数理解析研究所講究録
1149 巻 2000 年 76-79

DADE'S CONJECTURE FOR FINITE SPECIAL LINEAR GROUPS

鋤崎 英記
HIDEKI SUKIZAKI
(OSAKA UNIVERSITY)

1. DADE'S CONJECTURE

Let p be a prime number, and let G a finite group. A p-chain C of G is any strictly increasing chain

$$(1-1) \qquad C : U_0 < U_1 < \cdots < U_m$$

of p-subgroups U_i of G. We denote the length m of C by $|C|$. If K is any group acting (exponentially) as automorphisms of G, then any $g \in K$ sends the p-chain C to the p-chain

$$(1-2) \qquad C^g : U_0^g < U_1^g < \cdots < U_m^g$$

of G. The normalizer $N_K(C)$ of C in K is the subgroup of all $g \in K$ such that $C = C^g$, i.e.,

$$N_K(C) = \bigcap_{i=0}^{m} N_K(U_i).$$

We say that the p-chain C in (1-1) is radical (with respect with G) if U_0 is the largest normal p-subgroup $O_p(G)$ of G and

$$U_i = O_p(\bigcap_{j=0}^{i} N_G(U_j)) \quad \text{for } i = 1, 2, \cdots, m.$$

We denote by $\mathfrak{R}(G)$ the set of all radical p-chains of G. The set $\mathfrak{R}(G)$ is closed under the conjugation action (1-2) of G on its p-chains. We denote by $\mathfrak{R}(G)/G$ any complete representatives for the G-conjugacy classes in $\mathfrak{R}(G)$.

For a p-block B of G and a non negative integer d, we denote by $\mathrm{Irr}(H, B, d)$ the set of complex irreducible characters ψ of H such that
(i) the p-part of $|H|/\psi(1)$ is p^d, and
(ii) ψ lies in a p-block b of H such that $b^G = B$.

In [D1], E. C. Dade gives the following conjecture.

Conjecture 1 (Dade's ordinary conjecture). *If $O_p(G) = 1$ and the defect of B is positive, then*

$$\sum_{C \in \mathfrak{R}(G)/G} (-1)^{|C|} |\mathrm{Irr}(N_G(C), B, d)| = 0.$$

We mention a stronger conjecture.

Let E be a finite group such that $G \lhd E$. By the conjugation action of E on G, we define an action (1-2) of E on the p-chains C of G. So any such C has a normalizer $N_E(C)$ in E, and we have $N_G(C) \lhd N_E(C)$. Thus $N_E(C)$ acts by conjugation on $\mathrm{Irr}(N_G(C))$. For $\phi \in \mathrm{Irr}(N_G(C))$, we write

$$T_{N_E(C)}(\phi) = \{g \in N_E(C) | \phi^g = \phi\}.$$

For $\bar{F} \lhd E/G$, we denote by $\mathrm{Irr}(N_G(C), B, d, \bar{F})$ the set of $\phi \in \mathrm{Irr}(N_G(C), B, d)$ such that

(iii) $G \cdot T_{N_E(C)}(\phi)/G = \bar{F}$.

The following conjecture is given in [D2].

Conjecture 2 (Dade's invariant conjecture). *If $O_p(G) = 1$ and the defect of B is positive, then*

$$\sum_{C \in \mathfrak{R}(G)/G} (-1)^{|C|} |\mathrm{Irr}(N_G(C), B, d, \bar{F})| = 0.$$

Here, we treat a verification of Dade's invariant conjecture for $G = SL(n, q)$ and $E = GL(n, q)$ with $p|q$. This implies Dade's invariant conjecture for $G = PSL(n, q)$ and $E = PGL(n, q)$.

2. On radical p-chains of a Chevalley group

In this section, let G be a Chevalley group and let the definig field of G charactaristic p. Then $\mathfrak{R}(G)$ is the set of p-chains consisting of unipotent radicals of parabolic subgroups of G [BT] [BW]. Now we fix a Borel subgroup U. Then we may take $\mathfrak{R}(G)/G$ to be the set of p-chains consisting of unipotent radicals of parabolic subgroups of G cotaining U. Thus, for any $C \in \mathfrak{R}(G)/G$, $N_G(C)$ is some parabolic subgroup of G containing U.

It is well known that the set of all parabolic subgroups of G containing U is parametrized by the set of subsets of a fundamental root system I of G. Thus we denote by P_J the parabolic subgroup corresponding to $J \subseteq I$.

By the above argument and [W] [KR], Conjecture 2 is euqivalent to the following.

Conjecture 3. *If $O_p(G) = 1$ and the defect of B is positive, then*

$$\sum_{J \subseteq I} (-1)^{|I \setminus J|} |\mathrm{Irr}(P_J, B, d, \bar{F})| = 0.$$

3. The case for $G = SL(n,q)$ and $E = GL(n,q)$ $(p|q)$

We consider the case for $G = SL(n,q)$ and $E = GL(n,q)$ with $p|q$.

We take $I = \{1, 2, \cdots, n-1\}$ as a fundamental root system and take the subgroup U of lower triangular matrices in $GL(n,q)$ as a Borel subgroup of $GL(n,q)$. Then, if $J \subseteq I$ satisfying $I \backslash J = \{a_1, \cdots, a_k\}$, the parabolic subgroup P_J of $GL(n.q)$ is

$$\{(p_{ij}) \in GL(n.q)| \text{ If some } k \text{ satisfies } i \leq a_k \text{ and } j > a_k, \text{ then } \quad p_{ij} = 0\}.$$

Moreover $U \cap SL(n,q)$ is a Borel subgroup of $SL(n,q)$ and $P_J \cap SL(n,q)$ is a parabolic subgroup of $SL(n,q)$ containing $U \cap SL(n,q)$.

Here we restate Dade conjecture for $SL(n,q)$ to a statement on $GL(n,q)$. For a positive integer s, we denote by $\mathrm{Irr}(J, B, d, s)$ the set of irreducible characters ψ in $Irr(P_J \cap SL(n,q), B, d)$ such that the $GL(n,q)$-conjugacy class containing ψ has s elements. Because $GL(n,q)/SL(n,q)$ is cyclic and its order is relatively prime to p, Conjecture 3 for $G = SL(n,q)$ and $E = GL(n,q)$ is equivalent to the following: For any p-block B of $SL(n,q)$ whose defect is positive, any non negative integer d and any positive integer s,

$$\sum_{J \subseteq I} (-1)^{|I \backslash J|} |\mathrm{Irr}(J, B, d, s)| = 0.$$

For a positive integer s and a p-block $\bar{B}$ of $GL(n,q)$, we denote by $\widetilde{\mathrm{Irr}}(J, \bar{B}, d, s)$ the set of irreducible characters ϕ in $Irr(P_J, \bar{B}, d)$ such that the restriction of ϕ to $P_J \cap SL(n,q)$ has s irreducible constituents. Then, we have the following theorem on $GL(n,q)$, slightly stronger than the above statement.

Theorem[S]. *For any p-block $\bar{B}$ of $GL(n,q)$ whose defect is positive, any non negative integer d and positive integer s, the following holds:*

$$\sum_{J \subseteq I} (-1)^{|I \backslash J|} |\widetilde{\mathrm{Irr}}(J, \bar{B}, d, s)| = 0.$$

The proof of this theorem is an extention of the proof of Dade's ordinary conjecture for $GL(n,q)$ [OU].

Thus, we have

Corollary. *If $p|q$, Conjecture 3 for $G = SL(n,q)$ and $E = GL(n,q)$ is true. Moreover conjecture 3 for $G = PSL(n,q)$ and $E = PGL(n,q)$ is true.*

References

[BT] A.Borel and J.Tits: *Eléments unipotents et sous-groupes paraboliques des groupes réductifs, I*, Invent.Math. **12** (1971), 95–104.

[BW] N.Burgoyne and C.Williamson: *On a theorem of Borel and Tits for finite Chevalley groups*, Arch.Math. **27** (1976), 489–491.

[D1] E.C.Dade: *Counting characters in blocks, I*, Invent. Math. **109** (1992), 187–210.

[D2] E.C.Dade: *Counting characters in blocks, 2.9*, Representation Theory of Finite Groups (R.Solomon, ed.), Walter de Gruyter & Co., Berlin · New York, 1997, p. 45–59.

[KR] R.Knörr and G.Robinson: *Some remarks on a conjecture of Alperin*, J.London Math.Soc. **(2)39** (1989), 48–60.

[OU] J.B.Olsson and K.Uno: *Dade's conjecture for general linear groups in the defining characteristic*, Proc.London Math.Soc. **(3)72** (1996), 359–384.

[S] H.Sukizaki: *Dade's conjecture for special linear groups in the defining characteristic* (submitted to J.Algebra).

[W] P.J.Webb *Subgroup complexes*, The Arcara conference on Representaions of Finite Groups (Proceeding of Symposia in Pure Mathematics 47 (American Mathematical Society, Providence, R.I. 1987)) (P.Fong, ed.), p. 349–365.