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# CONTACT INVARIANTS OF ORDINARY DIFFERENTIAL EQUATIONS

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## 1. Characteristic Cartan connection for systems of ODE's

The geometric approach to the study of differential equations goes back to Sophus Lie and Elie Cartan. According to the modern interpretation of this approach, based on the notion of jet space, we consider a differential equation as a submanifold in the jet space with induced geometric structure.

Using the methods of filtered manifolds developed in works of Tanaka [3, 4] and Morimoto [2], we construct a *characteristic* Cartan connection, naturally associated with any system of $m$ equations of the $(n+1)$-th order whenever $m \geq 2, n \geq 1$ or $m = 1, n \geq 2$. Then we compute the compete set of fundamental invariants which appear as coefficients of the curvature tensor. Here by fundamental invariants of ordinary differential equations we understand relative invariants with respect to the contact transformations which generate the set of all invariants of a given ODE.

Note that in the case of single second order ODE there is a classical result of Sophus Lie showing that all ODE's of the second order are contact equivalent and have an infinite dimensional symmetry algebra, which makes it impossible to construct a characteristic connection in this particular case.

**Theorem 1** ([1]). *With any system of $m$ ordinary differential equations of the $(n+1)$-th order, where $m \geq 2$, $n \geq 1$ or $m = 1$, $n \geq 2$, there is naturally associated a Cartan connection with model $G/H$, where*

*for $m = 1$, $n = 2$:*

$$G = SP(4, \mathbb{R}), \ H \text{ is the Borel subgroup of } G;$$

*for* $m \geq 2$, $n = 1$*:*

$$G = SL(m+2,\mathbb{R}),\ H = \left\{ \left( \begin{smallmatrix} x & * & * \\ 0 & y & * \\ 0 & 0 & Z \end{smallmatrix} \right) \,\middle|\, x, y \in \mathbb{R}^*,\ Z \in GL(m,\mathbb{R}) \right\};$$

*for* $m = 1$, $n \geq 2$ *or* $m \geq 2$, $n \geq 3$*:*

$$G = (SL(2,\mathbb{R}) \times GL(m,\mathbb{R})) \ltimes (E_n \otimes \mathbb{R}^m),\ H = ST(2,\mathbb{R}) \times GL(m,\mathbb{R}),$$

*where* $ST(2,\mathbb{R})$ *is the Borel subgroup of* $SL(2,\mathbb{R})$, $E_n$ *is a* $(n+1)$*-dimensional irreducible* $SL(2,\mathbb{R})$*-module, and* $\mathbb{R}^m$ *is the natural* $GL(m,\mathbb{R})$*-module.*

In all the cases above the Lie algebra $\mathfrak{g}$ of the Lie group $G$ is naturally supplied with the gradation $\mathfrak{g} = \sum_i \mathfrak{g}_i$ such that the subalgebra $\mathfrak{h}$ is equal to $\sum_{i\geq 0} \mathfrak{g}_i$. Let $\mathfrak{g}_- = \sum_{i<0} \mathfrak{g}_i$ be the negative part of $\mathfrak{g}$ and let $H^p(\mathfrak{g}_-, \mathfrak{g}) = \sum_q H^p_q$ be the $p$-th generalized Spencer cohomology space, which naturally inherits the gradation from $\mathfrak{g}$.

The complete system of invariants of the characteristic Cartan connection can be derived from the finite set of fundamental invariants by means of the covariant derivatives. The fundamental invariants are described by the positive part $\sum_{q>0} H^2_q$ of the second cohomology space [4]. In cases, when $\mathfrak{g}$ is semisimple, i.e., for one ODE of third order or for system of second order ODE's, these cohomology spaces where computed by Yamaguchi [5]. In the next section we compute the cohomology space $H^2(\mathfrak{g}_-, \mathfrak{g})$ in the non-semisimple case.

## 2. Computation of cohomology spaces

The symbol algebra $\mathfrak{g}$ of a system of $m$ ODE's of $(n+1)$-th order is isomorphic to the semidirect product of $\mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{gl}(m,\mathbb{R})$ and an abelian ideal $V = E_n \otimes \mathbb{R}^m$, where $E_n$ is the irreducible $\mathfrak{sl}(2,\mathbb{R})$-module isomorphic to $S^n(\mathbb{R}^2)$ (here $\mathbb{R}^2$ is considered as the canonical $\mathfrak{gl}(2,\mathbb{R})$-module) and $\mathbb{R}^m$ is the natural $\mathfrak{gl}(m,\mathbb{R})$-module.

In the sequel we assume that $m = 1, n \geq 3$ or $m \geq 2, n \geq 2$, so that we consider only single ODE's of order $\geq 4$ or systems of ordinary differential equations on order $\geq 3$.

Let us fix the standard basis $x, y, h$ of $\mathfrak{sl}(2,\mathbb{R})$:

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Put $e_i = f_1^{n-i} f_2^i / i!$, where $f_1, f_2$ is the standard basis in $\mathbb{R}^2$. We denote also by $\{E_1, \ldots, E_m\}$ and $\{E_j^i\}$ the natural bases of $\mathbb{R}^m$ and $\mathfrak{gl}(m, \mathbb{R})$ respectively.

The gradation of $\mathfrak{g}$ is defined as follows:

$$\begin{aligned}
\mathfrak{g}_1 &= \mathbb{R}y, \\
\mathfrak{g}_0 &= \mathbb{R}h \oplus \mathfrak{gl}(m, \mathbb{R}), \\
\mathfrak{g}_{-1} &= \mathbb{R}x \oplus \mathbb{R}e_n \otimes \mathbb{R}^m, \\
\mathfrak{g}_{-i} &= \mathbb{R}e_{n+1-i} \otimes \mathbb{R}^m \quad \text{for all } i = 2, \ldots, n+1,
\end{aligned}$$

and $\mathfrak{g}_n = \{0\}$ for all other $n \in \mathbb{Z}$.

We compute the cohomology space $H^2(\mathfrak{g}_-, \mathfrak{g})$ by means of the Serre–Hochschild spectral sequence, determined by the subalgebra $V$ of $\mathfrak{g}_-$. Namely, since $V$ is an ideal, the second term $E_2$ of this spectral sequence has the form: $E_2 = \bigoplus_{p,q} E_2^{p,q}$, where

$$E_2^{p,q} = H^p(\mathbb{R}x, H^q(V, \mathfrak{g})) \quad \text{for all } p, q \geq 0.$$

Since the algebra $\mathbb{R}x$ is one-dimensional, we see that $E_2^{p,q} = \{0\}$ for all $p > 1$. Therefore, the differential

$$d_2^{p,q} \colon E_2^{p,q} \to E_2^{p+2,q-1}$$

is trivial and the spectral sequence is stabilized in the second term.

Therefore, we get the following intermediate result

**Lemma 1**. *The second cohomology space $H^2(\mathfrak{g}_-, \mathfrak{g})$ is naturally isomorphic with the subspace $E_2^{1,1} \oplus E_2^{0,2}$ of the Serre-Hochschild spectral sequence determined by the ideal $V \subset \mathfrak{g}_-$.*

*Moreover, we have*

$$\begin{aligned}
E_2^{1,1} &= H^1(\mathbb{R}x, H^1(V, \mathfrak{g})), \\
E_2^{0,2} &= H^0(\mathbb{R}x, H^2(V, \mathfrak{g})) = \operatorname{Inv}_x H^2(V, \mathfrak{g}).
\end{aligned}$$

Let $\mathfrak{a}$ be the subalgebra of $\mathfrak{gl}(V)$ corresponding to the action of $\mathfrak{sl}(2, \mathbb{R}) \times \mathfrak{gl}(m, \mathbb{R})$ on $V$. Then the cohomology spaces $H^k(V, \mathfrak{g})$ are precisely the classical Spencer cohomology spaces determined by the subalgebra $\mathfrak{a} \subset \mathfrak{gl}(V)$. The first and second cohomology spaces are be easily computed in terms of the Spencer operator

$$S^k\colon \operatorname{Hom}(\wedge^k V, \mathfrak{a}) \to \operatorname{Hom}(\wedge^{k+1} V, V),$$

$$S^k(\phi)(v_1 \wedge v_2 \wedge \cdots \wedge v_{k+1}) = \sum_{i=1}^{k+1} (-1)^i \phi(v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge v_{k+1}) v_i.$$

**Lemma 2.** *We have $H^0(V, \mathfrak{g}) = V$ and*

$$H^k(V, \mathfrak{g}) = \ker S^k \oplus \operatorname{Hom}(\wedge^k V, V) / \operatorname{im} S^{k-1}$$

*for all $k \geq 1$.*

*Proof.* Indeed, let us represent an arbitrary cocycle $c \in C^k(V, \mathfrak{g})$ as $c = c_{\mathfrak{a}} + c_V$, where $c_{\mathfrak{a}} \in \operatorname{Hom}(\wedge^k V, \mathfrak{a})$ and $c_V \in \operatorname{Hom}(\wedge^k V, V)$. Since $V$ is commutative Lie algebra, we have

$$(\partial c) = S^k(c_{\mathfrak{a}}) \in \operatorname{Hom}(\wedge^{k+1}, V, V).$$

This immediately implies the statement of the lemma. □

For $k = 1, 2$ the mappings $S^k$ are easily described explicitly.

**Lemma 3.**

*1. The mapping $S^1$ is injective for $m = 1, n \geq 3$ and $m = 2, n \geq 2$.*

*2. The mapping $S^2$ is injective for $m = 1, n \geq 5$, or $m = 2, n \geq 4$, or $m \geq 3, n \geq 3$.*

*3. In all other cases the structure of the $\mathfrak{sl}(2, \mathbb{R})$-module $\ker S^2$ is given in the following table:*

| $n \setminus m$ | 1 | 2 | $\geq 3$ |
|---|---|---|---|
| 2 | – | $E_2 + E_0 \otimes S^2(\mathbb{R}^2)^*$ | $E_0 \otimes S^2(\mathbb{R}^m)^*$ |
| 3 | $E_2 + E_4$ | $E_0$ | 0 |
| 4 | $E_0$ | 0 | 0 |
| $\geq 5$ | 0 | 0 | 0 |

,

*where $E_l$ is an $(l+1)$-dimensional irreducible $\mathfrak{sl}(2, \mathbb{R})$-module.*

*Proof.*

1. Let us note that $\ker S^1$ is precisely the first prolongation $\mathfrak{a}^{(1)}$ of the subalgebra $\mathfrak{a} \subset \mathfrak{gl}(V)$. Suppose that $\mathfrak{a}^{(1)} \neq \{0\}$. Then the algebra $V + \mathfrak{a} + \sum_{i=1}^{\infty} \mathfrak{a}^{(i)}$ is an irreducible graded Lie algebra of order $\geq 2$. All these algebras are described in [6]. In particular, Lemma 7.3 of [6] formulates necessary conditions on the highest root of $\mathfrak{a}$ and the highest

weight the $\mathfrak{a}$-module $V$. It is easy to check that in the cases $m = 1, n \geq 3$ and $m \geq 2, n \geq 2$ the subalgebra $\mathfrak{a}$ does not satisfy these conditions. This proves that $\ker S^1 = \mathfrak{a}^{(1)} = \{0\}$.

2. We consider only the case $m = 1$. All other cases can be dealt in the same manner. Let us denote for simplicity the elements $e_i \otimes E_1$ of $V$ also by $e_i$ and the element $E_1^1$ of $\mathfrak{gl}(1, \mathbb{R})$ by $z$.

Let $\alpha$ be an arbitrary element of $\ker S^2$. Put $\alpha_{ij} = \alpha(e_i, e_j)$ for all $0 \leq i < j \leq n$. Let us show that $\alpha_{ij} = 0$ for all $i, j \geq 3$ and $i, j \leq n-3$. Indeed, for $i, j \geq 3$ we have

$$\alpha_{ij}e_0 - \alpha_{0j}e_i + \alpha_{0i}e_j = 0.$$

But for any element $X \in \mathfrak{a}$ and any $i = 0, \ldots, n$ we have $Xe_i \subset \langle e_{i-1}, e_i, e_{i+1} \rangle$. Hence, $\alpha_{ij}e_0 = 0$, that is

$$\alpha_{ij} \subset \langle x, h - z \rangle \tag{1}$$

Similarly,

$$\alpha_{ij}e_1 - \alpha_{1j}e_i + \alpha_{1i}e_j = 0.$$

From (1) we see that $\alpha_{ij}e_1 \subset \langle e_0, e_1 \rangle$. Therefore, $\alpha_{ij}e_1 = 0$, which is only possible if $\alpha_{ij} = 0$. In the same way we can prove that $\alpha_{ij} = 0$ for all $i, j \leq n - 3$.

Consider now the following subspace $W \subset \wedge^2 V$:

$$W = \{w \in \wedge^2 V \mid \alpha(w) = 0 \text{ for all } \alpha \in \ker S^2\}.$$

It is clear that $W$ is a submodule of the $\mathfrak{sl}(2, \mathbb{R})$-module $\wedge^2 V$. As we have just proved, $e_i \wedge e_j \subset W$ for all $i, j \geq 3$ and $i, j \leq n - 3$. Hence, $W$ contains also the submodule generated by these elements. But it is easy to check that these elements generate whole $V$ for $n \geq 6$. In the remaining case $n = 5$ they generate the submodule of codimension 1, complimentary to the submodule $\mathbb{R}(e_0 \wedge e_5 - e_1 \wedge e_4 + e_2 \wedge e_3)$. Therefore, any non-trivial element $\alpha$ of $\ker S^2$ must be of the form:

$$\alpha\colon e_0 \wedge e_5 \mapsto X,\ e_1 \wedge e_4 \mapsto -X,\ e_2 \wedge e_3 \mapsto X, \quad X \in \mathfrak{a},$$

and $\alpha(e_i \wedge e_j) = 0$ in all other cases. Then we have $\alpha(e_0 \wedge e_5)e_i = Xe_i = 0$ for $i = 1, \ldots, 4$, which is possible only if $X = 0$.

3. This table is obtained by direct computation. □

Let $E_m$ be an arbitrary irreducible $\mathfrak{sl}(2,\mathbb{R})$-module of dimension $m+1$. Then the cohomology spaces $H^k(\mathbb{R}x, E_m)$ have the form:

**Lemma 4.** *The space $H^k(\mathbb{R}x, E_m)$ is trivial for $k \geq 2$ and is one-dimensional for $k = 0, 1$.*

*Let $v_0$ and $v_m$ be highest and lowest vectors of $E_m$ (that is $h.v_0 = mv_0$ and $h.v_m = -mv_m$). Then $H^0(\mathbb{R}x, E_m)$ is generated by $v_0$, and $H^1(\mathbb{R}x, E_m)$ is generated by $[\alpha\colon x \to v_m]$.*

*Proof.* Immediately follows from the explicit description of irreducible $\mathfrak{sl}(2,\mathbb{R})$-modules. □

Thus, description of $H^2(\mathfrak{g}_-, \mathfrak{g})$ reduces essentially to the decomposition of $\mathfrak{sl}(2,\mathbb{R})$-modules $\mathrm{Hom}(V,V)/\mathfrak{a}$ and $\mathrm{Hom}(\wedge^2 V, V)/S(\mathrm{Hom}(V,\mathfrak{a}))$ into sums of irreducible submodules.

The gradation of $H^2(\mathfrak{g}_-, \mathfrak{g})$ can be derived by means of the following result.

**Lemma 5.** *Let $[c] \in H^k(\mathfrak{g}_-, g)$ and $h.c = \alpha c$, $z.c = \beta c$. Then $[c] \subset H^k_p(\mathfrak{g}_-, g)$, where*
$$p = -\frac{\alpha n + \beta(n+2)}{2n}.$$

In particular, let $E_l$ be an irreducible submodule of $\mathrm{Hom}(V,V)/\mathfrak{a}$. Then the subspace $H^1(\mathbb{R}x, E_l) \subset H^2(\mathfrak{g}_-, \mathfrak{g})$ has degree $(l+2)/2$. Similarly, let $E_l$ be an irreducible submodule of $\mathrm{Hom}(\wedge^2 V, V)/S(\mathrm{Hom}(V,\mathfrak{a}))$. Then the subspace $H^0(\mathbb{R}x, E_l) \subset H^2(\mathfrak{g}_-, \mathfrak{g})$ has degree $(n+2-l)/2$.

**Example 1.** Let us compute $H^2(\mathfrak{g}_-, \mathfrak{g})$ for $n = 3$. From Lemma 3 we have:

$$H^1(V, \mathfrak{g}) = \mathrm{Hom}(V,V)/\mathfrak{a} \cong (E_3 \otimes E_3)/(E_0 \oplus E_2) \cong E_6 \oplus E_4;$$
$$H^2(V, \mathfrak{g}) = \ker S^2 \oplus \mathrm{Hom}(\wedge^2 V, V)/\operatorname{im} S^1,$$

where $\ker S^2 = E_2 \oplus E_4$, and

$$\begin{aligned}\mathrm{Hom}(\wedge^2 V, V)/\operatorname{im} S^1 &= (\wedge^2 E_3 \otimes E_3)/(E_3 \otimes (E_2 \oplus E_0)) \cong \\ &((E_4 \oplus E_0) \otimes E_3)/(E_3 \otimes (E_2 \oplus E_0)) \cong \\ &(E_7 \oplus E5 \oplus 2E_3 \oplus E_1)/(E_5 \oplus 2E_3 \oplus E_1) \cong E_7.\end{aligned}$$

Hence, we see that the space $E_2^{1,1}$ is two-dimensional, and by Lemma 5 the corresponding two elements of $H^2(\mathfrak{g}_-, \mathfrak{g})$ have degrees 3 and 4. Similarly, the space $E_2^{0,2}$ is three-dimensional, and the only element, corresponding to $\operatorname{Hom}(V,V)/\operatorname{im} S^1$ has degree $-1$. Let us find degrees of two elements corresponding to $\ker S^2$. Let $v_0$ be the highest vector of the submodule $E_2 \subset \ker S^2$. Then we have $h.v_0 = 2v_0$ and $z.v_0 = -6v_0$. Hence, by Lemma 5 the corresponding element of $H^2(\mathfrak{g}_-, \mathfrak{g})$ is of degree 4. In the same way we compute that the element corresponding to $E_4 \subset \ker S^2$ is of degree 3.

Hence, we see that the space $H^2(\mathfrak{g}_-, \mathfrak{g})$ is 5-dimensional and is spanned by one element of degree $-1$, two elements of degree 3 and two elements of degree 4.

**Example 2.** Let us compute dimension and degree of $H^2(\mathfrak{g}_-, \mathfrak{g})$ for $n = 4$. From Lemma 3 we have the following decompositions:

$$\operatorname{Hom}(V,V)/\mathfrak{a} \cong E_4 \otimes E_4/(E_2 + E_0) \cong E_8 \oplus E_6 \oplus E_4,$$

and

$$\begin{aligned}\operatorname{Hom}(\Lambda^2 V, V)/S(\operatorname{Hom}(V,\mathfrak{a})) &\cong \Lambda^2 E_4 \otimes E_4 / \big(E_4 \otimes (E_2 \oplus E_0)\big) = \\ &(E_6 \oplus E_2) \otimes E_4/(E_6 \oplus 2E_4 \oplus E_2) = \\ &(E_{10} \oplus E_8 \oplus E_6 \oplus E_4 \oplus E_2 \oplus E_6 \oplus E_4 \oplus E_2)/(E_6 \oplus 2E_4 \oplus E_2) = \\ &E_{10} \oplus E_8 \oplus E_6 \oplus E_2.\end{aligned}$$

Let us also find the degree of $H^0(\mathbb{R}x, \mathbb{R}[\alpha]) = \mathbb{R}[\alpha]$, where $\alpha$ is the $\mathfrak{sl}(2,\mathbb{R})$ invariant mapping $\Lambda^2 V \to \mathfrak{a}$. We have $h.\alpha = 0$, $z.\alpha = -8\alpha$. Hence, by Lemma 5 the element $[\alpha]$ has degree 6.

Summarizing all these computations we see that $H^2(\mathfrak{g}_-, \mathfrak{g}) = E_2^{1,1} \oplus E_2^{0,2}$, where $E_2^{1,1}$ has dimension 3 and is generated by elements of degree 5, 4, and 3. The space $E_2^{0,2}$ has dimension 5 and is generated by elements of degree $-2$, $-1$, 0, 2 and 6. Thus, the positive part of $H^2(\mathfrak{g}_-, \mathfrak{g})$ is 5-dimensional and is generated by elements of degree 2, 3, 4, 5 and 6.

## 3. Explicit formulas for fundamental invariants

In the table below we give the form of fundamental invariants in case of one ordinary differential equation of order $\geq 4$. This result was obtained by explicit computation using Maple V. The fundamental

| Degree | Invariant |
| --- | --- |
| 4th order equation | |
| 3 | $L_3$ |
| 3 | $I_1 = f_{333}$ |
| 4 | $L_4$ |
| 4 | $I_2 = 6f_{233} + f_{33}^2 \mod \langle I_1 \rangle$ |
| 5th order equation | |
| 2 | $I_1 = f_{44}$ |
| 3 | $L_3$ |
| 4 | $L_4$ |
| 5 | $L_5$ |
| 6 | $I_2 = 6f_{234} - 4f_{333} - 3f_{34}^2 \mod \langle I_1, L_3 \rangle$ |
| 6th order equation | |
| 2 | $I_1 = f_{55}$ |
| 3 | $L_3$ |
| 3 | $I_2 = f_{45} \mod \langle I_1 \rangle$ |
| 4 | $L_4$ |
| 5 | $L_5$ |
| 6 | $L_6$ |
| equation of order $n+1 \geq 7$ | |
| 2 | $I_1 = f_{n,n}$ |
| 3 | $L_3$ |
| 3 | $I_2 = f_{n,n-1} \mod \langle I_1 \rangle$ |
| 4 | $L_4$ |
| 4 | $I_3 = f_{n,n-2} \mod \langle I_1, I_2, L_3 \rangle$ |
| $5 \leq i \leq n+1$ | $L_i$ |

invariants of third order ODE's were obtained by S.-S. Chen [7], and for systems of second order ODE's by M. Fels [8].

We use the following notation:

**equation:** $y^{(n+1)} = f(x, y, y', \dots, y^{(n)})$;

**partial derivatives:** $F_i = \frac{\partial F}{\partial y_i}$ for $i = 0, \dots, n$, where $y_0 = y$;

**total derivative:** $F_x = \frac{\partial F}{\partial x} + \sum_{i=0}^{n-2} y_{i+1} F_i + f(x, y, y_1, \dots, y_{(n)}) F_n$;

**linear invariants:** by $L_i$, $i = 3, \dots, n+1$ we denote $n-1$ invariants, corresponding to the term $E^{1,1}$ in the decomposition of $H^2(\mathfrak{g}_-, \mathfrak{g})$ given in Lemma 1. It appears that they are expressed in terms only of $f_0, \dots, f_n$ and their total derivatives and can be obtained from corresponding linear invariants of an $n$-th order linear ODE as described in the classical work of Wilczynski [9] (see also the work of Se-ashi [10]) by substituting the usual derivative by total derivative.

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