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YANG-MILLS THEORY IN EINSTEIN-WEYL GEOMETRY AND AFFINE GEOMETRY

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§1. Introduction.

This is an expository paper which explains our recent work on Yang-Mills theory, Weyl geometry, and affine differential geometry, based on [U1], [U2], [U3], [U4], [IFU], [DIU].

Yang-Mills theory and the other variational theory as Seiberg-Witten theory have been developed greatly and influenced to topology and physics, especially in the case of 4-dimensional manifolds. The aim of this paper is to see relationships between Yang-Mills theory and differential geometry, and to give a new insight on Yang-Mills theory, and apply it to the theory of Weyl geometry and affine differential geometry.

§2. Yang-Mills theory in differential geometry.

2.1. Yang-Mills theory appeared in differential geometry as Riemannian manifolds with harmonic curvature (cf. [Bo], [Be, p. 443]). This means that Riemannian manifolds (M, g) of which curvature tensor R of the Levi-Civita connection ∇ satisfies $\delta R = 0$, i.e., ∇ is a Yang-Mills connection, taking $E = TM$, the tangent bundle of M, and $h = g$ as in §4. For recent works, see [De1], [De2], [O], [KN], [Um].

2.2. In this subsection, we see the ralationship between Ynag-Mills theory and Kähler geometry. In 1985, Donaldson showed a stable holomorphic vector bundle over a projective surface admits a unique hermitian Yang-Mills connection (cf. [Do]). Kobayashi formulated this theorem for a holomorphic vector bundle E over a compact Kähler manifold (M, g) with a hermitian metric h as follows (cf. [Ko]). A connection D of (E, h) is *hermitian* if it satisfies that

(i) $D_{\overline{X}} s = \overline{\partial}_{\overline{X}} s, \quad \overline{X} \in \Gamma(T^{0,1}M),\ s \in \Gamma(E),$

(ii) $Xh(s,t) = h(D_X s,t) + h(s, D_{\overline{X}} t), \quad X \in \Gamma(T^{\mathbf{C}}M),\ s,t \in \Gamma(E),$
where $\overline{\partial}$ is the holomorphic structure of E, $T^{\mathbf{C}}M$ is the complexification of TM which decomposed into $T^{\mathbf{C}}M = T^{1,0}M \oplus T^{0,1}M$, $\overline{X}$ is the complex conjugate of $X \in T^{\mathbf{C}}M$, and $\Gamma(E)$ is the space of smooth sections of E. Then the curvature tensor R^D belongs to the space $A^{1,1}(\mathrm{End}(E))$ of 2 forms on M with values in $\mathrm{End}(E)$ of type (1,1). We can define the trace ΛR^D of R^D naturally by

$$\sqrt{-1}\Lambda R^D = \sum_{i=1}^{n} R^D(e_i, \overline{e_i}),$$

where $\{e_i\}_{i=1}^n$ is a basis of $T^{1,0}M$ satisfying $g(e_i, \overline{e_j}) = \delta_{ij}$. Kobayashi defined for a hermitian connection D, to be *Einstein-Hermitian* connection by

$$\sqrt{-1}\Lambda R^D = c\,\mathrm{Id}, \tag{2.1}$$

for some constant c, where Id is the identity operator of $\mathrm{End}(E)$, and showed that D is a Yang-Mills connection of (E,h) if D is Einstein-Hermitian (cf. [Ko]). Furthermore, it holds (cf. [Ko], [UY], and see also [Su]) that

Theorem 2.2. *Let E be a holomorphic vector bundle with a hermitian metric h over a compact Kähler manifold (M,g). Then there exists a unique Einstein-Hermitian connection D if and only if E is stable in the sense of algebraic geometry.*

2.3. In the case of odd dimensional manifolds, one can also formulate a similar theory. Let M be a smooth manifold of dimension $2n+1$. M is called to be a *CR manifold* if there exists an n-dimensional subbundle $\mathcal{S}$ of $T^{\mathbf{C}}M$ satisfying that

$$\text{(i)}\quad \mathcal{S} \cap \overline{\mathcal{S}} = \{0\}, \quad \text{and} \quad \text{(ii)}\quad [X,Y] \in \Gamma(\mathcal{S}) \quad \text{for all} \quad X, Y \in \Gamma(\mathcal{S}).$$

Then there exist a subbundle P of TM and a bundle map I of P satisfying that $P^{\mathbf{C}} = \mathcal{S} \oplus \overline{\mathcal{S}}$, $I^2 = -\mathrm{Id}$ and $\mathcal{S} = \{X - \sqrt{-1}IX;\ X \in P\}$. We assume a *contact* 1-form θ on M whose anihilater in T_xM coincides with P_x for all $x \in M$, and $\omega = -d\theta$ is non-degenerate everywhere on M. There exists a unique vector field ξ on M satisfying $\theta(\xi) = 1$, $\omega(\xi, \bullet) = 0$, and $[\xi, X] \in \Gamma(P)$ for all $X \in \Gamma(P)$. Then $T_xM = \mathbf{R}\xi \oplus P_x$, $x \in M$. A contact CR manifold (M,θ) is *strongly pseudoconvex* if the *Levi form* L defined by $L(X,Y) = \omega(IX,Y)$, $X, Y \in P_x$, $x \in M$, is positive definite everywhere on M. Putting $L(\xi, \bullet) = 0$, we can define a Riemannian metric g by

$$g(X,Y) = L(X,Y) + \theta(X)\theta(Y),\ X,\ Y \in T_xM,\ x \in M.$$

In 1975, Tanaka (cf. [T]) introduced the notion of holomorphic vector bundle over this strongly pseudoconver CR manifold (M,g). A complex vector bundle E over M is *holomorphic* if there exists a differential operator $\overline{\partial}$ of E satisfying that

$$\text{(i)}\quad \overline{X}(fs) = \overline{X}f\,s + f\,\overline{X}s, \quad f \in C^\infty(M),\ \overline{X} \in \Gamma(\overline{\mathcal{S}}),$$
$$\text{(ii)}\quad [\overline{X},\overline{Y}]s = \overline{X}(\overline{Y}s) - \overline{Y}(\overline{X}s), \quad \overline{X},\ \overline{Y} \in \Gamma(\overline{\mathcal{S}}).$$

Then one can define by the same way, the notion of *hermitian connection* as the case of Kähler manifolds. Furthermore, Tanaka (cf. [T]) showed

Theorem 2.3. *There exists a unique hermitian connection (called* Tanaka's connection *) D on a holomorphic vector bundle E with a hermitian metric h over a compact strongly pseudoconvex CR manifold (M, g) satisfying that*

$$\sqrt{-1}\Lambda R^D = \sum_{i=1}^{n} R^D(e_i, \overline{e_i}) = 0,$$

where $\{e_i\}_{i=1}^n$ is a basis of S_x satisfying $g(e_i, \overline{e_j}) = \delta_{ij}$ $(x \in M)$.

Then we obtain (cf. [U1])

Theorem 2.4. *Assume that (E, h) is as in Theorem 2.3 and D is a hermitian connection whose curvature R^D is of (1,1) type, i.e., $R^D \in \Gamma(S^* \otimes \overline{S}^* \otimes \mathrm{End}(E))$. Then D is a Yang-Mills connection if and only if D is Tanaka's connection.*

The moduli space theory of Yang-Mills connections over compact strongly pseudoconvex CR manifolds (M, g) can be obtained as in the case of Kähler manifolds (cf. [Ko], [U1]).

§3 Affine differential geometry and Weyl geometry.

Weyl geometry was formulated by H.Weyl to initiate the gauge theory, and affine differential geometry was initiated by W.Blaschke, and recently they have been developed extensively (cf. [NS]). Due to Rao and Amari (cf. [R], [A]), it turns out that the affine differential geometry is closely related to statistics .

Following [NS], we first explain affine differential geometry. Let $f : M^n \to \mathbf{R}^{n+1}$ be an immersion, and take a transversal vector field ξ on M, i.e.,

$$T_{f(x)}\mathbf{R}^{n+1} = f_* T_x M + \mathbf{R}\xi_x, \quad x \in M.$$

We denote by D_0 the standard affine connection on $\mathbf{R}^{n+1}$. Then we have

$$(D_0)_X f_* Y = f_*(D_X Y) + h(X, Y)\xi, \quad X, Y \in \Gamma(TM),$$

where D is a torsion free affine connection on M and h is a symmetric bilinear form on M, called the *affine second fundamental form.* We always assume that h is non-degenerate. An immersion $f : M \to \mathbf{R}^{n+1}$ is called *centro-affine* if the transversal vector field ξ is given by $\xi_x = f(x)$, $x \in M$.

Recently, Shima showed (cf. [Sh]) that

Theorem 3.1. *Let $M = G/K$ be a homogeneous space. Then M admits a G-invariant projectively flat affine connection D if and only if there exists an equivariant centro-affine immersion $f : M^n \to \mathbf{R}^{n+1}$.*

Here D is called to be *projectively flat* if in a neighborhood of each point of M, D is projectively equivalent to an affine connection whose curvature tensor vanishes. Then we classified all Riemannian symmetric spaces admitting invariant projectively flat affine connections (cf. [U4]):

Theorem 3.2. *Let $M = G/K$ be a Riemannian symmetric space. Then M admits an invariant projectively flat affine connection if and only if M is one of the following:*

(1) $S^n = SO(n+1)/SO(n)$, $n \geqq 2$,
(2) $H^n = SO_0(n,1)/SO(n)$, $n \geqq 2$,
(3) $SL(n,\mathbf{R})/SO(n)$, $n \geqq 3$,
(4) $SL(n,\mathbf{C})/SU(n)$, $n \geqq 2$,
(5) $SL(n,\mathbf{H})/Sp(n) = SU^*(2n)/Sp(n)$, $n \geqq 3$,
(6) E_6/F_4 *(noncompact type of EIV).*

We also obtained (cf. [U4])

Theorem 3.3. *Let G be a real simple Lie group. Then G admits a left invariant projectively flat affine connection if and only if the Lie algebra $\mathfrak{g}$ is one of the following:*

(a) $\mathfrak{o}(3)$,
(b) $\mathfrak{sl}(n+1,\mathbf{R})$, $n \geqq 1$,
(c) $\mathfrak{su}^*(2n)$, $n \geqq 2$,
(d) $\mathfrak{su}(r,s)$ $(r+s : \textit{even} \geqq 4)$; $\mathfrak{o}(3,4)$, $\mathfrak{o}(1,9)$, $\mathfrak{o}(5,5)$, $\mathfrak{o}(3,11)$, $\mathfrak{o}(7,7)$.

Remark 3.4. In the cases (a) $\sim$ (c), G admits a left invariant projectively flat affine connection. We do not know whether G admits the one for the case (d).

Let us recall for a pair (D,g) of a torsion free affine connection D and a Riemannian metric g to be a *Weyl structure* if $D_X g = \omega(X)g$ for all $X \in \Gamma(TM)$, for some 1-form ω on M. A Weyl structure (D,g) is called to be *Einstein-Weyl* if the symmetrization of Ricci tensor of D coincides with g up to a multiple by a C^∞ function on M. It is known that

Theorem 3.5. (cf. [PPS]) *Let M be a 4 dimensional closed manifold, and (D,g) be a Weyl structure with $Dg = \omega \otimes g$ for some 1-form ω on M. Then the following two conditions are equivalent:*

(1) *The connection D attains the minimum, $4\pi^2|p_1(TM)|$, of the functional $(D,g) \mapsto \frac{1}{2}\int_M \|R^D\|^2 v_g$ among the set of Weyl structures.*

(2) *(D,g) is Einstein-Weyl and $d\omega = 0$.*

§4. Yang-Mills theory.

4.1. Let us recall the framework of Yang-Mills theory which has been introduced by physicists. Let E be a vector bundle with an inner product h over a Riemannian manifold (M,g). Let C_E^0 be the set of all connections D of E satisfying the *metric condition*, that is,

$$Xh(s,t) = h(D_X s,t) + h(s,D_X t), \quad s,\ t \in \Gamma(E),\ X \in \Gamma(TM). \tag{4.1}$$

We consider the *Yang-Mills functional* $\mathcal{YM}$ on C_E^0, which is given as usually (cf. [BL]) by

$$\mathcal{YM}(D) = \frac{1}{2}\int_M \|R^D\|^2 v_g, \tag{4.2}$$

where R^D is the curvature of D. Then a connection $D \in \mathcal{C}_E^0$ is a Yang-Mills connection, if for all smooth deformation D_t of D in $\mathcal{C}_E^0$ with $D_0 = D$,

$$\frac{d}{dt}\bigg|_{t=0} \mathcal{YM}(D_t) = 0. \tag{4.3}$$

It is well known (cf. [BL]) that the left hand side of (4.3) is calculated as

$$\frac{d}{dt}\bigg|_{t=0} \mathcal{YM}(D_t) = \int_M < d^D\beta, R^D > v_g = \int_M < \beta, \delta^D R^D > v_g,$$

where $\beta = \frac{d}{dt}\big|_{t=0} D_t \in A^1(\mathrm{End}(E))$.

Therefore, D is a Yang-Mills connection if and only if

$$\delta^D R^D = 0. \tag{4.4}$$

Here, $A^p(\mathrm{End}(E))$ is the space of p forms on M with valued in the vector bundle $\mathrm{End}(E)$ of endomorphisms of E, d^D is the exterior differentiation which is given by

$$(d^D\psi)(X_1, \ldots, X_{p+1}) = \sum_{k=1}^{p+1}(-1)^k (D_{X_k}\psi)(X_1, \ldots, \hat{X_k}, \ldots, X_{p+1}), \tag{4.5}$$

and δ^D is the formal adjoint of d^D, i.e., for $\psi \in A^p(\mathrm{End}(E))$ and $\varphi \in A^{p+1}(\mathrm{End}(E))$,

$$(\delta^D\varphi, \psi) = (\varphi, d^D\psi).$$

It holds that

$$(\delta^D\varphi)(X_1, \ldots, X_p) = -\sum_{j=1}^{n}(D_{e_j}\varphi)(e_j, X_1, \ldots, X_p),$$

$$\delta^D\varphi = (-1)^{p+1} *^{-1} d^D * \varphi = -(-1)^{np} * d^D * \varphi.$$

In particular,

$$\delta^D R^D(X) = *^{-1} d^D * R^D(X) = -\sum_{j=1}^{n}(D_{e_j}R^D)(e_j, X), \quad X \in \Gamma(TM).$$

Notice here that these calculations are valid only for connections $D \in \mathcal{C}_E^0$.

The following due to Atiyah, Hitchin and Singer is well known:

Theorem 4.6. *Let (M, g) be a four dimensional closed Riemannian manifold, and ∇, the Levei-Civita connection on $E = TM$. Then the following three conditions are equivalent:*

(1) ∇ is a minimizer of the functional $\mathcal{YM}$, i.e., $\mathcal{YM}(\nabla) = 4\pi^2|p_1(TM)|$, where $p_1(TM)$ is the first Pontryagin number of the tangent bundle TM.

(2) The Riemannian metric g is Einstein.

(3) The Levi-Civita connection ∇ of g is (anti-)self-dual, i.e., $*R^\nabla = \pm R^\nabla$.

4.2. Comparing Theorems 3.5 and 4.6, the condition (3) in Theorem 4.6 is missing in Theorem 3.5. In order to full this lack and apply Yang-Mills theory to affine geometry, we have to relax the metric condition in the frame work of Yang-Mills theory. To overcome the above difficulty, we consider the conjugate connection.

Definition 4.7. Let F be a vector bundle over a Riemannian manifold (M,g) admitting the inner product h and D, a connection of F. The *conjugate connection* (or the dual connection) $\overline{D}$ for D is the unique connection satisfying the condition (cf. [A] or [DNV]):

$$Xh(s,t) = h(D_X s,t) + h(s,\overline{D}_X t),\ s,\ t \in \Gamma(F),\ X \in \Gamma(TM). \tag{4.8}$$

The connection D on F together with the Levi-Civita connection ∇ of g in $\bigwedge^p T^*M$ induces a tensor product connection in $\bigwedge T^pM \otimes F$ which we denote by D. Using this connection, we define the exterior differentiation $d^D : A^p(F) \to A^{p+1}(F)$ as usual on the space $A^p(F) = \Gamma(\bigwedge^p T^*M \otimes F)$ of differential p-forms on M with values in F by the same way as (4.5).

We define an inner product $<,>$ in $\bigwedge^p T_x^*M \otimes F_x$ by

$$<\psi,\varphi> = \sum_{i_1<\cdots<i_p} h(\psi(e_{i_1},\cdots,e_{i_p}),\varphi(e_{i_1},\cdots,e_{i_p})),$$

where $\{e_1,\ldots,e_n\}$ is an orthonormal basis of T_xM with respect to g_x. Integrating this pointwise inner product over M with respect to the volume element v_g of g gives a global inner product $(\,,\,)$ on $A^p(F)$. Then we can again define the operator δ^D; $A^{p+1}(F) \to A^p(F)$ to be the formal adjoint of the operator d^D. Then we have

Proposition 4.9. *For $\varphi \in A^{p+1}(F)$ and $X_i \in \Gamma(TM)$, $i = 1,\ldots,p$,*

$$(\delta^D\varphi)(X_1,\ldots,X_p) = -\sum_{j=1}^{n}(\overline{D}_{e_j}\varphi)(e_j,X_1,\ldots,X_p), \tag{4.10}$$

$$\delta^D\varphi = (-1)^{p+1} *^{-1} d^{\overline{D}} * \varphi = -(-1)^{np} * d^{\overline{D}} * \varphi, \tag{4.11}$$

where $\overline{D}$ is the conjugate connection of D and $*$; $A^q(F) \to A^{n-q}(F)$ is the star operator with respect to g.

Proof. Let $\{\theta^i\}_{i=1}^n$ be the dual basis to an orthonormal local frame field $\{e_i\}_{i=1}^n$ on M with respect to g. Then each $\xi \in A^p(F)$ can be written as $\xi = \sum_I \theta^I \otimes u_I$, where $\theta^I = \theta^{i_1} \wedge \cdots \wedge \theta^{i_p}$ with $u_I = u_{i_1\ldots i_p} \in \Gamma(F)$ and also $\eta \in A^{p+1}(F)$ can be written as $\eta = \sum_J \theta^J \otimes v_J$, where $\theta^J = \theta^{j_1} \wedge \cdots \wedge \theta^{j_{p+1}}$ with $v_J \in \Gamma(F)$. Let us define $<\xi \wedge *\eta> = \sum_{I,J} h(u_I,v_J)\theta^I \wedge *\theta^J \in A^{n-1}(M)$, where $h(u_I,v_J)$ is a function defined locally on M. Then we have, by the definition of $\overline{D}$,

$$d(h(u_I,v_J)) = h(Du_I,v_J) + h(u_I,\overline{D}v_J).$$

Therefore, we have

$$d<\xi \wedge *\eta> = <d^D\xi \wedge *\eta> + (-1)^p <\xi \wedge *(*^{-1}d^{\overline{D}}(*\eta))> .$$

Integrating this over M and δ^D being (4.11), we have

$$0 = \int_M d<\xi \wedge *\eta> v_g = (d^D\xi,\eta) - (\xi,\delta^D\eta).$$

Calculating (4.11), we have (4.10). □

Let $F = \mathrm{End}(E)$ be the endomorphism bundle of a given vector bundle E with the inner product h. The conncetion D of E induces a natural connection on $\mathrm{End}(E)$ by

$$(\nabla_X\varphi)(\sigma) = \nabla_X(\varphi(\sigma)) - \varphi(\nabla_X\sigma))$$

for $X \in \Gamma(TM)$, $\varphi \in \Gamma(\mathrm{End}(E))$ and $\sigma \in \Gamma(E)$. Furthermore, the inner product h on E can be extended to $\mathrm{End}(E)$ by $h(\psi,\varphi) = \sum_{i=1}^{r} h(\psi(\sigma_i),\varphi(\sigma_i))$, for two sections ψ and φ of $\mathrm{End}(E)$ and an orthonormal basis $\{\sigma_i\}_{i=1}^{r}$ of E_x with respect to h_x, $x \in M$, where r is rank of E.

We define the connections D and $\overline{D}$ for $\psi \in \Gamma(\mathrm{End}(E))$, by

$$(D_X\psi)(Y) = D_X(\psi(Y)) - \psi(D_XY), \quad (\overline{D}_X\psi)(Y) = \overline{D}_X(\psi(Y)) - \psi(\overline{D}_XY).$$

Then the connection $\overline{D}$ is conjugate to D, i.e.,

$$Xh(\psi,\varphi) = h(D_X\psi,\varphi) + h(\psi,\overline{D}_X\varphi), \quad \psi,\,\varphi \in \Gamma(\mathrm{End}(E)),\, X \in \Gamma(TM).$$

4.3 Now we define the *Yang-Mills functional* on the space $\mathcal{C}_E$ of all connections of E by

$$\mathcal{YM}(D) = \frac{1}{2}\int_M \|R^D\|^2 v_g, \tag{4.12}$$

where $\| \ \|$ is the pointwise norm induced from the above pointwise inner product $<,>$ of the bundle $\bigwedge^2 TM \otimes \mathrm{End}(E)$ over M, and $R^D \in A^2(\mathrm{End}(E))$ is the curvature tensor of D. For a fixed $D \in \mathcal{C}_E$ and a smooth one-parameter family of connections D^t, $-\epsilon < t < \epsilon$, such that $D^0 = D$, we write $D^t = D + A^t$, where $A^t \in A^2(\mathrm{End}(E))$ for $|t| < \epsilon$ and $A^0 = 0$. Then the curvature R^{D^t} is given by

$$R^{D^t}(X,Y) = R^D(X,Y) + d^DA^t(X,Y) + \frac{1}{2}[A^t \wedge A^t](X,Y), \tag{4.13}$$

where $[\psi\wedge\varphi](X,Y) := [\psi(X),\varphi(Y)] - [\psi(Y),\varphi(X)]$.

Theorem 4.14. *The first variation of the Yang-Mills functional is given by*

$$\left.\frac{d}{dt}\right|_{t=0} \mathcal{YM}(D^t) = \int_M < d^D\beta, R^D > v_g = \int_M < \beta, \delta^D R^D > v_g,$$

where $\beta = \frac{d}{dt}\big|_{t=0} D^t = \frac{d}{dt}\big|_{t=0} A^t \in A^1(\mathrm{End}(E))$.

Consequently, D is a Yang-Mills connection if and only if

$$\delta^D R^D = 0. \tag{4.15}$$

In the case of a non compact or semi-Riemannian manifold (M,g), we take any relatively compact open domain U in M, and consider the functional

$$\mathcal{YM}_U(D) = \frac{1}{2}\int_U \|R^D\|^2 v_g.$$

For a fixed $D \in \mathcal{C}_E$ and smooth one-parameter family of connections D^t, $-\epsilon < t < \epsilon$, such that $D^0 = D$, and $D^t = D + A^t$, where $A^t \in A^1(\mathrm{End}(E))$ have all their support in U for $|t| < \epsilon$ and $A^0 = 0$,

$$\left.\frac{d}{dt}\right|_{t=0} \mathcal{YM}_U(D^t) = \int_U < d^D\beta, R^D > v_g = \int_M < \beta, \delta^D R^D > v_g,$$

where $\beta = \frac{d}{dt}\big|_{t=0} A^t \in A^1(\mathrm{End}(E))$ with support in U. Therefore, D is a Yang-Mills connection if and only if $\delta^D R^D = 0$ everywhere on M.

Since the second Bianchi identity for D, $d^D R^D = 0$, (4.15) is equivalent to

(4.16) $$\Delta^D R^D = 0,$$

where the Laplacian Δ^D on $A^2(\mathrm{End}(E))$ is given by $\Delta^D = d^D\delta^D + \delta^D d^D$.

§5. Four dimensional manifolds.

For four dimensional closed Riemannian manifold (M,g), we define for a connection D of E to be *(anti-)self-dual* if $*R^D = \pm R^{\overline{D}}$, where $*$ is the Hodge star operator and $R^{\overline{D}}$ is the curvature of the conjugate connection $\overline{D}$.

Note that the (anti-)self-dual connection D is a Yang-Mills connection, because

$$\delta^D R^D = *^{-1} d^{\overline{D}} * D = \pm *^{-1} d^{\overline{D}} R^{\overline{D}} = 0,$$

since the second Bianchi identity for $\overline{D}$, $d^{\overline{D}} R^{\overline{D}} = 0$.

For a torsion free affine connection D, we consider the affine connection $\hat{D}$ defined by $\hat{D} = \frac{1}{2}(D + \overline{D})$. Then we have (cf. [DIU])

Proposition 5.1. *Assume that* $\dim M = 4$. *Let* (D,g) *be a Weyl structure with* $Dg = \omega \otimes g$ *for some 1-form* ω. *Then the following are equivalent:*

(1) $$*R^D = \pm R^{\overline{D}},$$

(2) $$*R^{\hat{D}} = \pm R^{\hat{D}} \quad \textit{and} \quad * d\omega \pm d\omega = 0,$$

(3) $$*R^{\hat{D}} = \pm R^{\hat{D}} \quad \textit{and} \quad d\omega = 0,$$

where the sign $\pm$ *corresponds to each other, respectively.*

We obtain (cf. [DIU])

Theorem 5.2. *Let* M *be a 4 dimensional closed manifold, and* (D,g) *be a Weyl structure with* $Dg = \omega \otimes g$ *for some 1-form* ω *on* M. *Then the following four conditions are equivalent:*

(1) $\mathcal{YM}(D) = 4\pi^2|p_1(TM)|$,
(2) $\mathcal{YM}(\hat{D}) = 4\pi^2|p_1(TM)|$ *and* $d\omega = 0$,
(3) $*R^D = \pm R^{\overline{D}}$,
(4) $\hat{D}$ *is (anti-)self-dual and* $d\omega = 0$.

§6. Yang-Mills theory in affine differential geometry.

Furthermore, our Yang-Mills theory can be applied to affine differential geometry. It is known (cf. [NS]) that

Theorem 6.1. *Let $f : M \to \mathbf{R}^{n+1}$ be a nondegenerate affine hypersurface ($n \geqq 2$). Then we can choose a transversal vector field ξ on M for D, h and S satisfying the following seven conditions:*

(1) (Gauss) $R^D(X,Y)Z = h(Y,Z)SX - h(X,Z)SY$,
(2) (Codazzi for h) $(D_X h)(Y,Z) = (D_Y h)(X,Z)$,
(3) (Codazzi for S) $(D_X S)(Y) = (D_Y S)(X)$,
(4) (Ricci) $h(SX,Y) = h(X,SY)$,
(5) (equiaffine condition) $D\theta = 0$, *i.e.*, $\tau = 0$,
(6) (volume condiiton) $\theta = \omega_h$, *and*
(7) (apolarity condition) $D\omega_h = 0$.

Here θ is the induced volume form on M by the immersion f, ξ and the standard volume form on $\mathbf{R}^{n+1}$, and ω_h is the volume form on M corresponding to h.

Proposition 6.2. ([NS]) *The apolarity condition* (7) *in Theorem 6.1 is equivalent to the following condition* (7'):

(7') *For all $X \in T_xM$, the trace $\mathrm{Tr}_h(D_X h)$ vanishes.*

Here $\mathrm{Tr}_h(D_X h)$ is defined by

$$\mathrm{Tr}_h(D_X h) := \mathrm{Trace}\{(Y,Z) \mapsto (D_X h)(Y,Z)\} = \sum_{j=1}^{n} \epsilon_j (D_X h)(e_j, e_j),$$

where $g(e_i, e_j) = \epsilon_i \delta_{ij}$, and $\epsilon_{ij} = \pm 1$.

Now our theorem is (cf. [DIU])

Theorem 6.3. *Let $f : M \to \mathbf{R}^{n+1}$ be a non-degenerate affine immersion, D, the induced connection on M from the standard connection of $\mathbf{R}^{n+1}$ via f, and S, the affine shape operator. Assume that a transversal vector field ξ is chosen such as for D, h and S satisfy all the conditions in Theorem 6.1. Then D is a Yang-Mills connection with respect to h if and only if $\overline{D}_X(SY) = S(D_X Y)$ for X, $Y \in \Gamma(TM)$. Here $\overline{D}$ is the conjugate connection of D with respcet to h.*

As an application, we have (cf. [DIU])

Corollary 6.4. *Let $f : M \to \mathbf{R}^{n+1}$ be a non-degenerate affine immersion and ξ satisfy as in Theorem 6.1. Furthermore, assume that $S = \lambda\,\mathrm{Id}$ for some non-zero constant λ. Then D is a Yang-Mills connection with respect to h if and only if $f(M)$ is a quadratic hypersurface of $\mathbf{R}^{n+1}$.*

References

[A] T. Amari, *Differential Geometrical Methods in Statistics, Lecture Notes in Statistics*, vol. 26, Springer, Berlin, Heidelberg, NewYork, 1985.

[AHS] M. F. Atiyah, N. J. Hitchin and I. M. Singer, *Self-duality in four-dimensional Riemannian geometry*, Proc. R. Soc. London **A. 362** (1978), 425–461.

[Be] A. L. Besse, *Einstein Manifolds*, Springer-Verlag, 1987.

[Bo] J. P. Bourguignon, *Metrics with harmonic curvature*, Global Riemannian Geometry (T. J. Willmore and N. J. Hitchin, eds.), 1984, pp. 18–26.

[BL] J. P. Bourguignon and H. B. Lawson, *Stability and isolation phenomena for Yang-Mills fields*, Commun. Math. Phys. **79** (1981), 189–230.

[De1] A. Derdziński, *Classification of certain compact Riemannian manifolds with harmonic curvature and non-parallel Ricci tensor*, Math. Z. **172** (1980), 273–280.

[De2] ———, *On compact Riemannian manifolds with harmonic curvature*, Math. Ann. **259** (1982), 245–152.

[DS] ——— and C-L. Shen, *Codazzi tensor fields, curvature and Pontryagin forms*, Proc. London Math. Soc. **47** (1983), 15–26.

[DNV] F. Dillen, K. Nomizu and L. Vranken, *Conjugate connections and Radon's theorem in affine differential geometry*, Monatsh. Math. **109** (1990), 221–235.

[Do] S. Donaldson, *Anti self-dual Yang-Mills connections over complex algebraic surfaces and stable vector bundles*, Proc. London Math. Soc. **50** (1985), 1–26.

[DU] S. Dragomir and H. Urakawa, *On the inhomogeneous Yang-Mills equation* $d_D^* R^D = f$ (to appear).

[DIU] S. Dragomir, T. Ichiyama and H. Urakawa, *Yang-Mills theory and the conjugate connections* (to appear).

[IFU] T. Ichiyama, H. Furuhata and H. Urakawa, *A conformal gauge invariant functional for Weyl structures and the first variation formula*, Tsukuba J. Math. **23** (1999), 551–564.

[KN] U-H. Ki and H. Nakagawa, *Submanifolds with harmonic curvature*, Tsukuba J. Math. **10** (1986), 285–292.

[KNU] ——— and M. Umehara, *On complete hypersurfaces with harmonic curvature in a Riemannian manifold of constant curvature*, Tsukuba J. Math. **11** (1987), 61–76.

[Ko] S. Kobayashi, *Differetial Geometry of Complex Vector Bundles*, Iwanami Shoten Publ. and Princeton University Press, Tokyo and New Jersey, 1987.

[NS] K. Nomizu and T. Sasaki, *Affine Differential Geometry – Geometry of Affine Immersion*, Cambridge University Press, 1994.

[O] E. Omachi, *Hypersurfaces with harmonic curvature in a space of constant curvature*, Kodai Math. J. **9** (1986), 170–174.

[PPS] H. Pedersen, Y. S. Poon and A. Swan, *Einstein-Weyl deformations and submanifolds*, Intern. J. Math. **7** (1996), 705–719.

[R] C. R. Rao, *Information and accuracy attainnable in the estimation of statistical parameters*, Bull. Calcutta Math. Soc. **37** (1945), 81–91.

[Sh] H. Shima, *Homogeneous spaces with invariant projectively flat affine connections* (to appear).

[Sm] U. Simon, *Zur Relativgeometrie: symmetrische Zusammenhänge auf Hyperflächen*, Math. Z. **106** (1968), 35–46.

[Su] Y. T. Siu, *Lectures on Hermitian-Einstein Metrics for Stable Bundles and Kähler-Einstein Metrics*, Birkhäuser, Basel, Boston, 1987.

[T] N. Tanaka, *A differential geometric study on strongly pseudoconvex manifolds*, Lect. Notes Math., Kyoto Univ. **9** (1975).

[Um] M. Umehara, *Hypersurfaces with harmonic curvature*, Tsukuba J. Math. **10** (1986), 70–88.

[U1] H. Urakawa, *Yang-Mills connections over compact strongly pseudoconvex CR manifolds*, Math. Z. **216** (1994), 541–573.

[U2] ———, *Yang-Mills connections and deformation theory over compact strongly pseudoconvex CR manifolds*, Geometric Complex Analysis (J. Noguchi, ed.), World Scientific Publ., Singapore, 1996, pp. 635–652.

[U3] ———, *Variational problems over strongly pseudoconvex CR manifolds*, Differential Goemetry, Proceedings on the Symposium in Honour of Professor Su Buchin on His 90th Birthday, World Scientific Publ., Singapore, 1993, pp. 233–242.

[U4] ______, *On invariant projectively flat affine connections*, Hokkaido Math. J. **28** (1999), 333–356.

[UY] K. Uhlenbeck and S. T. Yau, *On the existence of hermitian Yang-Mills connections in stable vecotr bundles*, Commun. Pure Appl. Math. **39** (1986), S257–S293.

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