_{i=1}^n \alpha_i > \sum_{i=1}^n \beta_i$ or $(\sum_{i=1}^n \alpha_i = \sum_{i=1}^n \beta_i$ and $\alpha \succ_{plex} \beta)$. ($\succ_{plex}$ is purely lexicographic order induced by $x_{i_1} \succ x_{i_2} \succ \cdots \succ x_{i_n}$.)*

Definition 2.5 *Let $\omega \in \mathbb{R}^n_{\ge 0}$ be a non-negative vector and $\succ$ an arbitrary term order. We define a* refinement *$\succ_\omega$ of ω with respect to $\succ$ as follows: for any α and β,*

$$\alpha \succ_\omega \beta \Longleftrightarrow \omega\cdot\alpha > \omega\cdot\beta \text{ or } (\omega\cdot\alpha = \omega\cdot\beta \text{ and } \alpha \succ \beta).$$

Definition 2.6 *Let I be an ideal in $k[x_1, \dots, k_n]$ and $\succ$ a term order. A finite subset $\mathcal{G} = \{g_1, \dots, g_s\} \subset I$ is a* reduced Gröbner basis *for I with respect to $\succ$ if $\mathcal{G}$ satisfies the following:*

1. *For any $f \in I$, there exists some $g_i \in \mathcal{G}$ such that $in_\succ(f)$ is divisible by $in_\succ(g_i)$.*
2. *For any i, the coefficient of g_i is 1.*
3. *For any i, any term of g_i is not divisible by $in_\succ(g_j)$ $(i \neq j)$.*

We give some properties of Gröbner basis.

Proposition 2.7 *The reduced Gröbner basis is unique for an ideal and a term order.*

Proposition 2.8 *For any term order $\succ$, a Gröbner basis for I with respect to $\succ$ is a basis for I.*

Definition 2.9 *We call a union of reduced Gröbner basis of I with respect to any term orders a* universal Gröbner basis *for I.*

Although there are infinite term orders, a universal Gröbner basis is finite.

Proposition 2.10 *Every ideal $I \subset k[x_1, \dots, x_n]$ has a finite universal Gröbner basis.*

We define "division" on multi-variable polynomial ring.

Theorem 2.11 *Fix a monomial order $\succ$ and a Gröbner basis $\mathcal{G} = \{g_1, \dots, g_s\}$ for I with respect to $\succ$. Then every $f \in k[x_1, \dots, x_n]$ can be written as*

$$f = a_1 g_1 + \cdots + a_s g_s + r, \ a_i, r \in k[x_1, \dots, x_n]$$

where either $r = 0$ or no term of r is divisible by any of $in_\succ(g_1), \dots, in_\succ(g_s)$. r is unique, and called normal form *of f by $\mathcal{G}$.*

2.2 Toric Ideals

In this section, we consider $A \in \mathbb{Z}^{d\times n}$ as a set of column vectors $\{\mathbf{a}_1, \dots, \mathbf{a}_n\}$. Each vector $\mathbf{a}_i$ is identified with a monomial $\mathbf{t}^{\mathbf{a}_i}$ in the Laurent polynomial ring $k[\mathbf{t}^{\pm 1}] := k[t_1, \dots, t_d, t_1^{-1}, \dots, t_d^{-1}]$.

Definition 2.12 *Consider the homomorphism*

$$\pi\colon k[x_1, \dots, x_n] \longrightarrow k[\mathbf{t}^{\pm 1}], \ x_i \longmapsto \mathbf{t}^{\mathbf{a}_i}.$$

The kernel of π is denoted I_A and called the toric ideal *of A.*

Every vector $\mathbf{u} \in \mathbb{Z}^n$ can be written uniquely as $\mathbf{u} = \mathbf{u}^+ - \mathbf{u}^-$ where $\mathbf{u}^+$ and $\mathbf{u}^-$ are non-negative and have disjoint support.

Lemma 2.13

$$I_A = \langle \mathbf{x}^{\mathbf{u}_i^+} - \mathbf{x}^{\mathbf{u}_i^-} : \mathbf{u}_i \in \ker(A) \cap \mathbb{Z}^n, i = 1, \dots, s\rangle$$

Furthermore, toric ideal is generated by finite binomials. (A binomial *is a polynomial which consists of two monomials.)*

Definition 2.14 *A binomial $\mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-} \in I_A$ is called* circuit *if the support of $\mathbf{u}$ is minimal with respect to inclusion in $\ker(A)$ and the coordinates of $\mathbf{u}$ are relatively prime. We denote the set of all circuits in I_A by $\mathcal{C}_A$.*

Definition 2.15 *A binomial $\mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-} \in I_A$ is called* primitive *if there exists no other binomial $\mathbf{x}^{\mathbf{v}^+} - \mathbf{x}^{\mathbf{v}^-} \in I_A$ such that both $\mathbf{u}^+ - \mathbf{v}^+$ and $\mathbf{u}^- - \mathbf{v}^-$ are non-negative. The set of all primitive binomials in I_A is called the* Graver basis *of A and written by Gr_A.*

Let $\mathcal{U}_A$ be the universal Gröbner basis of I_A.

Proposition 2.16 *$\mathcal{C}_A \subseteq \mathcal{U}_A \subseteq Gr_A$. If A is a unimodular matrix, then $\mathcal{C}_A = Gr_A$.*

2.3 Toric Ideals of Acyclic Tournament Graphs

Let D_n be an acyclic tournament graph with n vertices which have labels $1, 2, \ldots, n$ such that each edge (i, j) $(i < j)$ is directed from i to j. Let $m = \binom{n}{2}$ be the number of edges in D_n. We associate each edge (i, j) with a variable x_{ij}, and we consider the polynomial ring $k[x_{ij} : 1 \leq i < j \leq n]$. We analyze the toric ideal I_{A_n} of incidence matrix A_n of D_n. This ideal is not homogeneous with respect to the standard grading $\deg(x_{ij}) = 1$, but is homogeneous with respect to the grading $\deg(x_{ij}) = j - i$.

Remark 2.17 *In this paper, we define a* circuit *of D_n as a simple cycle.*

Definition 2.18 *Let C be a circuit of D_n. If we fix a direction of C, we can partition the edges of C into two sets C^+ and C^- such that C^+ is the set of forward edges and C^- is the set of backward edges. Then the vector $\mathbf{x} = (x_{ij})_{1 \leq i < j \leq n} \in \mathbb{R}^m$ defined by*

$$x_{ij} = \begin{cases} 1 & \text{if } (i,j) \in C^+ \\ -1 & \text{if } (i,j) \in C^- \\ 0 & \text{if } (i,j) \notin C \end{cases} \qquad (i,j) \in E$$

is called the incidence vector *of C.*

Lemma 2.19 ([1]) *A binomial $\mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-} \in I_{A_n}$ is a circuit if and only if $\mathbf{u}$ is the incidence vector of a circuit of D_n.*

By Proposition 2.16, $\mathcal{C}_{A_n} = \mathcal{U}_{A_n} = Gr_{A_n}$ since the incidence matrix A_n is unimodular.

Corollary 2.20 *The universal Gröbner basis $\mathcal{U}_{A_n}$ is the set of binomials which correspond to the circuits of D_n.*

Corollary 2.21 *The number of elements in $\mathcal{U}_{A_n}$ is of exponential order with respect to n.*

3 Some Gröbner bases of I_{A_n}

In this section, we show that the elements in reduced Gröbner bases with respect to some specific term orders can be given in terms of graphs. As a corollary, we can show that there exist term orders for which reduced Gröbner bases remain in polynomial order.

We first show the term order for which the elements in reduced Gröbner basis correspond to the circuits of length three and some circuits of length four of D_n.

Theorem 3.1 *Let $\succ_1$ be a purely lexicographic order induced by the following variable ordering:*

$$x_{ij} \succ x_{kl} \iff i < k \text{ or } (i = k \text{ and } j < l).$$

Let

$$g_{ijk} := \underline{x_{ij}x_{jk}} - x_{ik} \ (1 \leq i < j < k \leq n)$$
$$g_{ijkl} := \underline{x_{ik}x_{jl}} - x_{il}x_{jk} \ (1 \leq i < j < k < l \leq n).$$

Then reduced Gröbner basis $\mathcal{G}_1$ of I_{A_n} with respect to $\succ_1$ is

$$\begin{aligned} \mathcal{G}_1 &= \{g_{ijk} : 1 \leq i < j < k \leq n\} \\ &\cup \{g_{ijkl} : 1 \leq i < j < k < l \leq n\} \end{aligned}$$

In particular, the number of elements in $\mathcal{G}_1$ equals $\binom{n}{3} + \binom{n}{4}$.

The set $\{g_{ijk} : 1 \leq i < j < k \leq n\}$ corresponds to all of the circuits of length three, and $\{g_{ijkl} : 1 \leq i < j < k < l\}$ corresponds to some of the circuits of length four(Figure 1).

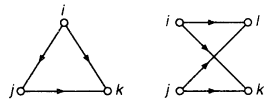

Figure 1: The circuit corresponding to g_{ijk} and the circuit corresponding to g_{ijkl}.

(**Proof**) For any circuit of length three defined by three vertices i, j, k $(i < j < k)$, the associated binomial equals $\underline{x_{ij}x_{jk}} - x_{ik}$, which is g_{ijk}.

The circuits defined by four vertices $i < j < k < l$ are $C_1 := i, j, k, l, i$, $C_2 := i, j, l, k, i$, $C_3 := i, k, j, l, i$ and their opposites. The binomial which corresponds to C_1 or its opposite is $\underline{x_{ij}x_{jk}x_{kl}} - x_{il}$, whose initial term is divisible by $in(g_{ijk})$. Similarly, the initial term of binomial which corresponds to C_2 or its opposite is divisible by $in(g_{ijl})$. The binomial which corresponds to C_3 or its opposite is g_{ijkl}.

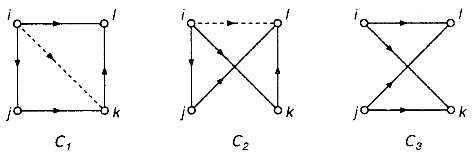

Figure 2: The circuits C_1, C_2, C_3.

Let C be a circuit of length more than 5. Let i_1 be the vertex whose label is minimum in C, and $C := i_1, i_2, \ldots, i_s, i_1$. Without loss of generality, we set $i_2 < i_s$. Let f_C be the binomial corresponding to C, then $in(f_C)$ is product of all variables whose associated edges have same direction with (i_1, i_2) on C. We show that $in(f_C)$ is divisible by initial term of a binomial in $\mathcal{G}_1$, which implies that $\mathcal{G}_1$ is Gröbner basis of I_{A_n} with respect to $\succ_1$.

If $i_2 < i_3$, then (i_1, i_2) and (i_2, i_3) have same direction on C. Thus the variables $x_{i_1 i_2}$ and $x_{i_2 i_3}$ appear in $in(f_C)$, and $in(f_C)$ is divisible by $in(g_{i_1 i_2 i_3})$ (Figure 3 left).

If $i_2 > i_3$, then since $i_3 < i_2 < i_s$, there exists k $(3 \leq k < s)$ such that $i_1 < i_k < i_2 < i_{k+1}$. Then the variables $x_{i_1 i_2}$ and $x_{i_k i_{k+1}}$ appear in $in(f_C)$, and $in(f_C)$ is divisible by $in(g_{i_1 i_k i_2 i_{k+1}})$ (Figure 3 right).

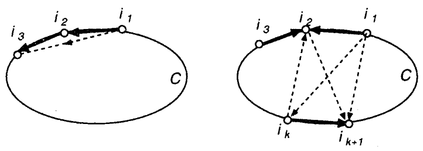

Figure 3: $x_{i_1 i_2}$ and $x_{i_2 i_3}$ (left) or $x_{i_1 i_2}$ and $x_{i_k i_{k+1}}$ (right) appear in $in(f_C)$.

Any term of g_{ijk} is not divisible by the initial term of any other binomials in $\mathcal{G}_1$, and so as g_{ijkl}. This implies that $\mathcal{G}_1$ is reduced. ∎

Next we show the term order for which the elements in reduced Gröbner basis correspond to the fundamental circuits for a certain spanning tree of D_n.

Theorem 3.2 *Let $\succ_2$ be a purely lexicographic order induced by the following variable ordering:*

$$x_{ij} \succ x_{kl} \iff i < k \text{ or } (i = k \text{ and } j > l).$$

For $1 \leq i < j - 1 < n$, let

$$g_{ij} := \underline{x_{ij}} - x_{i,i+1} x_{i+1,i+2} \cdots x_{j-1,j}$$

Then reduced Gröbner basis $\mathcal{G}_2$ of I_{A_n} with respect to $\succ_2$ is

$$\mathcal{G}_2 = \{g_{ij} : 1 \leq i < j - 1 < n\}.$$

In particular, the number of elements in $\mathcal{G}_2$ equals $\binom{n}{2} - (n-1)$.

The elements of reduced Gröbner basis $\mathcal{G}_2$ correspond to the fundamental circuits of D_n for the spanning tree $T := \{(i, i+1) : 1 \leq i < n\}$.

(**Proof**) Let C be a circuit which is not the fundamental circuit of T. Let i_1 be the vertex whose label is minimum in C, and $C := i_1, i_2, \ldots, i_s, i_1$. Without loss of generality, we set $i_2 < i_s$. Then the variable $x_{i_1 i_s}$ appears in the initial term of associated binomial f_C. Thus $in(f_C)$ is divisible by $in(g_{i_1 i_s})$.

The initial term of g_{ij} corresponds to an edge which is not contained in T, and other term corresponds to several edges which are contained in T. Thus any term of g_{ij} is not divisible by the initial term of other binomial in $\mathcal{G}_2$, which implies that $\mathcal{G}_2$ is reduced. ∎

We last show that there exist two term orders for which reduced Gröbner bases are same as $\mathcal{G}_1$.

Theorem 3.3 *Let $\prec_3$ be a purely lexicographic order induced by the following variable ordering:*

$$x_{ij} \succ x_{kl} \iff j < l \text{ or } (j = l \text{ and } i < k).$$

Then reduced Gröbner basis of I_{A_n} with respect to $\prec_3$ is same as $\mathcal{G}_1$ in Theorem 3.1.

(**Proof**) For the circuits of length less than four, we can show similarly as the proof of Theorem 3.1.

Let C be a circuit of length more than five. Let i_1 be the vertex whose label is minimum in C, and $C := i_1, i_2, \ldots, i_s, i_1$. Without loss of generality, we set $i_2 < i_s$. Let f_C be the associated binomial.

Let $T_C := \{i_s \in C : i_{s-1} < i_s\} \cup \{i_s \in C : i_{s+1} < i_s\}$. (We set $i_{s+1} = i_1$) This is the set of vertices which are the terminal points of edges in C. Let i_k be the vertex whose label is minimum in T_C.

If $k = 2$, then the variable $x_{i_1 i_2}$ is the maximum variable in f_C with respect to $\prec_3$. Then $in(f_C)$ is product of all variables whose associated edges have same direction with (i_1, i_2) on C. In this case, we can show that $\mathcal{G}_1$ is the reduced Gröbner basis with respect to $\prec_3$ by similar way as Theorem 3.1.

Let $k \neq 2$. If $i_{k-1} < i_k < i_{k+1}$ (Figure 4 left), the variable $x_{i_{k-1} i_k}$ is the maximum variable in f_C by the choice of k. Then the variables $x_{i_{k-1} i_k}$ and $x_{i_k i_{k+1}}$ appear in $in(f_C)$, and $in(f_C)$ is divisible by $in(g_{i_{k-1} i_k i_{k+1}})$. Similarly we can show for the case of $i_{k-1} > i_k > i_{k+1}$.

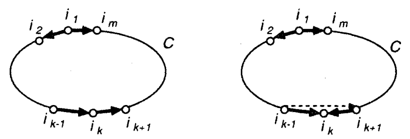

Figure 4: The cases $i_{k-1} < i_k < i_{k+1}$ (left) and $i_{k-1} < i_{k+1} < i_k$ (right).

Let $i_{k-1} < i_k$ and $i_{k+1} < i_k$ (Figure 4 right). If $i_{k-1} < i_{k+1}$, then the variable $x_{i_{k-1} i_k}$ is the maximum variable in f_C. Thus the variable $x_{i_{k-1} i_k}$ appears in $in(f_C)$. By the choice of k, it can be shown that $i_{k-1} < i_{k+1} < i_k < i_{k+2}$. (We set $i_{m+2} = i_2$.) In fact, if $i_{k+2} < i_{k+1}$ (Figure 5 left), then $i_{k+2} < i_{k+1} < i_k$. Thus i_{k+1} is the vertex whose label is minimum in T_C, which implies i_{k+1} contradicts the choice of k. If $i_{k+1} < i_{k+2} < i_k$ (Figure 5 right), then i_{k+2} contradicts the choice of k.

Since $i_{k-1} < i_{k+1} < i_k < i_{k+2}$, the variables $x_{i_{k-1} i_k}$ and $x_{i_{k+1} i_{k+2}}$ appear in $in(f_C)$. Thus $in(f_C)$ is divisible by $in(g_{i_{k-1} i_{k+1} i_k i_{k+2}})$. If $i_{k-1} >$

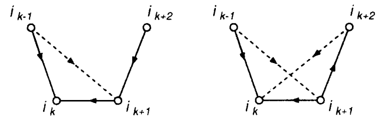

Figure 5: i_{k+1} (left) or i_{k+2} (right) contradict the choice of k.

i_{k+1}, similarly we can show that $in(f_C)$ is divisible by $in(g_{i_{k+1}i_{k-1}i_ki_{k-2}})$. Thus $\mathcal{G}_1$ is the Gröbner basis of I_{A_n} with respect to $\prec_3$.

The proof that $\mathcal{G}_1$ is reduced is same as the proof of Theorem 3.1. ∎

Theorem 3.4 *Let $\prec_4$ be a degree lexicographic order induced by the following variable ordering:*

$$x_{ij} \succ x_{kl} \Longleftrightarrow i < k \text{ or } (i = k \text{ and } j < l).$$

Then reduced Gröbner basis of I_{A_n} with respect to $\prec_4$ is same as $\mathcal{G}_1$ in Theorem 3.1.

(**Proof**) For the circuits of length less than four, we can show similarly as the proof of Theorem 3.1.

Let C be a circuit of length more than five. Let i_1 be the vertex whose label is minimum in C, and i_2 be the vertex adjacent to i_1 in C satisfying the following: let C_1 be the set of edges in C whose direction in C are same as (i_1, i_2) and C_2 be the set of edges in C which do not contained in C_1, then the cardinality of C_1 is more than that of C_2, or if the cardinality equals, then i_2 is the vertex adjacent to i_1 in C whose label is minimum. We write $C := i_1, i_2, \cdots, i_s, i_1$. Let f_C be the associated binomial. Then $in(f_C)$ is product of all variables whose associated edges are contained in C_1.

If there exists k which satisfies $i_{k-1} < i_k < i_{k+1}$ (we set $i_{s+1} = i_1$), then the variables $x_{i_{k-1}i_k}$ and $x_{i_ki_{k+1}}$ appears in $in(f_C)$. Thus $in(f_C)$ is divisible by $in(g_{i_{k-1}i_ki_ki_{k+1}})$.

If there does not exist such k, then between any two edges which are contained in C_1, there exists at least one edge which are contained in C_2. Then by the choice of i_2, the cardinality of C_1 equals that of C_2. Thus $i_3 < i_2 < i_s$ by hypothesis, and there exists k $(3 \leq k < s)$ such that $i_1 < i_k < i_2 < i_{k+1}$. Then the variables $x_{i_1i_2}$ and $x_{i_ki_{k+1}}$ appear in $in(f_C)$, and $in(f_C)$ is divisible by $in(g_{i_1i_ki_2i_{k+1}})$.

The proof that $\mathcal{G}_1$ is reduced is same as the proof of Theorem 3.1. ∎

4 Bounds for Size of Gröbner Bases for Various Term Orders

In this section, we deal with the number of elements of reduced Gröbner bases with respect to various term orders. The number of elements for general toric ideals are not well understood. For the case of the toric ideals of acyclic tournament graphs, since those vertex-edge incidence matrices are unimodular, the size of reduced Gröbner bases may be bounded.

For the number of elements in reduced Gröbner bases, we can get lower bound by Proposition 2.8.

Theorem 4.1 *The minimum number of elements in reduced Gröbner bases for I_{A_n} is $\binom{n}{2} - (n-1)$. The basis we have shown in Theorem 3.2 is the example achieving this bound.*

(**Proof**) Because of Proposition 2.8, the number of elements in reduced Gröbner basis is more than the number of elements in the basis for I_{A_n}. Since I_{A_n} corresponds to the cycle space of D_n, the number of elements in the basis for I_{A_n} equals the dimension of the cycle space, which is $\binom{n}{2} - (n-1)$. ∎

To analyze the upper bound for the number of elements in reduced Gröbner bases, we calculate all reduced Gröbner bases for small n using TiGERS [8]. TiGERS is a software system implemented in C which computes the state polytope of a homogeneous toric ideal [9]. Table 1 is the result for $n = 4, 5, 6, 7$.

n	# variables	# GB	max. of elements	min. of elements
4	6	10	5	3
5	10	211	15	6
6	15	48312	37	10
7	21	$\geq$ 37665	$\geq$ 75	15

Table 1: The number of reduced Gröbner basis, maximum of the number of elements and minimum of the number of elements.

For $n \leq 5$, the reduced Gröbner basis in Theorem 3.1 is the example achieving maximum elements, but it is not for $n \geq 6$. For $n = 6$, the Gröbner bases of size 37 are not the bases with respect to purely lexicographic orders. Thus the reduced Gröbner bases which achieve the maximum number of elements seem to be complicated and difficult to characterize.

5 Application to Integer Programming

In this section, we apply the toric ideals I_{A_n} to the minimum cost flow problem.

5.1 Conti-Traverso Algorithm

Conti and Traverso [2] introduced an algorithm based on Gröbner basis to solve integer programs. We describe the condensed version of Conti-Traverso Algorithm(See [11]). This version is useful for highlighting the main computational step involved. For the original version of Conti-Traverso Algorithm, see [2].

Let $A \in \mathbb{Z}^{d\times n}$, $b \in \mathbb{Z}^d$, $c \in \mathbb{R}^n_{\geq 0}$. We consider the integer program

$$IP_{A,c}(b) := minimize\{c \cdot x\colon Ax = b, x \in \mathbb{N}^n\}.$$

Conti-Traverso Algorithm is the algorithm which solves $IP_{A,c}(b)$ using the toric ideal I_A.

Algorithm 5.1 (Conti-Traverso Algorithm)
Input: $A \in \mathbb{Z}^{d\times n}$, $b \in \mathbb{Z}^d$, $c \in \mathbb{R}^n_{\geq 0}$
Output: *An optimal solution* $\mathbf{u}'$ *for* $IP_{A,c}(b)$

1. *Compute the reduced Gröbner basis* $\mathcal{G}_{\succ_c}$ *of* I_A.
2. *For any solution* $\mathbf{u}$ *of* $IP_{A,c}(b)$, *compute the normal form* $\mathbf{x}^{\mathbf{u}'}$ *of* $\mathbf{x}^{\mathbf{u}}$ *by* $\mathcal{G}_{\succ_c}$.
3. *Output* $\mathbf{u}'$. $\mathbf{u}'$ *is the unique optimal solution of* $IP_{A,c}(b)$.

5.2 Application to Minimum Cost Flow Problem

Using Algorithm 5.1, reduced Gröbner bases for I_{A_n} can be applied to minimum cost flow problems on D_n or the subgraphs of D_n.

The *minimum mean cycle-canceling algorithm* [7] is known as a strongly polynomial time algorithm which depends only on the number of vertices and edges. Using this algorithm, from any feasible flow, we can obtain the minimum cost flow by canceling minimum mean cycle at most $O(nm^2 \log n)$ times.

If reduced Gröbner basis with respect to the term order which corresponds to the cost vector is known, we can obtain the minimum cost flow by canceling the cycles which correspond to the elements of reduced Gröbner basis. Thus it is interesting to analyze the size of reduced Gröbner bases and the complexity of canceling the cycles for the case of acyclic tournament graphs.

6 Conclusions

In this paper, we have studied the reduced Gröbner bases for toric ideals of acyclic tournament graphs and applied them to minimum cost flow problems.

The universal Gröbner basis of acyclic tournament graph is of exponential size. We have shown two reduced Gröbner bases whose size is of polynomial order. And we showed the experimental result for the size of reduced Gröbner bases. But the upper bound for the number of elements is not known. Analyzing the upper bound for the number of elements should be a future work.

We also showed the application to minimum cost flow problems. We can apply the reduced Gröbner bases of acyclic tournament graphs to the minimum cost flow problems using Conti-Traverso Algorithm. This algorithm is similar to the minimum mean cycle-canceling algorithm. But the complexity of canceling cycles are not known. Analyzing the complexity of this algorithm should be also a future work.

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