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A Note on Decomposition Numbers for $SU(3, q^2)$

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This is a joint work with Prof. Okuyama. Let q be a power of a prime p and r is a prime which divides $q+1$. So there are s and a such that $q+1 = r^a s$ and s isn't divided by r any more. F is an algebraically closed field of the characteristic r.

We denote G $SU(3,q^2) = \left\{A \in SL(3,q^2) \mid A\omega\overline{A}^t = \omega\right\}$ where $\omega = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. Then the order of G is $q^3(q^3+1)(q^2-1)$.

In [1], Geck determined the decomposition numbers of the principal block of G as the following.

Theorem 0.1

		I_G	φ_S	φ_T
1	θ_1	1		
1	θ_{q^2-q}		1	
1	θ_{q^3}	1	α	1
r^a-1	θ_{q^2-q+1}	1	1	
r^a-1	θ_{q^2-q+1}		$\alpha-1$	1
$(r^a-1)(r^a-2)/6$	θ_{q^2-q+1}	1	$\alpha-2$	1

where $2 \leq \alpha \leq \frac{r^a+1}{3}$

Let we denote I_G, S, T simple FG-modules which are corresponding to above irreducible Brauer characters.

In this paper, we determine α in case the center of G is trivial.

1 Notation

For any elements a in the finite field $GF(q^2)$, $\overline{a} = a^q$. Then three kinds of elements t, $h(x)$, $u(a,b)$ in G denote $\begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} x & 0 & 0 \\ 0 & x^{q-1} & 0 \\ 0 & 0 & x^{-q} \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & -\overline{a} & 1 \end{pmatrix}$. From these elements, we can construct subgroups of G.

- $H = \{h(x) | x \in GF(q^2)^\times\}$

- $U = \{u(a,b) | a\bar{a} + b + \bar{b} = 0\}$
- $U_0 = \{u(0,b) | b + \bar{b} = 0\}$
- $B = H \ltimes U$
- $B_0 = H \ltimes U_0$

And the order of each subgroups is $q^2 - 1$, q^3, q, $q^3(q^2-1)$, $q(q^2-1)$. Let R be Sylow r-subgroup of B then the order of R is r^a.

2 About Subgroups

It is easy to check the following.

Lemma 2.1 $G = B \cup BtU$

Lemma 2.2 *The center of* U *is* U_0.

Lemma 2.3 *For any non-trivial subgroup* R' *of* R, $N_B(R') = B_0$.

Lemma 2.4 *Any subgroups* R' *of* R *is TI(Trivial Intersection) set.*

Let L denotes $B_0 \cup B_0tU_0$, then the number of elements in L is $q(q^2-1) + q^2(q^2-1) = q(q+1)(q^2-1)$ and a next lemma is followed.

Lemma 2.5 *L is a subgroup of G and it is isomorphic to* $U(2, q^2)$.

For any subsets S of G, we define $\widehat{S} = \sum_{s \in S} s$. We fix the element b in $GF(q^2)$ with the condition $b + \bar{b} \neq 0$. For this b, we define $\gamma(b) = \sum_a \widehat{B}tu(a,b)$ where a runs over $a\bar{a} + b + \bar{b} = 0$. Then we can get the following lemma.

Lemma 2.6 *For the element b, let* $a_0 \in GF(q^2)$ *with* $a_0\overline{a_0} + b + \bar{b} = 0$.

i) If $b_0 = (a_0\overline{a_0})^{-2}b$, *then* ${a_0}^{-1}\overline{{a_0}^{-1}} + b_0 + \overline{b_0} = 0$.

ii) If $g = u(-\overline{{a_0}^{-1}}, b_0)tu(a_0, b)$, *then* $\gamma(b) = \widehat{BLg}$.

Proof : (i)

$$\begin{aligned}\overline{{a_0}^{-1}}{a_0}^{-1} + b_0 + \overline{b_0} &= \overline{{a_0}^{-1}}{a_0}^{-1} + (a_0\overline{a_0})^{-2}b + (a_0\overline{a_0})^{-2}\bar{b} \\ &= (a_0\overline{a_0})^{-2}\left\{a_0\overline{a_0} + b + \bar{b}\right\} = 0\end{aligned}$$

(ii) Since $L = B_0 \cup B_0 t U_0$, $BL = B \cup BtU_0$. So $\widehat{BL} = \widehat{B} + \widehat{BtU_0} = \widehat{B} + \widehat{B}t\widehat{U_0}$. The equation : $tu(a,b)t = h(b)u(-a\overline{b^{-1}}b, \overline{b})tu(-a\overline{b^{-1}}, b^{-1})$ shows that

$$\begin{aligned}
\widehat{BL}g &= \widehat{B}g + \widehat{B}t\widehat{U_0}g \\
&= \widehat{B}u(-\overline{a_0{}^{-1}}, b_0)tu(a_0,b) + \widehat{B}t\widehat{U_0}u(-\overline{a_0{}^{-1}}, b_0)tu(a_0,b) \\
&= \widehat{B}tu(a_0,b) + \widehat{B}t\textstyle\sum_{b'} u(0,b')u(-\overline{a_0{}^{-1}}, b_0)tu(a_0,b) \\
&= \widehat{B}tu(a_0,b) + \widehat{B}t\textstyle\sum_{b'} u(-\overline{a_0{}^{-1}}, b_0+b')tu(a_0,b) \\
&= \widehat{B}tu(a_0,b) + \widehat{B}\textstyle\sum_{c} tu(-\overline{a_0{}^{-1}}, c)tu(a_0,b) \\
&= \widehat{B}tu(a_0,b) + \widehat{B}\textstyle\sum_{c} tu(\overline{a_0{}^{-1}c^{-1}}, c^{-1})u(a_0,b) \\
&= \widehat{B}tu(a_0,b) + \widehat{B}\textstyle\sum_{c} tu(a_0 + \overline{a_0{}^{-1}c^{-1}}, b)
\end{aligned}$$

where b' runs over $b' + \overline{b'} = 0$ and c runs over $\overline{a_0{}^{-1}}a_0{}^{-1} + c + \overline{c} = 0$. Now, if we put $a = a_0 + \overline{a_0{}^{-1}c^{-1}}$,

$$\begin{aligned}
a\overline{a} &= (a_0 + \overline{a_0{}^{-1}c^{-1}})\overline{a_0 + \overline{a_0{}^{-1}c^{-1}}} \\
&= (a_0 + \overline{a_0{}^{-1}c^{-1}})(\overline{a_0} + a_0{}^{-1}c^{-1}) \\
&= a_0\overline{a_0} + c^{-1} + \overline{c^{-1}} + a_0{}^{-1}c^{-1}\overline{a_0{}^{-1}c^{-1}} \\
&= a_0\overline{a_0} + c^{-1}\overline{c^{-1}}\left(c + \overline{c} + a_0{}^{-1}\overline{a_0{}^{-1}}\right) \\
&= a_0\overline{a_0}
\end{aligned}$$

So we can check that a runs over $a\overline{a} + b + \overline{b} = 0$ and $a \neq a_0$. ∎

3 Calculations of Modules

We denote k_H the trivial character of H and $k_H{}^B$ induced character of k_H to B. Since the restriction of the irreducible character φ_S to the Borel subgroup B is irreducible as an ordinary character, this irreducible character $\varphi = \varphi_{SB}$ is also irreducible as a Brauer character.

Let $\mathcal{B}$ be a block which contains φ. Let $\widetilde{S}$ be a simple FB-module which is corresponding to the character φ. This $\widetilde{S}$ is only simple module which belongs in block $\mathcal{B}$. The defect group $\delta(\mathcal{B})$ of block $\mathcal{B}$ is a cyclic group with its order r^a. So any indecomposable modules in block $\mathcal{B}$ is uniserial. Moreover the projective cover $P(\widetilde{S})$ is uniserial of Loewy length r^a. Let we denote $\{\theta_{q^2-q}^{(0)}, \theta_{q^2-q}^{(s)}, \ldots, \theta_{q^2-q}^{(s(r^a-1))}\}$ r^a ordinary irreducible characters in block $\mathcal{B}$.

Lemma 3.1 *For any non-projective indecomposable FB_0-module M, M^B has only one non-projective indecomposable summand.*

Proof : This is Green correspondence of M. So this lemma is followed by lemma 2.4. ∎

For FB-module M, $\mathcal{B}$-part of M means direct summands of M which belong in $\mathcal{B}$.

Lemma 3.2 *i)* $I_H{}^B = I_B \oplus Y \oplus Z$ *where*

$$Y = \mathcal{B}\text{-part of } I_H{}^B = \begin{pmatrix} \tilde{S} \\ \tilde{S} \\ \vdots \\ \tilde{S} \end{pmatrix}$$

is uniserial with Loewy length $r^a - 1$ *and* Z *is projective.*

ii) Let Y_i *be a submodule of* Y *with Loewy length* i*, then dim* $Inv_H(Y_i) = i$

iii) dim $Inv_H(Z) = q + 2 - r^a$

Proof : From calculations of characters,

- $k_H{}^{B_0} = k_{B_0} + \theta_0$ (θ_0 is an irreducible Brauer character.)
- $k_{B_0}{}^B = k_B + \theta_{q^2-1}$ (θ_{q^2-1} is an irreducible projective charactar.)
- $\theta_0{}^B = \sum_{u=1}^{q} \theta_{q^2-q}^{(u)}$

Since $\theta_{q^2-q}^{(0)}$ isn't in $\theta_0{}^B$, (i) is followed by lemma 3.1.

The restriction of short exact sequence

$$0 \to \tilde{S} \to P(\tilde{S}) \to Y \to 0$$

to B_0 shows that the Green correspondence of $\tilde{S}$ is uniserisal FB_0-module with Loewy length $r^a - 1$ and all composition factors are isomorphic to simple module $\widetilde{S_0}$ corresponding to θ_0. Let $\mathcal{B}_0$ be a block which contains θ_0.

Lemma 3.1 shows that the $\mathcal{B}_0$-part of the restriction of Y_i to B_0 is a direct sum of the uniserisal module Y_i' with Loewy length $r^a - i$ and $i-1$ projective indecomposable modules which are isomorphic to the projective cover $P(\widetilde{S_0})$.

From $k_H{}^{B_0} = k_{B_0} + \theta_0$,

$$\begin{aligned} \text{dim Inv}_H(Y_i) &= \text{dim Hom}_H(I_H, Y_{iH}) \\ &= \text{dim Hom}_{B_0}(I_H{}^{B_0}, Y_{iB_0}) \\ &= \text{dim Hom}_{B_0}(\widetilde{S_0}, Y_{iB_0}) \\ &= \text{dim Hom}_{B_0}(\widetilde{S_0}, Y_i') + (i-1)\text{dim Hom}_{B_0}(\widetilde{S_0}, P(\widetilde{S_0})) \\ &= 1 + (i-1) = i \end{aligned}$$

So (ii) is proved. Finally,

$$\begin{aligned} \text{dim Inv}_H(I_H{}^B) &= \text{dim Hom}_H(I_H, I_H{}^B{}_H) \\ &= \text{dim Hom}_{B_0}(I_H{}^B, I_H{}^B) \\ &= q + 2 \end{aligned}$$

So from (i),

$$\begin{aligned}\dim \mathrm{Inv}_H(Z) &= \dim \mathrm{Inv}_H({I_H}^B) - \dim \mathrm{Inv}_H(Y) - 1 \\ &= (q+2) - (r^a - 1) - 1 = q + 2 - r^a\end{aligned}$$

∎

4 The Number α

Theorem 4.1 *If the center of $SU(3, q^2)$ is trivial, then α in Theorem 0.1 is 2.*

Proof : From theorem 0.1, composition factors of ${I_B}^G$ is $2 \times I_G + \alpha \times S + T$. Remember the homomorphism $f : {I_L}^G \to {I_B}^G$ in section 5 of [1], the composition factors of $\mathrm{Im}(f)$ is $I_G + \alpha/2 \times S + T$. The correspondence between notations of [1] and one of this paper about ${I_L}^G = \hat{L}FG$, ${I_B}^G = \hat{B}FG$ are the following. $v_\infty \leftrightarrow \hat{B}$, $v_{0,0} \leftrightarrow \hat{B}t$, $\delta(v_\infty, v_{0,0}) \leftrightarrow \hat{L}$, and a set $\delta(v_\infty, v_{0,0})$ has elements $\{\langle v_\infty \rangle, \langle v_\infty tu \rangle \, | u \in U_0\}$. This correspondence shows that

$$\begin{aligned} f(\hat{L}) &\leftrightarrow f(\delta(v_\infty, v_{0,0})) \\ &= v_\infty + \textstyle\sum_{u \in U_0} v_\infty tu \\ &\leftrightarrow \hat{B} + \textstyle\sum_{u \in U_0} \hat{B}tu \\ &= \hat{B} + \hat{B}t\widehat{U_0} \\ &= \widehat{BL} \end{aligned}$$

Thus, $\mathrm{Im}(f) = f({I_L}^G) = f(\hat{L})FG = \widehat{BL}FG$. From lemma 3.2 i), $\hat{B}FG_B = {I_B}^G{}_B = I_B \oplus {I_H}^B = I_B \oplus I_B \oplus Y \oplus Z$. Since the composition factors of $\left(\hat{B}FG/\mathrm{Im}(f)\right)_B$ are I_B and $\alpha/2 \; \tilde{S}$,

$$\mathrm{Im}(f)_B = I_B \oplus Y_{r^a - 1 - \alpha/2} \oplus Z.$$

From lemma 2.6, both $\widehat{BL}$ and $\sum_{h \in H_0 \backslash H} \gamma(b)h$ are in $\mathrm{Im}(f) = \widehat{BL}FG$. Since the action of H on these linearly independent elements is trivial, $\dim \mathrm{Inv}_H(\mathrm{Im}(f)) \geq q + 1$. So from lemma 3.2 ii) and iii),

$$1 + r^a - 1 - \alpha/2 + q + 2 - r^a \geq q + 1.$$

We can get $\alpha = 2$ from $\alpha \geq 2$ in theorem 0.1. ∎

References

[1] Meinolf Geck, Irreducible Brauer characters of the 3-dimensional special unitary groups in non-defining characteristic, Communications in Algebra 18(2), 563–584, 1990

[2] Tetsuro Okuyama, Katsushi Waki, Decomposition Numbers of $Sp(4, q)$ J. Alg.,1997