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SIMPLE MODULES IN THE AUSLANDER-REITEN QUIVERS OF FINITE GROUP ALGEBRAS

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0. Introduction

Let G be a finite group, and let B be a block algebra of G over an algebraically closed field k of characteristic p, where p is a prime. We denote the stable Auslander–Reiten quiver (AR quiver for short) of B by $\Gamma_s(B)$. Each connected component Θ of $\Gamma_s(B)$ (AR component) has a tree class, which is determined up to quiver isomorphisms, and Θ is obtained from it and the AR translates.

A p-block B is wild if its defect group is neither cyclic, dihedral, generalized quaternion nor semidihedral. It is known that, if B is wild, then any AR component of $\Gamma_s(B)$ has tree class A_∞.([E3]) That is, any AR component Θ is isomorphic to either $\mathbf{Z}A_\infty$ or $\mathbf{Z}A_\infty/<\tau^m>$, where τ is the AR translate. The latter is called an infinite tube of rank m. Since group algebras are symmetric, the AR translate τ is equal to the composite Ω^2 of two Heller translates. Thus a module lies in a tube if and only if it is Ω-periodic. We may call it simply periodic in this paper. In the case where an AR component Θ has tree class A_∞, we say that a modules X in Θ lies at the end if there is only one arrow in Θ which goes into (or goes from) X. Moreover, a module X in Θ is said to lie in the i-th row from the end, if there is a subquiver $X = X_i \to X_{i-1} \to \cdots \to X_1$ of Θ such that X_1 lies at the end and that $X_{j+2} \not\cong \Omega^2(X_j)$ for all j with $1 \le j \le i-2$. For wild blocks, we consider the following problems on simple modules.

Question 1. *Do all simple B-modules lie at the end of its AR component Θ ?*

Question 2. *How many simple modules can lie in one AR component of B ?*

Question 3. *What is the rank of a tube which contains a simple module ?*

We report recent results concerning the above questions in Section 1. Sketches of the proofs of some of the results are given in the sections thereafter.

Notation is standard. See, for example, [Fe] or [NgT].

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1. Results

The first result concerning Question 1 is the following.

Theorem 1. (Kawata [Ka3]) *If G is p-solvable or $O_p(G) \neq \{ 1 \}$, then Question 1 has an affirmative answer. Here $O_p(G)$ denotes the maximal normal p-subgroup of G.*

Moreover, Kawata shows in Section 1 of [Ka3] the following.

Lemma 2. (Kawata [Ka3]) *Suppose that there exists a simple kG-module S such that it lies in an AR component Θ whose tree class is A_∞, it lies in the i-th row from the end of Θ with $i \geq 2$, and that there is no simple modules in the j-th row from the end of Θ for any j with $1 \leq j \leq i-1$. Then there is a simple module T with the following properties.*

(i) *T lies at the end of its AR component and there is a subquiver*

$$\Omega S = X_i \to X_{i-1} \to \cdots \to X_2 \to X_1 = T$$

of this AR component.

(ii) *The modules $T_r = \Omega^{2(r-1)}T$, where $1 \leq r \leq i-1$, are non-isomorphic simple modules.*

(iii) *The projective covers $P(T_r)$ of T_r have the following Loewy structure.*

$$P(T_1) = \begin{array}{c} T_1 \\ S \\ T_{i-1} \\ T_{i-2} \\ \vdots \\ T_2 \\ T_1 \end{array}, \quad P(T_2) = \begin{array}{c} T_2 \\ T_1 \\ S \\ T_{i-1} \\ T_{i-2} \\ \vdots \\ T_3 \\ T_2 \end{array}, \quad \ldots, \quad P(T_{i-1}) = \begin{array}{c} T_{i-1} \\ T_{i-2} \\ \vdots \\ T_2 \\ T_1 \\ S \\ T_{i-1} \end{array}$$

In particular, they are uniserial.

(iv) *The Cartan matrix of kG whose row and column entries are corresponding to T_1, T_2 $\cdots$, T_{i-1}, S, $\cdots$ in this order is*

$$\begin{bmatrix} 2 & 1 & \cdots & \cdots & 1 & 0 & \cdots & 0 \\ 1 & 2 & \ddots & & \vdots & \vdots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & & \vdots \\ \vdots & & \ddots & 2 & 1 & 0 & \cdots & 0 \\ 1 & \cdots & \cdots & 1 & * & \cdots & \cdots & * \\ 0 & \cdots & \cdots & 0 & \vdots & & & \vdots \\ \vdots & & & \vdots & \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 & * & \cdots & \cdots & * \end{bmatrix}$$

Thus, if one knows the Cartan matrix or the structure of projective indecomposable modules, then by just comparing them with (iii) and (iv) above, it may be possible to conclude that Question 1 has an affirmative answer. This gives the following results.

Observation 3. *For any of the groups and primes in the following list, Question 1 has an affirmative answer.*

(i) *Finite groups of Lie type in non-defining characteristics.*

$Sp(4,q)$, $(p \nmid q)$, $2.F_4(2)$, $(p=3,7)$, ${}^2F_4(2)'$, $(p=3,5)$, $G_2(q)$, $(p \neq 3,\ p \nmid q)$, $O_8^+(3).S_3$, $(p=5\ )$

(ii) *Sporadic simple groups and related groups.*

M_{11}, M_{12}, $(p=2,3)$, M_{22}, M_{23}, $PSL_3(4)$ *and its covering groups,* HS, Ru, Suz, $(p=3)$, $\widehat{3}M_{22}$, $(p=2)$, $2.J_2$, $(p=2,3,5\ )$, McL, $\widehat{3}McL$, Ru, $2.Ru$, Fi_{22}, $2.Fi_{22}$, $3.Fi_{22}$, $6.Fi_{22}$, $6.Fi_{22}.2$, Fi_{23}, $(p=5)$

As a matter of fact, much more blocks have been checked by this method, and Question 1 has an affirmative answer for them. This was done by Waki using GAP.

However, according to the 5-modular decomposition matrix for $F_4(2)$ given by Hiss [Hi4], Theorem 2.5, the group $F_4(2)$ has a simple module S with dimension 875823 which lies in the second row from the end of its AR component.

A similar example is found for the covering group $2.Ru$ of the sporadic Rudvalis simple group. By Hiss [Hi3], there is a 3-modular simple module with dimension 10528 which also lies in the second row of its AR component.

The next results consider finite groups of Lie type in the defining characteristic.

Theorem 4. (Kawata, Michler, U. [KaMiU1], [KaMiU2]) *Let G be a finite group of Lie type defined over a field k of characteristic p. Let B be a block of G with full defect of wild representation type. Then the following hold.*

(i) *Any simple kG-module S of B lies at the end of its AR component.*

(ii) *If G is perfect, then each AR component of $\Gamma_s(B)$ contains at most one simple module. Moreover, the same is true for AR components isomorphic to $\mathbf{Z}A_\infty$ without assuming that G is perfect.*

Theorem 5. (Chanter [Ch]) *Let G be* $\mathrm{SL}(2,p^n)$*, and B a wild p-block of G. Then, a periodic simple B-module lies in an infinite tube of rank $(p^n-1)/2$ if p odd, and that of rank 2^n-1 if $p=2$.*

For symmetric groups and related groups, we have the following.

Theorem 6. (Bessenrodt, U. [BsU]) (i) *Let G be a symmetric group, an alternating group, or the covering groups of them. Let B be a wild p-block of G, and assume that a defect group of B has order at least p^3. Then any simple kG-module B lies at the end of its AR component.*

(ii) *Let G be a symmetric group or an alternating group, B a wild p-block of G. Then each AR component of B isomorphic to $\mathbf{Z}A_\infty$ contains at most one simple module.*

Perhaps (i) of the above holds also for a wild p-block whose defect group has order p^2. However, it has not been proved yet.

The following result concerns the principal 2-block of G, when G has abelian Sylow 2-subgroups.

Theorem 7. (Kawata, Michler, U. [KaMiU2]) *Let G be a finite group with abelian Sylow 2-subgroups, and B the principal 2-block of G. If B is wild, then we have the following.*

(i) *Any simple kG-module S of B lies at the end of its AR component.*

(ii) *Each AR component of $\Gamma_s(B)$ contains at most one simple module.*

(iii) *If B contains a periodic simple module, then the rank of its AR component is $2^n - 1$, where n is the rank of a Sylow 2-subgroup.*

1. Cartan matrices

Details of Observation 3 in the previous section are as follows.

Observation 1.1. *For any of the groups and primes in the following list, there is no simple module T such that T does not belong to a finite block and the set of Cartan invariants $\{ c_{T,T'} \mid T' : \text{ simple } kG-\text{modules} \}$ consists of only one 2, and several 1's and 0's. In particular, Question 1 has an affirmative answer.*

$Sp(4,q)$, $p \nmid q$, ([Wh1], [Wh2], [Wh3], [Wk3], [OkWk]), $2.F_4(2)$, $p = 3,7$ ([Hi4]), ${}^2F_4(2)'$, $p = 3,5$ ([Hi1]), $G_2(q)$, $p = 3$, $p \nmid q$, ([Hi2], [HiSh]), $2.J_2$, $p = 2,3,5$ ([HiLu1]) , McL, $p = 5$ (p.1615 of [Le1], [HiLuP]), $\widehat{3}McL$, $p = 5$ ([Le2], [HiLuP]) , Ru, $p = 3$ ([Hi3]), Ru, $2.Ru$, $p = 5$ ([HiMü]) , Suz, $p = 3$ ([JsMü]), $\widehat{3}M_{22}$, $p = 2$ ([Le3]), Fi_{22}, $2.Fi_{22}$, Fi_{23}, $O_8^+(3).S_3$, $p = 5$ ([HiuL2]), $3.Fi_{22}$, $6.Fi_{22}$, $6.Fi_{22}.2$, $p = 5$ ([HiWh])

Observation 1.2. *For any of the groups and primes in the following list, there is no projective indecomposable module such that it does not belong to a finite block and that it is uniserial. In particular, Question 1 has an affirmative answer.*

M_{11}, $p = 2$, (See [Bn]), M_{12}, $p = 2,3$ ([Sc]), M_{22}, M_{23} *or* $PSL_3(4)$, $p = 3$ ([Wk1]), HS, $p = 3$ ([Wk2])

Observation 1.3. ([Wk3]) *For M_{11} and $p = 3$, there is no projective indecomposable module which satisfies* (iii) *of Lemma 2.*

2. Finite groups of Lie type in the defining characteristic

Let G be a finite groups of Lie type and suppose that char$k$ is the defining characteristic of G. Assume that G is defined over $\mathbf{F}_{p^m}$. If G is twisted type, then letting ℓ be the order of the "twist", write $m = n\ell$. Let σ be the generator of the Galois group of $\mathbf{F}_{p^n}$ over $\mathbf{F}_p$. Thus σ sends and a in $\mathbf{F}_{p^n}$ into a^p. In this section we give an outline of the proof of Theorem 4.

For simple modules, the following is well known.

Theorem 2.1. (Steinberg [St]) *Let M be a simple kG-module. Then we can write*

$$M = M_1 \otimes M_2^{\sigma} \otimes \cdots \otimes M_n^{\sigma^{n-1}},$$

where M_i's are restricted simple modules over the Lie algebra corresponding to G, lifted to kG-modules.

Lemma 2.2. *Let G be a perfect finite group of Lie type and P a Sylow p-subgroup of G. Let M and N be simple kG-modules. Then the following hold.*
(i) *P is a vertex of M.*
(ii) *M_P is a source of M.*
(iii) *M_P has a simple head and a simple socle.*
(iv) *$M \cong N$ if and only if $M_P \cong N_P$.*

The first, second and third statements are due to Dipper ([D1], [D2]). The forth statement is perhaps well known. But it seems that there is no reference for it. The proof is not so difficult and given in [KaMiU2].

It follows from Proposition 7.9 of [Gr], Lemma 2.2 (i) and (ii) together with standard facts on the AR sequence of the Green correspondences that the middle term of the AR sequence $\mathcal{A}(M)$ of M is indecomposable modulo projective if so is the middle term of $\mathcal{A}(M_P)$. However, by Lemma 1.4 of Erdmann [E2] and Lemma 2.2 (iii), it follows that M_P lies at the end of its AR component. Thus Question 1 has an affirmative answer.

From the above, we have

(2.3) Two simple modules M and N lie in the same AR component if and only if $\Omega^{2r}M \cong N$ for some r.

Concerning Question 2, the duality is useful. As is mentioned in 2.4 of [Hu], for a simple kG-module $L(\lambda)$ with weight λ, we have $L(\lambda)^* \cong L(-w_0\lambda)$, where w_0 is the longest element of the Weyl group of G. Moreover, it is known that $-w_0$ coincides with the graph automorphism except in type D_ℓ (ℓ even), where $-w_0$ is the identity. (See, for example, 15.3 of [Hu].) Thus, if G is not of type A_ℓ, D_ℓ (ℓ odd) or E_6, each simple module is self dual. Otherwise, $L(\lambda)^* \cong L(\lambda)^\rho$, where ρ is the automorphism induced from the graph automorphism of order two.
Now suppose that two simple kG-modules M and N satisfy $\Omega^{2r}M \cong N$ for some integer r. Then using the above remark, it is not difficult to prove that $\Omega^{4r}M \cong M$. In particular, if M is not periodic, then we have $r = 0$. Thus Question 2 has affirmative answer for non-periodic simple modules.

We now turn to periodic simple modules. Those modules are classified as in the following theorem.

Theorem 2.4. (Fleischmann, Janiszczak, Jantzen [FlJt], [JiJt]) *Let G be a finite group of Lie type defined over a field of characteristic p. Suppose that there exists a simple non-projective periodic kG-module. Then, G is of type A_1, ${}^2A_2$ or ${}^2B_2$.*

In the above, if G is simply connected, then types A_1, ${}^2A_2$ and ${}^2B_2$ mean $\mathrm{SL}(2,p^n)$, $\mathrm{SU}(3,p^{2n})$ and $\mathrm{Suz}(2^n)$, respectively. We use the notation $\mathrm{SU}(3,p^{2n})$ for the 3 dimensional unitary group which consists of matrices in $\mathrm{SL}(3,p^{2n})$ fixed by the automorphism $\rho\sigma^n$, where ρ is the automorphism of $\mathrm{SL}(3,p^{2n})$ induced by the non-trivial graph automorphism of the Dynkin diagram of A_2 and σ is the one sending any (a_{ij}) in $\mathrm{SL}(3,p^{2n})$ to (a_{ij}^p). If the characteristic of k is 2, then the group $\mathrm{Sp}_4(k)$ has an automorphism ρ arising from a symmetry of the Dynkin diagram, such that ρ^2 is the standard Frobenius map sending any (a_{ij}) in $\mathrm{Sp}_4(k)$ to (a_{ij}^2). The group $\mathrm{Suz}(2^n)$ for an odd n is the subgroup of $\mathrm{Sp}_4(k)$ consisting of the elements fixed by ρ^n. Thus ρ gives an automorphism of $\mathrm{Suz}(2^n)$ of order n. Except for $A_1(2)$, $A_1(3)$, ${}^2A_2(4)$ and ${}^2B_2(2)$, they are perfect.

Moreover, all the periodic simple modules are classified. To state it, we use the following notation.

Let G be either $\mathrm{SL}(2,p^n)$, $\mathrm{SU}(3,p^{2n})$ or $\mathrm{Suz}(2^n)$, and σ be the Galois automorphism sending any α to α^p. Let M be a simple module. We write

$$M = M_1 \otimes M_2^{\sigma} \otimes \cdots \otimes M_n^{\sigma^{n-1}},$$

where M_i's are restricted simple modules over the corresponding Lie algebra, lifted to kG-modules.

(2.5) Let S be as follows.
If G is $\mathrm{SL}(2,p^n)$, then S is the simple module with weight $(p-1)\lambda$, where $\{\lambda\}$ is a basis of the weight lattice. (Note : $\dim_k S = p$.)
If G is $\mathrm{SU}(3,p^{2n})$, then S is the simple module with weight $(p-1)(\lambda_1+\lambda_2)$, where $\{\lambda_1,\lambda_2\}$ form a basis of the weight lattice. (Note : $\dim_k S = p^3$.)
If G is $\mathrm{Suz}(2^n)$, then S is the natural 4-dimensional module.

Then, simple periodic modules are described as follows.

Proposition 2.6. (Jeyakumar, Fleischmann, Jantzen) *Let G, M and S be as above. Then M is non-projective periodic if and only if there exists a unique i with $1 \le i \le n$ such that $M_j \cong S$ for all j with $j \ne i$ and $M_i \not\cong S$, and moreover, in the case of* $\mathrm{SU}(3,q^2)$, *M_i must have weight either $(p-1)\lambda_1 + r\lambda_2$, $r\lambda_1 + (p-1)\lambda_2$ or $r\lambda_1 + (p-2-r)\lambda_2$ for some r with $0 \le r \le p-2$.*

For the proof, see [Je], Theorems 3.3 and 4.1 of [Fl2] and Section 3 of [FlJt]. See also [Fl1].

Proposition 2.6 is proved by looking at the rank varieties of modules. The rank variety is defined as follows. Let X be a kG-module. For an elementary abelian p-subgroup $E = < x_1, \cdots, x_n >$ of G and $\mathbf{a} = (a_1, \cdots, a_n)$ in k^n, define an element $u_{\mathbf{a}}$ of the group algebra kE of E over k as follows.

$$u_{\mathbf{a}} = 1 + \sum_{i=1}^{n} a_i(x_i - 1).$$

Then $u_{\mathbf{a}}$ is a unit of order p if and only if $\mathbf{a} \ne 0$. The rank variety $V_E(X)$ is defined by

$$V_E(X) = \{\mathbf{a} \in k^n \mid X_{<u_{\mathbf{a}}>} \text{ is not projective } \} \cup \{0\}.$$

Note that $V_E(X)$ not only depends on E but also on the generators of E. The rank variety is a homogeneous affine variety in k^n and it is known that X is periodic if and only if $\dim_k V_E(X) = 1$

Moreover, it is well known that $V_E(\Omega X) = V_E(X)$ and that $V_E(X \otimes_k X') = V_E(X) \cap V_E(X')$ for any kG-modules X and X'. (For the proof, see [Cl2].) In particular, if two modules lie in the same AR component, then their rank varieties coincide (fixing E).

Using the above and explicit computation of $V_E(M)$ given by Theorem 4.1 of [Fl2], Proposition (3.2) of [FlJt], we have the following.

Proposition 2.7. *Let G and $M = M_1 \otimes M_2^\sigma \otimes \cdots \otimes M_n^{\sigma^{n-1}}$ be as above, but we exclude the case* $\mathrm{SU}(3, p^{2n})$, *p odd. Let S be as in (2.5). Then fixing a certain maximal elementary abelian p-subgroup E of G, there exists an element u of k such that $\{u, u^\sigma, \cdots, u^{\sigma^{n-1}}\}$ is a normal basis of $\mathbf{F}_{p^n}$ over $\mathbf{F}_p$ and the following hold.*

(i) *The rank variety $V_E(S)$ of S can be described as*

$$V_E(S) = \{ \ (a_0, \cdots, a_{n-1}) \in k^n \mid \sum_{\mu=0}^{n-1} a_\mu u^{p^\mu} = 0 \ \}.$$

(ii) *Fix i. Denote by $\tilde{M}_i$ the module M such that $M_j \cong S$ for all j with $j \neq i$, and $M_i \not\cong S$. Then we have*

$$V_E(\tilde{M}_i) = \{ \ (a_0, \cdots, a_{n-1}) \in k^n \mid \sum_{\mu=0}^{n-1} a_\mu u^{p^{\mu+j-1}} = 0 \text{ for all } j \text{ with } j \neq i\}$$

(iii) *We have $V_E(\tilde{M}_i) = V_E(\tilde{M}_{i'})$ if and only if $i = i'$.*

Thus it follows from (iii) of the above that the exceptional index i can be determined by looking at the rank varieties.

In case of $\mathrm{Suz}(2^n)$, this gives the affirmative answer, since the possibility for M_i is only the trivial module.

For $\mathrm{SL}(2, p^n)$, the module M_i can be any simple module T with $1 \leq \dim_k T \leq p-1$, which is a restricted simple module over Lie algebra corresponding to G. Then $\dim_k \tilde{M}_i = p^{n-1} \dim_k T$. Thus there is only one possibility for such a T, if $\dim_k \tilde{M}_i$ is given. On the other hand, note that any simple module has dimension less than or equal to p^n and that two indecomposable modules U and V with $\tau^r U \cong V$ must satisfy $\dim_k U \equiv \dim_k V \mod p^n$. Thus two such modules $\tilde{M}_i$ with distinct M_i must lie in different AR components.

For $\mathrm{SU}(3, p^{2n})$ with odd p, we use a result of Carlson which asserts that

Lemma 2.8. (Carlson [Cl1]) *Let G be a finite p-group, M a periodic indecomposable kG-module. Let E be a maximal elementary abelian subgroup of G. Then the Ω-period of M divides $2|G : E|$.*

Applying the above to the source M_P of M, we can show that $\Omega^{2r} M_P \cong M_P$ and thus it follows that $M_P \cong N_P$. Hence $M \cong N$ by Lemma 2.2 (iv).

For $\mathrm{SU}(3, 2^{2n})$, there are three possibilities for the module M_i. That is, the trivial module, three dimensional natural module and its dual. In this case, we need to study those modules in detail by using the result of Sin [Si], and finally we can conclude that any two of the above possible modules can not lie in the same AR component. (See [KaMiU2] for detail.)

3. Symmetric groups and related groups

In this section, we consider the groups mentioned in the title of this section. Let S_n be the symmetric group of degree n and $\tilde{S}_n$ its covering group, A_n the alternating group of degree n and $\tilde{A}_n$ its covering group. The covering groups (of S_n or of A_n) are not unique but they are isoclinic. Let $G = S_n$. As is well known, simple modules over a field of characteristic zero are parameterized by partitions of n and those over a field of characteristic p are parameterized by p-regular partitions. For a partition λ, we denote by χ_λ the ordinary irreducible character of S_n corresponding to λ, and if λ is p-regular, the corresponding Brauer character of S_n is denoted by φ_λ. We usually identify a partition with the corresponding Young diagram. Let $d_{\lambda\mu}$ be the decomposition number corresponding χ_λ and φ_μ. Thus $\chi_\lambda = \sum_\mu d_{\lambda\mu}\varphi_\mu$ holds on the set of p-regular elements of G. It is known that, if λ is p-regular, then the decomposition number $d_{\lambda\lambda} = 1$. (See 6.3.50 of [JmKe].)

Let B be a p-block of S_n. Then there are integers ℓ and w such that $n = pw + \ell$, the defect group of B is isomorphic to a Sylow p-subgroup of S_{pw} and that B is induced up from the block $b_0 \times b$, where b_0 is the principal block of S_{pw} and b is a block of S_ℓ which is defect zero. This w is called the weight of B. It is known that, if the character χ_λ belongs to B, then we can remove a p-hook (a hook of length p) from λ, and after repeating these processes w times we finally obtain a partition $\mathrm{Core}(\lambda)$ of ℓ which does not contain any p-hook. The partition $\mathrm{Core}(\lambda)$ does not depend on the order of p-hooks which are removed from the partitions each time and is called the p-core of λ. Two irreducible characters χ_{λ_1} and χ_{λ_2} belong to the same p-block if and only if the p-cores of λ_1 and λ_2 coincide. With using the notion of p-quotients, this "removal of a hook" can be described as follows. Let $(\lambda_{[0]}, \lambda_{[1]}, \cdots, \lambda_{[p-1]})$ be the p-quotient of λ. Then here $\lambda_{[0]}$, $\lambda_{[1]}$, $\cdots$, $\lambda_{[p-1]}$ are partitions of n_0, n_1, $\cdots$, n_{p-1}, respectively, with $n_0 + n_1 + \cdots + n_{p-1} = w$, which are uniquely determined by λ and p. (Conversely, the p-core and the p-quotient determine λ.) Let λ be a partition of n and suppose that λ has a p-hook. Remove a p-hook from λ and denote the resulting partition by ν. Then, there exists the unique r with $0 \leq r \leq p-1$ such that the p-quotient $(\nu_{[0]}, \nu_{[1]}, \cdots, \nu_{[p-1]})$ of ν satisfies $\nu_{[i]} = \lambda_{[i]}$ for all i with $i \neq r$ and $\nu_{[r]}$ is obtained by removing one node from $\lambda[r]$. For those facts, see for example [Ol].

The following general result is quite useful in proving Theorem 6.

Proposition 3.1. ([Jm] 21.7) *If ν is a partition of $n-p$, then the generalized character of S_n corresponding to*

$$\sum (-1)^i \chi_\lambda$$

is zero on all classes except those containing a p-cycle, where the sum is taken over all λ such that $\lambda \setminus \nu$ is a skew p-hook of leg length i.

The above can be regarded as a version of Murnaghan-Nakayama Rule. If an element x in S_n is p-regular, then it does not contain a p-cycle, and thus the above sum is zero at x.

Concerning removals of hooks we remark the following.

Lemma 3.2. *Let ν_1 and ν_2 be distinct partitions of $n-p$. For $i = 1, 2$, let S_i be the set of partitions μ of n such that a removal of one p-hook from μ gives ν_i. Then $S_1 \cap S_2$ is either empty or consists only of one element.*

The proof of the above is easy by looking at p-quatients of ν_1 and ν_2 instead of themselves. The elements in $\mathcal{S}$, ν_1 and ν_2 have almost the same p-quotients. They differ always by one node.

Next we see what happens for a p-regular partition of n which has only one p-hook. The above is equivalent to the following.

(3.3) λ is a p-regular partition of n such that its p-quotient $(\lambda_{[0]}, \lambda_{[1]}, \cdots, \lambda_{[p-1]})$ satisfies that $\lambda_{[i]}$ is empty for all but one i, say r. Moreover, $\lambda_{[r]}$ is a rectangle, that is $\lambda_{[r]} = (a, a, \cdots, a)$ for some positive integer a.

Concerning the removal of the unique p-hook from a partition satisfying (3.3), the following holds.

Lemma 3.4. *Let λ be a partition satisfying* (3.3) *and ν the partition of $n-p$ which can be obtained by removing the unique p-hook from λ. Let $\mathcal{S}$ be the set of partitions μ of n such that a removal of one p-hook from μ gives ν. Let w be the weight of the p-block containing χ_λ.*

(i) *If $w = 1$, we have $\mathcal{S} = \{\lambda, \mu_1, \mu_2, \cdots, \mu_{p-1}\}$. Here μ_i is a partition whose p-quotients satisfy the following.*

$\mu_{i[i-1]}$ is the partitions (1) of 1 for $1 \le i \le r$.

$\mu_{i[i]}$ is the partitions (1) of 1 for $r+1 \le i \le p-1$.

The others in p-quotients are empty.

(ii) *If $w = 2$, we have $\mathcal{S} = \{\lambda, \mu, \mu_0, \mu_1, \cdots, \mu_{r-1}, \mu_{r+1}, \cdots, \mu_{p-1}\}$. Moreover, their p-quotients satisfy the following.*

$\{\lambda_{[r]}, \mu_{[r]}\} = \{(1,1), (2)\}$ and $\mu_{i[i]} = \mu_{i[r]} = (1)$.

$\lambda_{[j]}$, $\mu_{[j]}$ and $\mu_{i[j]}$ are empty for all i and j with $0 \le i, j \le p-1$ and $i \ne r \ne j$.

(iii) *If $w \ge 3$, then $|\mathcal{S}| \ge p$ and λ is the unique partition in $\mathcal{S}$ which satisfies* (3.3).

Again it is easy to see the above by looking at p-quotients of the partitions in $\mathcal{S}$.

Let us now consider the Cartan matrix of S_n.

Proposition 3.5. *Let λ be a p-regular partition of n.*

(i) *Suppose that λ has m p-hooks. Then we have $d_{\mu\lambda} \ne 0$ for at least $m+1$ partitions μ of n. In particular, we have $c_{\lambda\lambda} \ge m+1$.*

(ii) *Suppose that χ_λ belongs to a p-block with weight $w \ge 3$ and that λ has only one p-hook. Then we have $d_{\mu\lambda} \ne 0$ for at least 3 partitions μ of n. In particular, we have $c_{\lambda\lambda} \ge 3$.*

Proof. (i) Let ν_1, ν_2, $\cdots$, ν_m be distinct partitions of $n-p$ which are obtained by removing one p-hook form λ. For each i with $1 \le i \le m$, let $\mathcal{S}_i$ be the set of partitions μ of n such that a removal of one p-hook from μ gives ν_i. Then, since $d_{\lambda\lambda} \ne 0$, Proposition 3.1 yields that there exists a partition μ_i in $\mathcal{S}_i$ such that $\mu_i \ne \lambda$ and $d_{\mu_i\lambda} \ne 0$. Moreover, Lemma 3.2 yields that $\mu_i \ne \mu_j$ if $i \ne j$. Hence we obtain the desired consequence.

(ii) Recall that λ satisfies (3.3) Then, letting ν and $\mathcal{S}$ be as in Lemma 3.4, λ is the unique partition in $\mathcal{S}$ which satisfies (3.3) by Lemma 3.4 (iii). On the other hand, it follows by applying Proposition 3.1 to the character χ_ν that there is a partition μ_1 in $\mathcal{S} \setminus \{\lambda\}$ with $d_{\mu_1\lambda} \ne 0$. Since μ_1 does not satisfy (3.3), it has a

p-hook such that the removal of this p-hook from μ_1 gives a partition ν_1 different from ν. Let $\mathcal{S}'$ be the set of partitions μ of n such that a removal of one p-hook from μ gives ν_1. Then $\mathcal{S} \cap \mathcal{S}' = \{\mu_1\}$ by Lemma 3.2. By applying Proposition 3.1 to the character χ_{ν_1}, we can find μ_2 in $\mathcal{S}'$ such that $d_{\mu_2\lambda} \neq 0$ and $\mu_2 \neq \mu_1$. Notice that $\mu_2 \neq \lambda$. Therefore, $d_{\lambda\lambda}$, $d_{\mu_1\lambda}$ and $d_{\mu_2\lambda}$ are all nonzero. This completes the proof.

By using the above and Lemma 2, we can prove Theorem 6 for S_n.

Now let us consider the case of p odd and $G = \tilde{S}_n$. The set of partitions of n into distinct parts is denoted by $D(n)$. We write $D^+(n)$ and $D^-(n)$ for the sets of partitions λ in $D(n)$ with $n - \ell(\lambda)$ even and odd, respectively. Here $\ell(\lambda)$ is the number of parts in λ. The associate classes of spin characters of $\tilde{S}_n$ are labeled canonically by the partitions in $D(n)$. For each $\lambda \in D^+(n)$ there is a self-associate spin character $\eta_\lambda = \mathrm{Sgn}\eta_\lambda$, and to each $\lambda \in D^-(n)$ there is a pair of associate spin characters $\eta_\lambda, \eta_{\lambda'} = \mathrm{Sgn}\eta_\lambda$. We write η_λ^o for a choice of associate, and $\hat{\eta}$ for η_λ if $\lambda \in D^+(n)$ and for $\eta_\lambda + \eta_{\lambda'}$ if $\lambda \in D^-(n)$. The role played by hooks and hook partition in the case of S_n characters is taken on by bars and bar partitions in the case of the spin characters of the covering groups. Here we mean by a bar partition (or bar diagram) just a partition (resp. diagram) of the form $(a-b, b)$, $0 \leq b \leq \left[\frac{a-1}{2}\right]$, i.e. a partition with at most two distinct parts. After removing all p-bars recursively from λ, we obtain a partition called the $\overline{p}$-core of λ. Moreover, we can define the $\overline{p}$-quotient $(\lambda_{\overline{[0]}}, \lambda_{\overline{[1]}}, \cdots, \lambda_{\overline{[t]}})$ of λ. Here $t = (p-1)/2$ and $\lambda_{\overline{[0]}}$, $\lambda_{\overline{[1]}}, \cdots, \lambda_{\overline{[t]}}$ are partitions of $n_0, n_1, \cdots, n_t$ with $n_0 + n_1 + \cdots + n_t = w$. This w is also called the weight of the p-block to which η_λ belongs and plays an important role when investigating the block structure. It is known that η_{λ_1} and η_{λ_2} belong to the same p-block if and only if λ_1 and λ_2 have the same $\overline{p}$-core. The $\overline{p}$-quotient of λ describes removals of p-bars from λ. Therefore, essentially the same method can be used to prove that Question 1 has an affirmative answer for p-blocks of $\tilde{S}_n$ with weight $w \geq 3$. When doing it, the most crucial assertion is an analogue of Proposition 3.1, which is stated as follows and proved in [BsU].

Proposition 3.6. *Let p be an odd prime with $p \leq n$. Let $\nu \in D(n-p)$ be a partition. If p is not a part of ν, then set $\lambda_0 = \nu \cup \{p\}$ and let $b_0 = \lambda_0 \setminus \nu$ be the corresponding p-bar of λ_0. Then the generalized character χ of $\tilde{S}_n$ given by*

$$\chi = \begin{cases} \sum_{\substack{\lambda \in D(n), \lambda \neq \lambda_0 \\ \lambda \setminus \nu = b: p-bar}} (-1)^{L(b)} \hat{\eta}_\lambda + (-1)^{L(b_0)} \eta_\lambda^o, & \text{if } \; p \notin \nu \in D^-(n-p) \\ \sum_{\substack{\lambda \in D(n) \\ \lambda \setminus \nu = b: p-bar}} (-1)^{L(b)} \hat{\eta}_\lambda, & \text{otherwise} \end{cases}$$

is zero on all classes except possibly those corresponding to partitions with p as a part. Here $L(b)$ is the leg length of a p-bar b.

Character relations in the nature of Propositions 3.1 and 3.6 are related to the equations among reduced (Q-)Schur functions. See [ANkY] and [NkY]. In any case, remark however that there is no known parameterization of simple $k\tilde{S}_n$-modules which is valid for all primes p.

For A_n or $\tilde{A}_n$ when p is an odd prime, we use a reduction lemma, which gives information on the relationship between simple kG-modules and those for a normal subgroup N of G. It is obtained by using several standard results on Clifford theory of AR components. (See [Ka2] and [U2].)

Concerning Question 2, it suffices to remark that all the simple kS_n-modules are self dual. This follows from the facts that all the ordinary irreducible kS_n-modules are self dual and that the decomposition matrix is lower triangular if partitions are ordered appropriately. Since it is impossible that two self dual modules lie in the same τ-orbit of an AR component which is not a tube, Question 2 has an affirmative answer for non-periodic simple kS_n-modules. (Notice that if an AR component contains a self dual module, then the duality gives a graph automorphism of it which reverses the direction of all the arrows.) Finally, by using standard Clifford theory on AR components, one can show that Question 2 has an affirmative answer for non-periodic simple kA_n-modules, too.

4. Groups with abelian Sylow 2-subgroups

Let G be a finite group with an abelian Sylow 2-subgroup, and B the principal 2-block of G. We suppose also that B is a wild block. Let $O(G)$ be the maximal normal subgroup of odd order. Then by the results of Walter, Janko, Bombieri, Thompson and Ward, we have the following.

Theorem 4.1. ([Wl], see also p.485 of [Go]) *Let G be a finite group with an abelian Sylow 2-subgroup. Then G contains a normal subgroup H containing $O(G)$ with odd index such that $H/O(G)$ is a direct product of groups of the following types:*
(a) *An abelian 2-group.*
(b) $\mathrm{PSL}(2,q)$, $q > 3$, $q \equiv 3$ *or* $5 \mod 8$.
(c) $\mathrm{PSL}(2,2^n)$, $n > 1$.
(d) J_1, *the smallest Janko group of order 175560.*
(e) $R(q)$, *a simple group of Ree type with* $q = 3^{2n+1}$.

Since B is the principal 2-block, the elements of $O(G)$ act trivially on modules belonging to B. Because we consider only the modules in the principal block of G, in order to consider the questions, we may assume that

$$O(G) = \{1\}.$$

Let us write

$$H = H_1 \times H_2 \times \cdots \times H_m,$$

where H_1, H_2, $\cdots$, H_m are groups in Theorem 4.1. We assume that $|H_i| > 1$ for all i and that H_i is abelian for at most one i.

Let us look at individual groups appearing in Theorem 4.1.

(4.2) The case where G is an abelian 2-group.

There is only one simple module, namely, the trivial module. Because G is not cyclic, the trivial module is not periodic. Moreover, by [We] Theorem E, it lies at the end of its AR component in the case where kG is wild. Thus it is clear that all the questions have affirmative answers.

(4.3) The case where $G = \mathrm{PSL}(2,q)$ for $q \equiv 3$ or $5 \mod 8$.

Then the defect group is an elementary abelian of order 2^2 and the block is tame. There are 3 simple modules. If $q \equiv 3 \mod 8$, then all of them are not periodic. If $q \equiv 5 \mod 8$, then two non-trivial simple modules are periodic with Ω-period 3. In fact, the principal 2-block of $\mathrm{PSL}(2,q)$ for $q \equiv 3 \mod 8$ is Morita equivalent to the group algebra of the alternating group of degree 4 and that of $\mathrm{PSL}(2,q)$ for $q \equiv 5 \mod 8$ is Morita equivalent to the principal block of the alternating group of degree 5. Among simple modules, only periodic ones (which exist only when $q \equiv 5 \mod 8$) lie at the end of infinite tubes of rank three. Moreover, each infinite tube has at most one simple module. (See [E1].)

(4.4) The case where $G = \mathrm{PSL}(2,2^n)$, $n \geq 2$.

Then the defect group of the principal 2-block B of G is an elementary abelian of order 2^n. There are n simple modules. If $n \geq 3$, then all the questions have affirmative answers by the results in Section 2. If $n = 2$, then kG is isomorphic to the group algebra of the alternating group of degree 5. Thus as is remarked in (4.3), B has two periodic simple modules which lie at the end of distinct infinite tubes of rank three. Therefore, all the questions have affirmative answers, as well.

(4.5) The case where $G = J_1$.

Then the defect group of the principal 2-block B of G is an elementary abelian of order 2^3. There are 5 simple modules. They are all self dual. Among those, the simple B-module of dimension 20 is the unique periodic simple module, and it has Ω-period 7. (See [LaMi1].) By looking at Cartan matrix (see p. 209 of [LaMi1]), it follows from Lemma 2 that simple modules lie at the end. The unique periodic simple module lies at the end of an infinite tube of rank 7, and of course, this is the unique infinite tube which contains a simple module. On the other hand, since all simple modules are self dual, any AR component isomorphic to $\mathbf{Z}A_\infty$ can have at most one simple module. Therefore, all the questions have affirmative answers.

(4.6) The case where $G = R(q)$, $q = 3^{2n+1}$.

Then the defect group of the principal 2-block B of G is an elementary abelian of order 2^3. There are 5 simple modules. Three of them are self dual and the remaining two are dual to each other. However, the Green correspondents of those two have dimension 1. Among those 5 simple B-modules, the simple module of dimension $q^2 + 1 - m(q+1)$, where $m = 3^n$, is the unique periodic simple module, and it has Ω-period 7. (See [LaMi2].) By looking at Cartan matrix (see Theorem 3.9 (b) of [LaMi2]), it follows from Lemma 2 that simple kH-modules lie at the end. The unique periodic simple module lies at the end of an infinite tube of rank 7, and of course, this is the unique infinite tube which contains a simple module. For non-periodic simple modules, we can not use the same argument as in (4.5), since there exists a simple module which is not self dual. However, fortunately, the Green correspondents of those modules are simple, and one can show that they lie in distinct AR components. Since it is known that the Green correspondence gives a graph monomorphism, those simple B-modules also lie in distinct AR components. Finally, considering duality, one can conclude that all non-periodic simple modules lie in distinct AR components. Therefore, all the questions have affirmative answers.

In a general case, several reduction methods, which reduce the problems to those for normal subgroups and factor groups are necessary. Most of them are standard results in AR theory for group algebras, and found in [Bn], [G], [Ka1], [Ka2], [Ka3], [OkU1], [OkU2], [U1] and [U2]. Among others, we raise only one result which in fact reduces problems to almost simple cases.

Lemma 4.7. *Suppose that there exists a simple kG-module S such that it lies in an AR component Θ whose tree class is A_∞ and it lies in the i-th row from the end of Θ with $i \geq 2$. Let N be a normal subgroup of G, and assume further that any module in Θ are not N-projective. Then one of the following holds for any simple direct summand V of S_N.*

(i) *V belongs to a block of defect zero.*

(ii) *$p = 2$ and V belongs to a block whose defect group is cyclic of order 2.*

Finally we add a remark on the rank of infinite tubes which contain simple modules. See [KaMiU2] for detail. Note that we include also the case of $G \neq H$.

Remark 4.8. (i) *If there is a simple B-module, then the group H in the above argument must be simple.*

(ii) *The rank of an infinite tube which contains a simple kG-module in B is as follows. The case of $H = \mathrm{PSL}(2, 2^n)$, $n \geq 2$, the rank is always $2^n - 1$, and for $H = J_1$ or $R(q)$, the rank is always 7. Therefore, these ranks are always $2^m - 1$, where m is the rank of Sylow 2-subgroup.*

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