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A NATURAL EXISTENCE PROOF FOR JANKO'S SPORADIC GROUP J_1

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ABSTRACT. Using the second author's deterministic algorithm [8], which constructs all the finitely many simple groups G having a 2-central involution, say t, such that $C_G(t)$ is isomorphic to a given group H satisfying certain natural conditions, in this article we give a new and in some sense natural existence proof for Janko's first sporadic simple group J_1 [6].

1. INTRODUCTION

In [6] Z. Janko has proved the very remarkable

Theorem 1.1. *Let G be a finite group with following properties:*

(J1) *G contains a 2-central involution t with centralizer $C_G(t) \cong \langle t\rangle \times \mathrm{A}_5$, where A_5 denotes the alternating group of order 60.*

(J2) *G does not have a subgroup of index 2.*

Then G is isomorphic to the subgroup J of $\mathrm{GL}_7(11)$ generated by the two matrices

$$\begin{pmatrix} 0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1\\ 1&0&0&0&0&0&0 \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} -3&2&-1&-1&-3&-1&-3\\ -2&1&1&3&1&3&3\\ -1&-1&-3&-1&-3&-3&2\\ -1&-3&-1&-3&-3&2&-1\\ -3&-1&-3&-3&2&-1&-1\\ 1&3&3&-2&1&1&3\\ 3&3&-2&1&1&3&1 \end{pmatrix}$$

Moreover, G has order $|G| = 175560$, and up to isomorphism G has only one 7-dimensional irreducible representation over the prime field $\mathrm{GF}(11)$.

In [6] Z. Janko has also computed a character table of G, and he determined all the maximal subgroups of G.

It is the purpose of this article to give a new proof for the existence part of Janko's Theorem 1.1, based on the second author's deterministic algorithm of [8]. Starting from a given finite group H this algorithm constructs all the finite simple groups G having a 2-central involution t and the following properties:

(1) There exists an isomorphism $\tau : C_G(t) \to H$.

(2) There exists an elementary abelian normal subgroup A of a fixed Sylow 2-subgroup S of H of maximal order $|A| \geq 4$ such that

$$G = \langle C_G(t), N = N_G(\tau^{-1}(A)) \rangle.$$

(3) For some prime $p > 0$ not dividing $|H|\,|N|$ the group G has an irreducible p-modular representation M with multiplicity-free restriction $M_{|H}$.

In section 2 we apply this algorithm to a permutation group $H \cong 2 \times \mathrm{A}_5$ to obtain an existence proof for Janko's group J_1. Here the first author's algorithms and programs [7] for computing concrete character tables with matrix representatives of the conjugacy classes of a finite group have also been used, see Theorem 2.9.

Concerning notation and terminology we refer to the books by G. Butler [2], W. Feit [4] and B. Huppert [5]. All computations described in this article can easily be verified by means of MAGMA [1].

2. Existence proof

In order to construct a finite simple group G which contains a 2-central involution t having centralizer $C_G(t) \cong 2 \times \mathrm{A}_5$ we employ the construction method 4.6 of [8].

The two permutations $(1,2,3,4,5)$ and $(1,3,5)$ generate a finite group which is isomorphic to the alternating group A_5 of order 60. They both commute with the transposition $(6,7)$. Hence we know:

$$2 \times \mathrm{A}_5 \cong \langle (1,2,3,4,5), (1,3,5), (6,7) \rangle =: H \leq \mathrm{S}_7\,,$$

where S_7 denotes the symmetric group of degree 7. Just for the sake of convenience let us reduce the number of generators we have to work with. Obviously, we can achieve $H = \langle x, y \rangle$ by setting

$$x := (1,2,3,4,5) \quad \text{and} \quad y := (1,3,5)(6,7)\,.$$

Notation 2.1. *Within the group H we distinguish the following elements: $z := y^3 = (6,7)$, $a_1 := xy^2 = (1,2)(3,4)$, $a_2 := (x^2y)^2x = (1,3)(2,4)$, and $d := y^2x^2 = (1,2,4)$.*

Lemma 2.2. *Let $A := \langle z, a_1, a_2 \rangle$, and let $D := \langle z, a_1, a_2, d \rangle$. Then:*

(a) A is elementary abelian of order 8;

(b) A is a Sylow 2-subgroup of H;

(c) $z^d = z$, $a_1^d = a_2$, and $a_2^d = a_1a_2$;

(d) $N_H(A) = D \cong \langle z \rangle \times \langle a_1, a_2, d \rangle \cong 2 \times \mathrm{A}_4$.

Proof. Since $a_1a_2 = a_2a_1 = (1,4)(2,3)$ assertion (a) holds, and (b) follows immediately. The equations in (c) can be checked by hand; they show that A is normal in $D \cong \langle z \rangle \times \langle a_1, a_2, d \rangle \cong 2 \times \mathrm{A}_4$. Thus, verifying $N_H(A) \leq D$ by means of MAGMA completes the proof. ☐

According to step 2 of algorithm 4.6 of [8] we state

Proposition 2.3. *Using the notation established so far the following assertions hold:*

(a) $C := C_H(A) = A$.

(b) With respect to the basis $\{z, a_1, a_2\}$ *of the vector space* A *over* GF(2) *the conjugation action of* D *on* A *induces a group homomorphism*

$$\eta : D \to \Delta := \eta(D) = \langle \eta(d) \rangle \leq \mathrm{GL}_3(2)$$

with kernel $ker(\eta) = C$.

(c) Up to conjugacy there is a uniquely determined subgroup Φ *of* $\mathrm{GL}_3(2)$ *which acts naturally on the* GF(2)*-vector space* A *and satisfies the following conditions:*

(i) $\Delta = \mathrm{Stab}_\Phi(z)$;
(ii) $|\Phi : \Delta|$ *is odd;*
(iii) *Up to conjugacy in* $E_\Phi := A : \Phi$ *there is a uniquely determined embedding* μ *of* $D = A : \langle d \rangle$ *into the semidirect product* E_Φ *such that the diagram*

$$\begin{tikzcd}
1 \arrow[r] & A \arrow[r, "\mathrm{id}"'] \arrow[d, "\mathrm{id}"] & E_\Phi \arrow[r, "\lambda"] \arrow[d, "\mu"] & \Phi \arrow[r] \arrow[d, "\mathrm{id}"] & 1 \\
1 \arrow[r] & A \arrow[r, "\mathrm{id}"'] & D \arrow[r, "\eta"] & \Delta \arrow[r] & 1
\end{tikzcd}$$

commutes.

This group Φ *is isomorphic to the Frobenius group* $F_{21} = 7 : 3$.

(d) Up to isomorphism the free product $H *_D E_\Phi$ *with amalgamated subgroup* D *is uniquely determined by* H *and the identification of* D *with* $\mu(D)$ *via the monomorphism* μ.

Proof. Assertion (a) holds by Lemma 2.2.

Consider A as a 3-dimensional vector space over GF(2). With respect to the fixed basis $\{z, a_1, a_2\}$ of this vector space the conjugation action of D on its subgroup A can be described by the following elements of $\mathrm{GL}_3(2)$:

$$\eta(z) := \eta(a_1) := \eta(a_2) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\widehat{d} := \eta(d) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Linear extension yields a homomorphism

$$\eta : D \to \Delta = \langle \widehat{d} \rangle \leq \mathrm{GL}_3(2)$$

with kernel $\ker(\eta) = A = C$.

By Sylow's theorem $GL_3(2)$ has — up to conjugacy — only one subgroup which contains Δ with odd index, namely the Frobenius group $F_{21} = 7:3$. Computer search reveals twelve elements $\widehat{e} \in GL_3(2)$ such that $|\langle \widehat{e} \rangle| = 7$ and $\langle \widehat{e} \rangle$ is normalized by $\widehat{d}$. We may choose

$$\widehat{e} := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \in GL_3(2).$$

Then $\Phi := \langle \widehat{d}, \widehat{e} \rangle \leq GL_3(2)$ is isomorphic to F_{21}. Let $E_\Phi = A : \Phi$ be the semidirect product of A by Φ with respect to the action of Φ on A as a GF(2)-vector space. Then the canonical embedding $\Delta \hookrightarrow \Phi$ induces an embedding μ of $D = A : \langle d \rangle$ into E_Φ such that the diagram

$$\begin{tikzcd} 1 \arrow[r] & A \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"] & E_\Phi \arrow[r, "\kappa"] \arrow[d, "\mu"] & \Phi \arrow[r] \arrow[d, "\mathrm{id}"] & 1 \\ 1 \arrow[r] & A \arrow[r, "\mathrm{id}"] & D \arrow[r, "\eta"] & \Delta \arrow[r] & 1 \end{tikzcd}$$

commutes. Therefore, assertion (c) holds by the theorem of Schur and Zassenhaus; (d) follows immediately. □

From the information given in Lemma 2.2(a),(c) and from the observation that $\widehat{e}^{\widehat{d}} = \widehat{e}^2$ it is now easy to deduce a finite presentation of the group E_Φ:

$$\begin{aligned} E_\Phi = \langle \quad z, a_1, a_2, d, e \quad | \quad & z^2 = a_1^2 = a_2^2 = d^3 = e^7 = 1, \\ & [z, a_1] = [a_1, a_2] = [a_2, z] = 1, \\ & z^d = z,\ a_1^d = a_2,\ a_2^d = a_1 a_2, \\ & z^e = a_1,\ a_1^e = a_2,\ a_2^e = z a_1, \\ & e^d = e^2 \quad \rangle; \end{aligned}$$

Using MAGMA we build a faithful permutation representation ρ of E_Φ by computing its action on the (eight) right cosets of $\langle d, e \rangle$:

$$\begin{aligned} z &\mapsto (1,2)(3,5)(4,6)(7,8), \\ a_1 &\mapsto (1,3)(2,5)(4,7)(6,8), \\ a_2 &\mapsto (1,4)(2,6)(3,7)(5,8), \\ d &\mapsto (3,4,7)(5,6,8), \\ e &\mapsto (2,3,4,5,7,8,6). \end{aligned}$$

Let us identify E_Φ and $\rho(E_\Phi)$ in the sequel.

In order to apply step 3 of algorithm 4.6 of [8] we need the character tables of H, E_Φ and D. Realize that our *concrete* conjugacy class representatives provide

1. the essential link from the irreducible characters of D as a subgroup of $H \leq S_7$ to the irreducible characters of D as a subgroup of $E_\Phi \leq S_8$;
2. the fusions of the conjugacy classes of D into H and of D into E_Φ, respectively.

Character table 2.4. *of the group* $H \cong 2 \times A_5$

Class	1	2_1	2_2	2_3	3	5_1	5_2	6	10_1	10_2
Length	1	1	15	15	20	12	12	20	12	12
Repr.	1	z	a_1	za_1	d	x	a_2x	zd	zx	za_2x
χ_1	1	1	1	1	1	1	1	1	1	1
χ_2	1	-1	1	-1	1	1	1	-1	-1	-1
χ_3	3	-3	-1	1	0	α	$*\alpha$	0	$-\alpha$	$-*\alpha$
χ_4	3	-3	-1	1	0	$*\alpha$	α	0	$-*\alpha$	$-\alpha$
χ_5	3	3	-1	-1	0	α	$*\alpha$	0	α	$*\alpha$
χ_6	3	3	-1	-1	0	$*\alpha$	α	0	$*\alpha$	α
χ_7	4	-4	0	0	1	-1	-1	-1	1	1
χ_8	4	4	0	0	1	-1	-1	1	-1	-1
χ_9	5	-5	1	-1	-1	0	0	1	0	0
χ_{10}	5	5	1	1	-1	0	0	-1	0	0

where $\alpha = \frac{1}{2}(1+\sqrt{5})$, *and* $*\alpha = \frac{1}{2}(1-\sqrt{5})$.

Character table 2.5. *of the group* $E_\Phi \cong 2^3 : F_{21}$

Class	1	2	3_1	3_2	6_1	6_2	7_1	7_2
Length	1	7	28	28	28	28	24	24
Repr.	1	z	d	d^2	zd	zd^2	e	e^3
ϑ_1	1	1	1	1	1	1	1	1
ϑ_2	1	1	β	$\overline{\beta}$	β	$\overline{\beta}$	1	1
ϑ_3	1	1	$\overline{\beta}$	β	$\overline{\beta}$	β	1	1
ϑ_4	3	3	0	0	0	0	γ	$\overline{\gamma}$
ϑ_5	3	3	0	0	0	0	$\overline{\gamma}$	γ
ϑ_6	7	-1	1	1	-1	-1	0	0
ϑ_7	7	-1	β	$\overline{\beta}$	$-\beta$	$-\overline{\beta}$	0	0
ϑ_8	7	-1	$\overline{\beta}$	β	$-\overline{\beta}$	$-\beta$	0	0

where $\beta = \exp\left(\frac{2\pi i}{3}\right)$,
and $\gamma = \frac{1}{2}(-1+i\sqrt{7})$.

Character table 2.6. *of the group* $D \cong 2 \times A_4$

Class	1	2_1	2_2	2_3	3_1	3_2	6_1	6_2
Length	1	1	3	3	4	4	4	4
Repr.	1	z	a_1	za_1	d	d^2	zd	zd^2
δ_1	1	1	1	1	1	1	1	1
δ_2	1	-1	1	-1	1	1	-1	-1
δ_3	1	1	1	1	β	$\overline{\beta}$	β	$\overline{\beta}$
δ_4	1	-1	1	-1	β	$\overline{\beta}$	$-\beta$	$-\overline{\beta}$
δ_5	1	1	1	1	$\overline{\beta}$	β	$\overline{\beta}$	β
δ_6	1	-1	1	-1	$\overline{\beta}$	β	$-\overline{\beta}$	$-\beta$
δ_7	3	3	-1	-1	0	0	0	0
δ_8	3	-3	-1	1	0	0	0	0

where $\beta = \exp\left(\frac{2\pi i}{3}\right)$.

By means of Kratzer's algorithm [7] we calculate the finite set

$$\Pi = \{(\chi, \vartheta) \in \mathrm{mfchar}_{\mathbb{C}}(H) \times \mathrm{fchar}_{\mathbb{C}}(E_\Phi) \mid \chi_{|D} = \vartheta_{|D}\}$$

of *compatible pairs* (χ, ϑ) where $\mathrm{mfchar}_{\mathbb{C}}(H)$ denotes the set of all multiplicity-free faithful characters of H, and $\mathrm{fchar}_{\mathbb{C}}(E_\Phi)$ denotes the set of all faithful characters of E_Φ. The set Π is finite by Proposition 3.5 of [8].

For each $(\chi, \vartheta) \in \Pi$ the positive integer $\chi(1) = \vartheta(1)$ is called the *degree* of the compatible pair (χ, ϑ).

Using the identifiers introduced within the character tables above and printing faithful irreducible characters of any of the three groups H, D or E_Φ in bold face we state the result as

Lemma 2.7. *There are six compatible pairs* (χ, ϑ) *of multiplicity-free faithful characters* χ *of* H *and faithful characters* ϑ *of* E_Φ *with minimal degree* $\chi(1) = \vartheta(1) = 7$, *namely:*

(a) $(\boldsymbol{\chi_7} + \chi_5, \boldsymbol{\vartheta_6})$,

(b) $(\boldsymbol{\chi_7} + \chi_6, \boldsymbol{\vartheta_6})$,

(c) $(\chi_2 + \boldsymbol{\chi_3} + \chi_5, \boldsymbol{\vartheta_6})$,

(d) $(\chi_2 + \boldsymbol{\chi_3} + \chi_6, \boldsymbol{\vartheta_6})$,

(e) $(\chi_2 + \boldsymbol{\chi_4} + \chi_5, \boldsymbol{\vartheta_6})$,

(f) $(\chi_2 + \boldsymbol{\chi_4} + \chi_6, \boldsymbol{\vartheta_6})$.

In each case the common restriction to D *is* $\chi_{|D} = \delta_2 + \boldsymbol{\delta_8} + \delta_7 = \boldsymbol{\vartheta_6}_{|D}$.

Proof. Taking the three character tables 2.4, 2.5, 2.6 and the fusion patterns of D into H and D into E_Φ as input Kratzer's algorithm [7] yields that 7 is the minimal degree of all the compatible pairs $(\chi, \vartheta) \in \Pi$, and that the six faithful characters $\chi \in \mathrm{mfchar}_{\mathbb{C}}(H)$ of degree 7 stated in the assertion are compatible with the irreducible character ϑ_6 of E_Φ. In fact, the algorithm also proves that these are the only compatible pairs (χ, ϑ) of degree 7. □

The smallest prime $p > 0$ not dividing the product $|H|\,|E_\Phi|$ is $p = 11$. By the character tables 2.4 and 2.5 the prime field $\mathbb{F} = \mathrm{GF}(11)$ is a splitting field for all the irreducible constituents occurring in the six compatible pairs (χ, ϑ_6) listed in Lemma 2.7.

Lemma 2.8. *For each compatible pair (χ, ϑ_6) given in Lemma 2.7 there exists a 7-dimensional semisimple multiplicity-free representation $V \leftrightarrow \chi$ of H and an irreducible representation $W \leftrightarrow \vartheta_6$ of E_Φ over* GF(11) *such that*

(1) $H \cong \langle \mathcal{X}, \mathcal{Y} \rangle \leq \mathrm{GL}_7(11)$,

(2) $D \cong \langle \mathcal{Z}, \mathcal{A}_1, \mathcal{A}_2, \mathcal{D} \rangle$,

(3) $E_\Phi \cong \langle \mathcal{Z}, \mathcal{A}_1, \mathcal{A}_2, \mathcal{D}, \mathcal{E} \rangle$,

where $\mathcal{Z} = \mathcal{Y}^3$, $\mathcal{A}_1 = \mathcal{X}\mathcal{Y}^2$, $\mathcal{A}_2 = (\mathcal{X}^2\mathcal{Y})^2\mathcal{X}$, and $\mathcal{D} = \mathcal{Y}^2\mathcal{X}^2$. For each of the cases (a)–(f) of Lemma 2.7 the generating matrices $\mathcal{X}, \mathcal{Y}, \mathcal{E} \in \mathrm{GL}_7(11)$ are given in the appendix.

Proof. (1)&(2) By Lemma 2.2 the group $H = \langle x, y \rangle \leq S_7$ has a faithful permutation representation on $\Omega_H = \{1, 2, \dots, 7\}$. Let $\mathbb{F} = \mathrm{GF}(11)$, and let $P = \mathbb{F}\Omega_H$ be the corresponding permutation module of H over $\mathbb{F}$. Then MAGMA gives us the direct decomposition $P \cong \mathbb{1}_H^2 \oplus M_2 \oplus M_8$, where M_2 corresponds to the signum representation of H, and M_8 is an irreducible module of H affording $\chi_8 \in \mathrm{char}_{\mathbb{C}}(H)$. Hence $M_7 = M_2 \otimes M_8$ is the irreducible $\mathbb{F}H$-module corresponding to $\chi_7 \in \mathrm{char}_{\mathbb{C}}(H)$. By constructing tensor products like $M_7 \otimes M_7$ and splitting them into irreducible constituents we get all the irreducible $\mathbb{F}H$-modules M_i corresponding to the irreducible characters $\chi_i \in \mathrm{char}_{\mathbb{C}}(H)$ occurring in Lemma 2.7 (Note: *Here we fix 4 as a primitive 5-th root of unity in $\mathbb{F}$ — as* MAGMA *does!*). Thus, for each individual case (a)–(f) we now know how to set up blocked diagonal matrices $\mathcal{X}_0, \mathcal{Y}_0 \in \mathrm{GL}_7(11)$ corresponding to the permutations $x, y \in H$.

Let $V = V_\aleph$ be the semisimple $\mathbb{F}H$-module corresponding to the faithful character χ of H given in Lemma 2.7 ($\aleph$), $\aleph \in \{a, b, c, d, e, f\}$. The restriction $V_{|D}$ to the subgroup D of H can be computed easily, just confer Notation 2.1.

In particular, Lemma 2.7 says that $V_{|D}$ decomposes directly into the three pairwise non-isomorphic $\mathbb{F}D$-modules $V_{\aleph 2}$, $V_{\aleph 8}$ and $V_{\aleph 7}$ corresponding to the irreducible characters δ_2, δ_8 and δ_7 of $\mathrm{char}_{\mathbb{C}}(D)$, respectively. However, bases of $V_{\aleph 2}$, $V_{\aleph 8}$, $V_{\aleph 7}$ may be merged to a basis of V, and with respect to this new basis we transform the two matrices $\mathcal{X}_0^{(\aleph)}, \mathcal{Y}_0^{(\aleph)}$ constructed above into matrices $\mathcal{X}^{(\aleph)}, \mathcal{Y}^{(\aleph)}$.

For a complete summary of our explicit results at this stage the reader is referred to the first two columns of the table in the appendix. Since it becomes important later we just want to substantiate case (b) a little bit: Emulating Notation 2.1 here yields that a group isomorphic to D is generated by the following matrices:

$$\mathcal{Z}^{(b)} = \begin{pmatrix} 10 & & & & & & \\ & 10 & 0 & 0 & & & \\ & 0 & 10 & 0 & & & \\ & 0 & 0 & 10 & & & \\ & & & & 1 & 0 & 0 \\ & & & & 0 & 1 & 0 \\ & & & & 0 & 0 & 1 \end{pmatrix}$$

$$\mathcal{A}_1^{(b)} = \begin{pmatrix} 1 & & & & & & \\ & 10 & 10 & 10 & & & \\ & 0 & 0 & 1 & & & \\ & 0 & 1 & 0 & & & \\ & & & & 10 & 10 & 10 \\ & & & & 0 & 0 & 1 \\ & & & & 0 & 1 & 0 \end{pmatrix},$$

$$\mathcal{A}_2^{(b)} = \begin{pmatrix} 1 & & & & & & \\ & 0 & 0 & 1 & & & \\ & 10 & 10 & 10 & & & \\ & 1 & 0 & 0 & & & \\ & & & & 0 & 0 & 1 \\ & & & & 10 & 10 & 10 \\ & & & & 1 & 0 & 0 \end{pmatrix}$$

$$\mathcal{D}^{(b)} = \begin{pmatrix} 1 & & & & & & \\ & 0 & 0 & 1 & & & \\ & 0 & 1 & 0 & & & \\ & 10 & 10 & 10 & & & \\ & & & & 0 & 0 & 1 \\ & & & & 0 & 1 & 0 \\ & & & & 10 & 10 & 10 \end{pmatrix}$$

(3) It remains to construct the irreducible representation W of E_Φ corresponding to $\vartheta_6 \in \mathrm{char}_{\mathbb{C}}(E_\Phi)$. We remember that $E_\Phi = \rho(E_\Phi)$ is a permutation group acting on $\Omega_{E_\Phi} = \{1, 2, \ldots, 8\}$ with stabilizer $\mathrm{Stab}_{E_\Phi}(1) \cong \Phi$ and permutation character $(\mathbb{1}_\Phi)^{E_\Phi} = \mathbb{1}_{E_\Phi} + \vartheta_6$ by the character table 2.5. Let $P = \mathbb{F}\Omega_{E_\Phi}$ be the permutation module of E_Φ over $\mathbb{F} = \mathrm{GF}(11)$. Decomposition of P into irreducible $\mathbb{F}E_\Phi$-modules by means of MAGMA confirms $P \cong \mathbb{1}_{E_\Phi} \oplus W$.

We now have to match the groups $D \leq H$ and $\mu(D) \leq E_\Phi$ *effectively* for each case of Lemma 2.7.

With respect to some fixed basis of W the five permutations $\rho(z)$, $\rho(a_1)$, $\rho(a_2)$, $\rho(d)$, $\rho(e) \in \mathrm{S}_8$ can be represented by 7×7-matrices $\widetilde{\mathcal{Z}}$, $\widetilde{\mathcal{A}}_1$, $\widetilde{\mathcal{A}}_2$, $\widetilde{\mathcal{D}}$, $\widetilde{\mathcal{E}}$ over the field $\mathbb{F}$, respectively. Therewith $\mu(D) = \langle \widetilde{\mathcal{Z}}, \widetilde{\mathcal{A}}_1, \widetilde{\mathcal{A}}_2, \widetilde{\mathcal{D}} \rangle \leq \mathrm{GL}_7(11)$, and $\mu(D) \cong D = \langle \mathcal{Z}, \mathcal{A}_1, \mathcal{A}_2, \mathcal{D} \rangle \leq \mathrm{GL}_7(11)$. Hence there exists a group isomorphism ψ between those two subgroups induced by the corresponding base change. Employing MAGMA again we obtain a transformation matrix $\mathcal{L}_\psi \in \mathrm{GL}_7(11)$ such that

$$\begin{aligned}
\mathcal{Z} &= \mathcal{L}_\psi^{-1}\,\widetilde{\mathcal{Z}}\,\mathcal{L}_\psi\,, \\
\mathcal{A}_1 &= \mathcal{L}_\psi^{-1}\,\widetilde{\mathcal{A}}_1\,\mathcal{L}_\psi\,, \\
\mathcal{A}_2 &= \mathcal{L}_\psi^{-1}\,\widetilde{\mathcal{A}}_2\,\mathcal{L}_\psi\,, \\
\mathcal{D} &= \mathcal{L}_\psi^{-1}\,\widetilde{\mathcal{D}}\,\mathcal{L}_\psi\,.
\end{aligned}$$

(Note: *The transformation $\mathcal{L}_\psi$ is uniquely determined up to multiplication with elements of $C_{\mathrm{GL}_7(11)}(\mu(D))$!*) Obviously, setting $\mathcal{E} := \mathcal{L}_\psi^{-1}\widetilde{\mathcal{E}}\mathcal{L}_\psi$ ensures that we have $E_\Phi \cong \langle \mathcal{Z}, \mathcal{A}_1, \mathcal{A}_2, \mathcal{D}, \mathcal{E}\rangle$ in any of the cases (a)–(f). □

Finally, we are able to give an existence proof of Janko's sporadic simple group J_1 described in Theorem 1.1.

Theorem 2.9. *Let H be some finite group with a central involution $z \neq 1$ such that $H \cong \langle z\rangle \times \mathrm{A}_5$. Moreover, let A be a fixed Sylow 2-subgroup of H, $D = N_H(A)$, and $\eta : D \to \mathrm{GL}_3(2)$ be the homomorphism determined by the conjugation action of D on A. Then the following assertions hold:*

(a) Up to conjugacy there exists a unique subgroup $\Phi \cong F_{21}$ of $\mathrm{GL}_3(2)$ containing $\Delta = \eta(D) \cong 3$ with odd index and an embedding μ of D into the semidirect product $E_\Phi = A : \Phi$ such that the diagram

$$\begin{tikzcd}
1 \arrow[r] & A \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"] & E_\Phi \arrow[r, "\lambda"] \arrow[d, "\mu"] & \Phi \arrow[r] \arrow[d, "\mathrm{id}"] & 1 \\
1 \arrow[r] & A \arrow[r, "\mathrm{id}"] & D \arrow[r, "\eta"] & \Delta \arrow[r] & 1
\end{tikzcd}$$

commutes.

(b) The free product $H *_D E_\Phi$ of H and E_Φ with amalgamated subgroup D is uniquely determined by H up to isomorphism, and there is a unique 7-dimensional irreducible representation $\kappa : H *_D E_\Phi \to \mathrm{GL}_7(11)$ over the field $\mathrm{GF}(11)$ such that the group

$$J = \langle \kappa(H), \kappa(E_\Phi)\rangle \leq \mathrm{GL}_7(11)$$

has an involution $\mathcal{Z}$ with $C_J(\mathcal{Z}) \cong H$.

(c) J is a simple group of order $|J| = 175560$ generated by the matrices $\mathcal{X}, \mathcal{Y}$ in line (b) of the table in the appendix and the matrix

$$\mathcal{S} = \begin{pmatrix}
0 & 8 & 0 & 8 & 0 & 0 & 0 \\
0 & 5 & 5 & 0 & 2 & 1 & 1 \\
0 & 6 & 6 & 0 & 0 & 1 & 10 \\
0 & 5 & 5 & 0 & 9 & 10 & 10 \\
10 & 0 & 3 & 3 & 0 & 5 & 5 \\
1 & 0 & 3 & 3 & 0 & 6 & 6 \\
1 & 0 & 8 & 8 & 0 & 5 & 5
\end{pmatrix}$$

(d) Matrix representatives $\mathcal{W}$ for the conjugacy classes $(\mathcal{W})^J$ of J are given in the following table:

Order	$\mathcal{W}$	$\lvert J : C_J(\mathcal{W})\rvert$	$\lvert C_J(\mathcal{W})\rvert$
1	1	1	$2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$
2	$\mathcal{X}\mathcal{Y}^2$	1463	$2^3 \cdot 3 \cdot 5$
3	$\mathcal{Y}^2$	5852	$2 \cdot 3 \cdot 5$
5_1	$\mathcal{X}$	5852	$2 \cdot 3 \cdot 5$
5_2	$\mathcal{X}^2$	5852	$2 \cdot 3 \cdot 5$
6	$\mathcal{Y}$	29260	$2 \cdot 3$
7	$\mathcal{S}$	25080	7
10_1	$\mathcal{X}\mathcal{Y}$	17556	$2 \cdot 5$
10_2	$\mathcal{X}\mathcal{S}$	17556	$2 \cdot 5$
11	$\mathcal{X}^2\mathcal{Y}\mathcal{S}$	15960	11
15_1	$\mathcal{X}\mathcal{Y}\mathcal{S}$	11704	$3 \cdot 5$
15_2	$\mathcal{X}^4\mathcal{S}$	11704	$3 \cdot 5$
19_1	$\mathcal{X}\mathcal{Y}\mathcal{X}\mathcal{S}$	9240	19
19_2	$\mathcal{X}^2\mathcal{S}\mathcal{Y}\mathcal{S}$	9240	19
19_3	$\mathcal{S}\mathcal{X}^2\mathcal{Y}\mathcal{S}$	9240	19

(e) J has the same character table as Janko's sporadic simple group J_1 *given in the* ATLAS[3].

Proof. (a) follows immediately from Proposition 2.3. Hence the amalgamated free product $P = H *_D E_\Phi$ is uniquely determined by H up to isomorphism.

Let now (χ, ϑ_6) be any of the compatible pairs of faithful multiplicity-free characters χ of H and faithful irreducible character ϑ_6 of E_Φ determined in Lemma 2.7 of minimal degree $\chi(1) = 7 = \vartheta_6(1)$. According to step 4 of algorithm 4.6 of [8] identify H and E_Φ with their isomorphic images in $\mathrm{GL}_7(11)$ afforded by the faithful modules V and W over GF(11) corresponding to the characters χ and ϑ_6 of H and E_Φ, respectively.

For each of the cases (a)–(f) the matrix generators of H and E_Φ are given in Lemma 2.8. By Lemma 2.7 the compatible pairs (χ, ϑ_6) of the cases (c)–(f) have a faithful semi-simple multiplicity-free character χ with three irreducible constituents, and the common restriction $\chi_{|D} = \vartheta_{6|D}$ to $D = H \cap E_\Phi$ has three non-isomorphic irreducible constituents as well. Therefore Thompson's theorem [9] asserts that the free product $P = H *_D E_\Phi$ has only one irreducible 7-dimensional representation $\kappa : P \to \mathrm{GL}_7(11)$ in any of these four cases. Hence

$$\kappa(P) = \langle \mathcal{X}, \mathcal{Y}, \mathcal{E} \rangle \leq \mathrm{GL}_7(11),$$

where $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{E}$ are the matrices in $\mathrm{GL}_7(11)$ of Lemma 2.8. The explicit triple of generators for each particular case can be found in the corresponding line of the table in the appendix.

By step 4 c) of algorithm 4.6 of [8] we now have to check whether each Sylow 2-subgroup S of these four groups $\kappa(P)$ has exponent 2. This is not the case as we see from the following table of orders $\mathrm{ord}(\mathcal{M})$ of certain elements $\mathcal{M} \in \kappa(P)$:

Case	$\mathrm{ord}(\mathcal{X}\mathcal{E})$	$\mathrm{ord}(\mathcal{X}\mathcal{E}^3)$
(c)	37	60
(d)	60	15
(e)	366	60
(f)	133	132

In each of the cases (a) and (b) of Lemma 2.7 one has to examine ten different irreducible 7-dimensional representations $\kappa_i(P) \to \mathrm{GL}_7(11)$, parametrized by diagonal matrices $\mathcal{C}_i := \mathrm{diag}(i, 1, 1, 1, 1, 1, 1)$, $i \in \mathrm{GF}(11)^\times$, by Thompson's theorem [9].

Denote the three generating matrices of Lemma 2.8 by $\mathcal{X}_a$, $\mathcal{Y}_a$, $\mathcal{E}_a$ and $\mathcal{X}_b$, $\mathcal{Y}_b$, $\mathcal{E}_b$ in case (a) and (b), respectively. Then by step 4 c) of algorithm 4.6 of [8] we have to determine the exponent of a Sylow 2-subgroup of any of the following groups:

$$\begin{aligned} J_{a,i} &:= \langle \mathcal{X}_a, \mathcal{Y}_a, \mathcal{C}_i^{-1}\mathcal{E}_a\mathcal{C}_i \rangle \leq \mathrm{GL}_7(11) \, , \\ J_{b,i} &:= \langle \mathcal{X}_b, \mathcal{Y}_b, \mathcal{C}_i^{-1}\mathcal{E}_b\mathcal{C}_i \rangle \leq \mathrm{GL}_7(11) \, . \end{aligned}$$

Therefore we compute the orders of the elements

$$\begin{aligned} \mathcal{M}_{a,i} &= \mathcal{X}_a\mathcal{C}_i^{-1}\mathcal{E}_a\mathcal{C}_i \, , & \mathcal{M}_{b,i} &= \mathcal{X}_b\mathcal{C}_i^{-1}\mathcal{E}_b^2\mathcal{C}_i \, , \\ \mathcal{N}_{a,i} &= \mathcal{X}_a\mathcal{C}_i^{-1}\mathcal{E}_a^4\mathcal{C}_i \, , & \mathcal{N}_{b,i} &= \mathcal{X}_b\mathcal{C}_i^{-1}\mathcal{E}_b^4\mathcal{C}_i \end{aligned}$$

for all $i = 1, 2, \ldots 10$. The results are given in the following table:

i	$\mathrm{ord}(\mathcal{M}_{a,i})$	$\mathrm{ord}(\mathcal{N}_{a,i})$	$\mathrm{ord}(\mathcal{M}_{b,i})$	$\mathrm{ord}(\mathcal{N}_{b,i})$
1	354312	1771560	1320	1771550
2	111	8	60	133
3	7320	1948717	32210	486780
4	354312	1771560	1320	1771550
5	7320	1948717	32210	486780
6	32210	354312	1330	7320
7	7320	1330	1220	7320
8	32210	354312	1330	7320
9	333	132	6	5
10	7320	1330	1220	7320

The table shows that only in case (b) the choice $i = 9$ leads to a group $J_{b,9}$ with a possible Sylow 2-subgroup isomorphic to 2^3. In particular, assertion (b) holds by Thompson's theorem [9].

Let $J = J_{b,9} = \langle \mathcal{X}_b, \mathcal{Y}_b, \mathcal{S} \rangle \leq \mathrm{GL}_7(11)$, where $\mathcal{S} := \mathcal{C}_9{}^{-1}\mathcal{E}_a\mathcal{C}_9$ is the matrix stated in assertion (c). Employing MAGMA we can now construct the permutation representation $(\mathbb{1}_H)^J$ of J. Indeed, $(\mathbb{1}_H)^J$ is faithful, and $|J : H| = 1463$. Thus we may conclude $|J| = 175560$.

By application of Kratzer's algorithm [7] the conjugacy classes of J have representatives as given in table (d).

Using MAGMA and (d) it follows that J has the same character table as Janko's group J_1 which is given in the ATLAS [3]. In particular, J is simple.

Certainly $H = \langle \mathcal{X}_b, \mathcal{Y}_b \rangle \leq C_J(\mathcal{Z})$ for the involution $\mathcal{Z} = \mathcal{Y}_b^3$ ($= \mathcal{Z}^{(b)}$ as given in the proof of Lemma 2.8). Since $|C_J(\mathcal{Z})| = 120$ by the character table of J, we have $C_J(\mathcal{Z}) = H \cong 2 \times A_5$. □

3. Appendix: Generators of the local subgroups H and E_Φ of Lemma 2.8

Case	$\mathcal{X}$	$\mathcal{Y}$	$\mathcal{E}$
(a)	$\begin{pmatrix} 8&5&5&5&&& \\ 9&8&9&8&&& \\ 9&8&8&9&&& \\ 6&8&8&8&&& \\ &&&&0&9&5 \\ &&&&0&6&1 \\ &&&&1&10&2 \end{pmatrix}$	$\begin{pmatrix} 3&6&6&6&&& \\ 2&2&3&3&&& \\ 5&3&3&3&&& \\ 2&3&3&2&&& \\ &&&&0&5&10 \\ &&&&0&2&6 \\ &&&&10&1&9 \end{pmatrix}$	$\begin{pmatrix} 0&6&0&6&0&0&0 \\ 0&5&5&0&0&0&8 \\ 0&6&6&0&4&4&2 \\ 0&5&5&0&0&0&3 \\ 3&0&4&4&10&1&0 \\ 8&0&4&4&10&1&0 \\ 0&0&9&9&3&8&0 \end{pmatrix}$
(b)	$\begin{pmatrix} 8&5&5&5&&& \\ 9&8&9&8&&& \\ 9&8&8&9&&& \\ 6&8&8&8&&& \\ &&&&7&5&9 \\ &&&&9&7&5 \\ &&&&1&1&1 \end{pmatrix}$	$\begin{pmatrix} 3&6&6&6&&& \\ 2&2&3&3&&& \\ 5&3&3&3&&& \\ 2&3&3&2&&& \\ &&&&5&9&7 \\ &&&&1&1&1 \\ &&&&9&7&5 \end{pmatrix}$	$\begin{pmatrix} 0&6&0&6&0&0&0 \\ 0&5&5&0&2&1&1 \\ 0&6&6&0&0&1&10 \\ 0&5&5&0&9&10&10 \\ 6&0&3&3&0&5&5 \\ 5&0&3&3&0&6&6 \\ 5&0&8&8&0&5&5 \end{pmatrix}$
(c)	$\begin{pmatrix} 1&&&&&& \\ &0&9&5&&& \\ &0&6&1&&& \\ &1&10&2&&& \\ &&&&0&9&5 \\ &&&&0&6&1 \\ &&&&1&10&2 \end{pmatrix}$	$\begin{pmatrix} 10&&&&&& \\ &0&6&1&&& \\ &0&9&5&&& \\ &1&10&2&&& \\ &&&&0&5&10 \\ &&&&0&2&6 \\ &&&&10&1&9 \end{pmatrix}$	$\begin{pmatrix} 0&8&8&10&0&0&0 \\ 0&1&1&8&9&9&10 \\ 0&1&1&8&0&0&7 \\ 0&8&8&9&6&6&4 \\ 1&3&8&0&10&1&0 \\ 10&3&8&0&10&1&0 \\ 0&4&7&0&3&8&0 \end{pmatrix}$
(d)	$\begin{pmatrix} 1&&&&&& \\ &0&9&5&&& \\ &0&6&1&&& \\ &1&10&2&&& \\ &&&&7&5&9 \\ &&&&9&7&5 \\ &&&&1&1&1 \end{pmatrix}$	$\begin{pmatrix} 10&&&&&& \\ &0&6&1&&& \\ &0&9&5&&& \\ &1&10&2&&& \\ &&&&5&9&7 \\ &&&&1&1&1 \\ &&&&9&7&5 \end{pmatrix}$	$\begin{pmatrix} 0&6&6&2&0&0&0 \\ 0&1&1&8&0&9&2 \\ 0&1&1&8&7&9&9 \\ 0&8&8&9&1&1&0 \\ 8&4&7&0&0&5&5 \\ 3&4&7&0&0&6&6 \\ 3&7&4&0&0&5&5 \end{pmatrix}$
(e)	$\begin{pmatrix} 1&&&&&& \\ &7&5&9&&& \\ &9&7&5&&& \\ &1&1&1&&& \\ &&&&0&9&5 \\ &&&&0&6&1 \\ &&&&1&10&2 \end{pmatrix}$	$\begin{pmatrix} 10&&&&&& \\ &6&2&4&&& \\ &10&10&10&&& \\ &2&4&6&&& \\ &&&&0&5&10 \\ &&&&0&2&6 \\ &&&&10&1&9 \end{pmatrix}$	$\begin{pmatrix} 0&7&0&7&0&0&0 \\ 0&5&5&0&0&0&10 \\ 0&6&6&0&5&5&8 \\ 0&5&5&0&0&0&1 \\ 3&0&1&1&10&1&0 \\ 8&0&1&1&10&1&0 \\ 0&0&5&5&3&8&0 \end{pmatrix}$
(f)	$\begin{pmatrix} 1&&&&&& \\ &7&5&9&&& \\ &9&7&5&&& \\ &1&1&1&&& \\ &&&&7&5&9 \\ &&&&9&7&5 \\ &&&&1&1&1 \end{pmatrix}$	$\begin{pmatrix} 10&&&&&& \\ &6&2&4&&& \\ &10&10&10&&& \\ &2&4&6&&& \\ &&&&5&9&7 \\ &&&&1&1&1 \\ &&&&9&7&5 \end{pmatrix}$	$\begin{pmatrix} 0&3&0&3&0&0&0 \\ 0&5&5&0&5&8&8 \\ 0&6&6&0&0&8&3 \\ 0&5&5&0&6&3&3 \\ 7&0&10&10&0&5&5 \\ 4&0&10&10&0&6&6 \\ 4&0&1&1&0&5&5 \end{pmatrix}$

References

[1] W. Bosma and J. Cannon, MAGMA *Handbook*, Sydney, 1996.
[2] G. Butler, *Fundamental algorithms for permutation groups*, Lect. Notes in Computer Science **559**, Springer Verlag, Heidelberg, 1991.
[3] J.H. Conway, R.T. Curtis, S.P. Norton, R.A. Parker, R.A. Wilson, *Atlas of finite groups*, Clarendon Press, Oxford, 1985.
[4] W. Feit, *The representation theory of finite groups*, North-Holland, Amsterdam, 1982.
[5] B. Huppert, *Endliche Gruppen I*, Heidelberg, 1967.
[6] Z. Janko, *A new finite simple group with abelian Sylow 2-subgroups and its characterization*, J. Algebra **3**, (1966), 147-186.
[7] M. Kratzer, *Algorithms for constructing concrete character tables of finite matrix groups*, in preparation.
[8] G.O. Michler, *On the construction of the finite simple groups with a given centralizer of a 2-central involution*, Preprint.
[9] J. G. Thompson, *Finite-dimensional representations of free products with an amalgamated subgroup*, J. Algebra **69** (1981), 146-149.

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