

GEOMETRY OF HIGHER ORDER DIFFERENTIAL EQUATIONS OF
 FINITE TYPE ASSOCIATED WITH SYMMETRIC SPACES

KEIZO YAMAGUCHI AND TOMOAKI YATSUI

§1. Differential Equations of Finite Type

We will consider Higher order ODE

$$y^{(k)} = F(x, y, y', \dots, y^{(k-1)}).$$

Or more generally

$$\frac{\partial^k y}{\partial x_{i_1} \dots \partial x_{i_k}} = F_{i_1 \dots i_k}(x_1, \dots, x_n, y, p_1, \dots, p_n, \dots, p_{i_1 \dots i_{k-1}}) \quad (1 \leq i_1 \leq \dots \leq i_k \leq n),$$

where $p_{i_1 \dots i_\ell} = \frac{\partial^\ell y}{\partial x_{i_1} \dots \partial x_{i_\ell}}$. These equations define a submanifold R in k -jets space J^k .

$$J^k \supset R \rightarrow J^{k-1}; \text{Diffeomorphism} \quad (*)$$

We have the Contact system C^k on J^k

$$\begin{cases} \varpi = dy - \sum p_i dx_i, \\ \varpi_i = dp_i - \sum p_{ij} dx_j, \\ \dots\dots\dots, \\ \varpi_{i_1 \dots i_{k-1}} = dp_{i_1 \dots i_{k-1}} - \sum p_{i_1 \dots i_{k-1} j} dx_j. \end{cases}$$

C^k gives a foliation on R when R is integrable.

Through the diffeomorphism $(*)$, R defines a differential system E on J^{k-1} such that

$$C^{k-1} = E \oplus F, \quad F = \text{Ker}(\pi_{k-2}^{k-1})_*$$

where $\pi_{k-2}^{k-1} : J^{k-1} \rightarrow J^{k-2}$ is the bundle projection.

N.Tanaka introduced the notion of pseudo-product manifolds as follows.

Pseudo-Product Manifolds $(R; E, F)$

- (1) $E \cap F = 0$, and both E and F are completely integrable.
- (2) $D = E \oplus F$ is non-degenerate.
- (3) The full derived systems of D coincides with $T(R)$

N.Tanaka, *On affine symmetric spaces and the automorphism groups of product manifolds*, Hokkaido Math.J. 14 (1985), 277-351

§2. Geometry of Linear Differential Systems (Tanaka Theory)

We summarize here the basic notion for differential systems.

For a manifold M of dimension d , a subbundle $D \subset T(M)$ of rank r ($s + r = d$) is called a differential system of rank r (or codimension s).

$$D = \{ \omega_1 = \cdots = \omega_s = 0 \}.$$

(M, D) is completely integrable

$$\begin{aligned} &\iff D = \{ dx_1 = \cdots = dx_s = 0 \} \\ &\iff d\omega_i \equiv 0 \pmod{\omega_1, \dots, \omega_s} \quad (1 \leq i \leq s) \\ &\iff [D, D] \subset D \text{ where } D = \Gamma(D) \end{aligned}$$

For non-completely integrable system, we have

Derived System ∂D : $\partial D = D + [D, D]$.

Cauchy Characteristic System $Ch(D)$:

$$Ch(D)(x) = \{ X \in D(x) \mid X]d\omega_i \equiv 0 \pmod{\omega_1, \dots, \omega_s} \text{ for } i = 1, \dots, s \},$$

k -th **Derived System** $\partial^k D$:

$$\partial^k D = \partial(\partial^{k-1} D)$$

k -th **Weak Derived System** $\partial^{(k)} D$:

$$\partial^{(k)} D = \partial^{(k-1)} D + [D, \partial^{(k-1)} D],$$

Symbol Algebras

(M, D) is called **regular** iff

(S1) $\exists \mu > 0$ such that, for all $k \geq \mu$,

$$D^{-k} = \cdots = D^{-\mu} \supsetneq \cdots \supsetneq D^{-2} \supsetneq D^{-1} = D,$$

(S2) $[D^p, D^q] \subset D^{p+q}$ for all $p, q < 0$.

From now on, we will consider **regular** differential systems (M, D) such that $T(M) = D^{-\mu}$.

Symbol algebra $\mathfrak{m}(x)$ of (M, D) at x is defined as follows;

$\forall x \in M$,

$$\mathfrak{m}(x) = \bigoplus_{p=-1}^{-\mu} \mathfrak{g}_p(x).$$

$$\mathfrak{g}_{-1}(x) = D^{-1}(x), \quad \mathfrak{g}_p(x) = D^p(x)/D^{p+1}(x)$$

$$[X, Y] = \varpi_{p+q}([\tilde{X}, \tilde{Y}]_x),$$

$$\begin{cases} \tilde{X} \in \Gamma(D^p), X = \varpi_p(\tilde{X}_x) \in \mathfrak{g}_p(x), \\ \tilde{Y} \in \Gamma(D^q), Y = \varpi_q(\tilde{Y}_x) \in \mathfrak{g}_q(x). \end{cases}$$

$$\mathfrak{g}_p(x) = [\mathfrak{g}_{p+1}(x), \mathfrak{g}_{-1}(x)] \quad \text{for } p < -1.$$

Conversely given a

Fundamental Graded Lie Algebra :

$$\mathfrak{m} = \bigoplus_{p=-1}^{-\mu} \mathfrak{g}_p$$

i.e., Nilpotent GLA satisfying the generating condition :

$$\mathfrak{g}_p = [\mathfrak{g}_{p+1}, \mathfrak{g}_{-1}] \quad \text{for } p < -1$$

We have the notions of

Standard Differential System $(M(\mathfrak{m}), D_{\mathfrak{m}})$ of type \mathfrak{m}

Prolongation $\mathfrak{g}(\mathfrak{m})$ of $\mathfrak{m} = \bigoplus_{p < 0} \mathfrak{g}_p$

N.Tanaka, *On differential systems, graded Lie algebras and pseudo-groups*, J.Math. Kyoto Univ. **10** (1970), 1-82

§3. Symbol Algebra of (J^k, C^k)

$$C^k = \{\varpi = \varpi_i = \dots = \varpi_{i_1 \dots i_{k-1}} = 0\}$$

$$\mathfrak{C}^k(m, n) = \mathfrak{C}_{-(k+1)} \oplus \mathfrak{C}_{-k} \oplus \dots \oplus \mathfrak{C}_{-1}$$

where $\mathfrak{C}_{-(k+1)} = W$, $\mathfrak{C}_p = W \otimes S^{k+p+1}(V^*)$, $\mathfrak{C}_{-1} = V \oplus W \otimes S^k(V^*)$.

Coframe:

$$\{\varpi, \dots, \varpi_{i_1 \dots i_\ell}, \dots, dx_i, dp_{i_1 \dots i_k}\},$$

Dual Frame:

$$\left\{ \frac{\partial}{\partial y}, \dots, \frac{\partial}{\partial p_{i_1 \dots i_\ell}}, \dots, \frac{d}{dx_i}, \frac{\partial}{\partial p_{i_1 \dots i_k}} \right\}$$

where

$$\frac{d}{dx_i} = \frac{\partial}{\partial x_i} + \sum p_{ij_1 \dots j_\ell} \frac{\partial}{\partial p_{j_1 \dots j_\ell}}$$

K.Yamaguchi, *Contact Geometry of Higher Order*, Japan. J. Math. **8** (1982), 109-176

Pseudo-projective structures of order k of bidegree (m, n)

Starting from

$$\mathfrak{C}_{-1} = \mathfrak{e} \oplus \mathfrak{f},$$

where $\mathfrak{e} = V, \mathfrak{f} = W \otimes S^{k-1}(V^*)$.

This splitting represents the pseudo-product structure of k -th order equation R in the symbol level.

Put

$$\check{\mathfrak{g}}_0 = \{X \in \text{Der}_0(\mathfrak{C}^{k-1}(m, n)) : [X, \mathfrak{e}] \subset \mathfrak{e}, [X, \mathfrak{f}] \subset \mathfrak{f}\}$$

Pseudo-projective GLA

$$\mathfrak{g}^k(m, n) = \text{Prolongation of } (\mathfrak{C}^{k-1}(m, n), \check{\mathfrak{g}}_0)$$

Cartan Connections

T.Morimoto, *Geometric structures on filtered manifolds*, Hokkaido Math. J. **22**(1993), 263-347

§4. Pseudo-product GLA $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ of type (\mathfrak{l}, S)

Now we will give the notion of the pseudo-product GLA of type (\mathfrak{l}, S) .

$\mathfrak{l} = \mathfrak{l}_{-1} \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_1$: **reductive GLA**

(1) $\hat{\mathfrak{l}} = \mathfrak{l}_{-1} \oplus [\mathfrak{l}_{-1}, \mathfrak{l}_1] \oplus \mathfrak{l}_1$ is simple.

(2) $\mathfrak{z}(\mathfrak{l}) \subset \mathfrak{l}_0$.

S : **faithful irreducible \mathfrak{l} -module.**

$$S_{-1} = \{s \in S : \mathfrak{l}_1 \cdot s = 0\}$$

$$S_p = \text{ad}(\mathfrak{l}_{-1})^{-p-1} S_{-1} \text{ for } p < 0$$

Form the semi-direct product

$$\begin{aligned} \mathfrak{g} &= S \oplus \mathfrak{l}, & [S, S] &= 0 \\ \mathfrak{g}_p &= \mathfrak{l}_p \ (p \geq 0), & \mathfrak{g}_{-1} &= \mathfrak{l}_{-1} \oplus S_{-1}, \\ \mathfrak{g}_q &= S_q \ (q \leq -2). \end{aligned}$$

Then the following hold:

- (1) $S = \bigoplus_{p=-1}^{-\mu} S_p$;
- (2) $\mathfrak{m} = \bigoplus_{p < 0} \mathfrak{g}_p$ is generated by \mathfrak{g}_{-1}
- (3) $S_{-\mu} = \{s \in S : [\mathfrak{l}_{-1}, s] = 0\}$
- (4) $S_p \hookrightarrow W \otimes S^{\mu+p}(\mathfrak{l}_{-1}^*)$, $W = S_{-\mu}$
- (5) $S_{-1}, S_{-\mu}$: irreducible \mathfrak{l}_0 -modules

Thus, \mathfrak{m} is a symbol algebra of μ -th order differential equations of finite type.

We will ask the following questions.

Our Problem

- (1) When is \mathfrak{g} the prolongation of \mathfrak{m} or $(\mathfrak{m}, \mathfrak{g}_0)$?

(2) Find the fundamental invariants for equations of type m .

§5. Y. Se-ashi's Theory for Linear Differential Equations of Finite Type

For the linear differential equations of finite type, we have the following theory due to Y. Se-ashi.

Y. Se-ashi, *On differential invariants of integrable finite type linear differential equations*, Hokkaido Math.J., **17** (1988), 151-195

In particular, he established the **Rigidity Theorem** of Equations of type (l, S) for $M = L/L'$ other than projective spaces and quadrics

Utilizing this Rigidity Theorem, we have an **Application to Hypergeometric Equations**

T. Sasaki, K. Yamaguchi and M. Yoshida, *On the Rigidity of Differential Systems modelled on Hermitian Symmetric Spaces and Disproofs of a Conjecture concerning Modular Interpretations of Configuration Spaces*, Advanced Studies in Pure Math. **25** (1997), 318-354

§6. Generalized Spencer cohomology

$\mathfrak{a} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{a}_p$: graded Lie algebra

$V = \bigoplus_{p \in \mathbb{Z}} V_p$: graded \mathfrak{a} -module

Cohomology space $H^q(\mathfrak{a}, V)$ associated with $(C^q(\mathfrak{a}, V), \partial) = (\text{Hom}(\bigwedge^q \mathfrak{a}, V), \partial)$

here $\partial^q : C^q(\mathfrak{a}, V) \rightarrow C^{q+1}(\mathfrak{a}, V)$ is given by

$$\begin{aligned} \partial^q \omega(x_1, \dots, x_{q+1}) &= \sum_{i=1}^{q+1} (-1)^{i+1} x_i \cdot \omega(x_1, \dots, \hat{x}_i, \dots, x_{q+1}) \\ &\quad + \sum_{i < j} (-1)^{i+j} \omega([x_i, x_j], x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_{q+1}), \end{aligned}$$

where $x_i \in \mathfrak{a}$ and $\omega \in C^q(\mathfrak{a}, V)$.

Moreover

$$C^q(\mathfrak{a}, V) = \bigoplus_r C^q(\mathfrak{a}, V)_r,$$

$$C^q(\mathfrak{a}, V)_r = \{ \omega \in C^q(\mathfrak{a}, V) : \omega(\mathfrak{a}_{i_1} \wedge \dots \wedge \mathfrak{a}_{i_q}) \subset V_{i_1 + \dots + i_q + r} \}.$$

Cohomology group $H^*(\mathfrak{m}, \mathfrak{g})$ associated with the adjoint representation of \mathfrak{m} on \mathfrak{g}

Put $\mathfrak{b}_{-1} = S$, $\mathfrak{b}_0 = \mathfrak{l}$, $\mathfrak{b}_p = 0$ ($p \neq -1, 0$)

$$\mathfrak{g} = \bigoplus_p \mathfrak{b}_p = \mathfrak{b}_{-1} \oplus \mathfrak{b}_0, \quad \mathfrak{b}_- = \mathfrak{b}_{-1}$$

Cohomology group $H^*(\mathfrak{b}_-, \mathfrak{g})$ associated with the adjoint rep. of \mathfrak{b}_- on \mathfrak{g}

$H^i(\mathfrak{b}_-, \mathfrak{g})$ is a \mathfrak{l} -module

Theorem 6.1.

$H^q(\mathfrak{m}, \mathfrak{g}) \cong \bigoplus_{i=0}^q H^{q-i}(\mathfrak{l}_-, H^i(\mathfrak{b}_-, \mathfrak{g}))$ as a \mathfrak{l}_0 -module.

Explicitly for $q = 1$

$$H^1(\mathfrak{m}, \mathfrak{g}) \cong H^1(\mathfrak{l}_-, S) \oplus H^0(\mathfrak{l}_-, S \otimes S^*/\mathfrak{l}) \oplus H^0(\mathfrak{l}_-, \check{\mathfrak{b}}_1)$$

where $\check{\mathfrak{b}} = \bigoplus_{p \in \mathbb{Z}} \check{\mathfrak{b}}_p$ is the prolongation of $\mathfrak{g} = \mathfrak{b}_{-1} \oplus \mathfrak{b}_0 = S \oplus \mathfrak{l}$

Gradations of cohomology groups

Put $\mathfrak{g}_{p,q} = \mathfrak{g}_p \cap \mathfrak{b}_q$

$$C^q(\mathfrak{m}, \mathfrak{g})_{r,s} = \{\omega \in \text{Hom}(\bigwedge^q \mathfrak{m}, \mathfrak{g}) :$$

$$\omega(\mathfrak{g}_{i_1, j_1} \wedge \cdots \wedge \mathfrak{g}_{i_q, j_q}) \subset \mathfrak{g}_{i_1 + \cdots + i_q + r, j_1 + \cdots + j_q + s} \text{ for all } i_1, \dots, i_q, j_1, \dots, j_q\}$$

$$H^*(\mathfrak{m}, \mathfrak{g}) = \bigoplus_{q,r,s} H^q(\mathfrak{m}, \mathfrak{g})_{r,s}$$

$$C^q(\mathfrak{b}_-, \mathfrak{g})_s = \{\omega \in \text{Hom}(\bigwedge^q \mathfrak{b}_-, \mathfrak{g}) :$$

$$\omega(\mathfrak{b}_{j_1} \wedge \cdots \wedge \mathfrak{b}_{j_q}) \subset \mathfrak{b}_{j_1 + \cdots + j_q + s} \text{ for all } j_1, \dots, j_q < 0\}$$

$$H^*(\mathfrak{b}_-, \mathfrak{g}) = \bigoplus_{q,s} H^q(\mathfrak{b}_-, \mathfrak{g})_s$$

Thus

$$H^q(\mathfrak{m}, \mathfrak{g})_{r,s} \cong \bigoplus_{i=0}^q H^{q-i}(\mathfrak{l}_-, H^i(\mathfrak{b}_-, \mathfrak{g})_s)_r$$

Utilizing the following theorems

Theorem A (Kostant) Let $\mathfrak{s} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{s}_p$ be a simple GLA of type (X_l, Δ_1) and $M(\omega)$ be an irreducible \mathfrak{s} -module with lowest weight ω . Then

$$\text{ch}_{\mathfrak{s}_0}(H^j(\mathfrak{s}_-, M(\omega))) = \sum_{w \in W_1^j} \text{ch}_{\mathfrak{s}_0}(m(w(\omega - \rho) + \rho)),$$

where ρ is the half sum of positive roots.

Theorem B (Kobayashi-Nagano)

S : faithful irreducible \mathfrak{l} -module.

If $\check{\mathfrak{b}}_1 \neq \{0\}$,

$$(1) \dim \check{\mathfrak{b}} < \infty$$

$$\check{\mathfrak{b}} = \mathfrak{b}_{-1} \oplus \mathfrak{b}_0 \oplus \check{\mathfrak{b}}_1: \text{ simple}$$

$$\mathfrak{b}_{-1} = S, \quad \mathfrak{b}_0 = \mathfrak{l}, \quad \check{\mathfrak{b}}_1 = S^*$$

$$(2) \dim \check{\mathfrak{b}} = \infty$$

$$\mathfrak{l} = \mathfrak{gl}(S) \text{ or } \mathfrak{csp}(S)$$

We have the following answer for our problem (1) cited in §4.

Theorem 6.2. Let $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ be a pseudo-product GLA of type (l, S) .
Except for (1), (2), (3),

$$\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p \cong \mathfrak{g}(\mathfrak{m}),$$

where $\mathfrak{m} = \bigoplus_{p < 0} \mathfrak{g}_p$

(1) $0 < \dim \check{\mathfrak{b}} < \infty$ ($\check{\mathfrak{b}}$: simple)

| $\mathcal{D}(l)$ | λ | $\check{\mathfrak{b}} = (Y_{\ell+1}, \Sigma_1)$ |
|---|-----------------------|---|
| $(A_i \times A_{\ell-i}, \{\alpha_j\}) (j \leq i)$ | $\varpi_1 + \varpi_i$ | $(A_{\ell+1}, \{\alpha_j, \alpha_{i+1}\})$ |
| $(B_\ell, \{\alpha_1\}) (\ell \geq 3)$ | ϖ_1 | $(B_{\ell+1}, \{\alpha_1, \alpha_2\})$ |
| $(A_\ell, \{\alpha_i\}) (\ell \geq 2)$ | $2\varpi_\ell$ | $(C_{\ell+1}, \{\alpha_i, \alpha_{\ell+1}\})$ |
| $(D_\ell, \{\alpha_\ell\}) (\ell \geq 4)$ | ϖ_1 | $(D_{\ell+1}, \{\alpha_1, \alpha_{\ell+1}\})$ |
| $(D_\ell, \{\alpha_1\}) (\ell \geq 4)$ | ϖ_1 | $(D_{\ell+1}, \{\alpha_1, \alpha_2\})$ |
| $(A_\ell, \{\alpha_i\}) (\ell \geq 4) (1 < i < \ell)$ | $\varpi_{\ell-1}$ | $(D_{\ell+1}, \{\alpha_i, \alpha_{\ell+1}\})$ |
| $(D_5, \{\alpha_5\})$ | ϖ_5 | $(E_6, \{\alpha_1, \alpha_3\})$ |
| $(D_5, \{\alpha_4\})$ | ϖ_5 | $(E_6, \{\alpha_1, \alpha_2\})$ |
| $(D_5, \{\alpha_1\})$ | ϖ_5 | $(E_6, \{\alpha_1, \alpha_6\})$ |
| $(E_6, \{\alpha_1\})$ | ϖ_6 | $(E_7, \{\alpha_1, \alpha_7\})$ |
| $(E_6, \{\alpha_6\})$ | ϖ_6 | $(E_7, \{\alpha_6, \alpha_7\})$ |

(2) $\dim \check{\mathfrak{b}} = \infty$

| $\mathcal{D}(l)$ | λ | $\mathfrak{g}(\mathfrak{m}, \mathfrak{g}_0)$ |
|-----------------------------|------------|---|
| $(A_\ell, \{\alpha_i\})$ | ϖ_1 | $(A_{\ell+1}, \{\alpha_i, \alpha_{\ell+1}\})$ |
| $(C_\ell, \{\alpha_\ell\})$ | ϖ_1 | \mathfrak{g} |

In $(C_\ell, \{\alpha_\ell\})$ -case, $\mu = 2$

$$S_{-2} = V^*, \quad S_{-1} = V, \quad l_{-1} = S^2(V^*),$$

$$l_0 = V \otimes V^* \oplus \mathbb{C}, \quad l_1 = S^2(V)$$

(3) \mathfrak{g} is a pseudo-projective GLA, i.e., $\mathcal{D}(l) = (A_\ell \times A_n, \{\alpha_1\})$ and $\lambda = k\varpi_1 + \pi_1$
 $\mu = k + 1$ and $\dim W = n + 1$

$$S_{-\mu} = W, \quad S_p = W \otimes S^{\mu+p}(V^*) \quad (-\mu < p < 0),$$

$$l_{-1} = V, \quad l_0 = \mathfrak{gl}(V) \oplus \mathfrak{sl}(W), \quad l_1 = V^*$$

§7. Gradation of Simple Lie Algebras.

We summarize here the relevant terminology to express the contents of Theorem 6.2.

\mathfrak{s} : Simple Lie Algebra over \mathbb{C}

\mathfrak{h} : Cartan Subalgebra ; $\Phi \subset \mathfrak{h}^*$: Root System

$\Delta = \{\alpha_1, \dots, \alpha_\ell\}$: Simple Root System

$$\mathfrak{s} = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha \oplus \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_{-\alpha},$$

$\Delta_1 \subset \Delta$: Fix, $\Phi^+ = \bigcup_{p \geq 0} \Phi_p^+$,

$$\Phi_p^+ = \left\{ \alpha = \sum_{i=1}^{\ell} n_i \alpha_i \mid \sum_{\alpha_i \in \Delta_1} n_i = p \right\},$$

$$\begin{cases} \mathfrak{s}_p = \bigoplus_{\alpha \in \Phi_p^+} \mathfrak{g}_\alpha, & (p > 0) \\ \mathfrak{s}_0 = \bigoplus_{\alpha \in \Phi_0^+} \mathfrak{g}_\alpha \oplus \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi_0^+} \mathfrak{g}_{-\alpha}, \\ \mathfrak{s}_{-p} = \bigoplus_{\alpha \in \Phi_p^+} \mathfrak{g}_{-\alpha}, \end{cases}$$

Then

$$[\mathfrak{s}_p, \mathfrak{s}_q] \subset \mathfrak{s}_{p+q} \quad \text{for } p, q \in \mathbb{Z}.$$

Generating Condition: $\mathfrak{m} = \bigoplus_{p < 0} \mathfrak{s}_p$

$$(*) \quad \mathfrak{s}_p = [\mathfrak{s}_{p+1}, \mathfrak{s}_{-1}] \quad \text{for } p < -1$$

$$\Delta_1 \subset \Delta \implies (X_\ell, \Delta_1): \quad \mathfrak{s} = \bigoplus_{p=-\mu}^{\mu} \mathfrak{s}_p$$

where $\mu = \sum_{\alpha_i \in \Delta_1} n_i(\theta)$, $\theta = \sum_{i=1}^{\ell} n_i(\theta) \alpha_i$

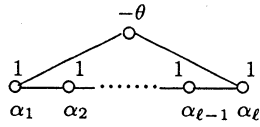
Theorem 7.1. $\mathfrak{s} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{s}_p$: Simple Graded Lie Algebra over \mathbb{C} satisfying $(*)$.
 X_ℓ : Dynkin Diagram of \mathfrak{s} .

\implies

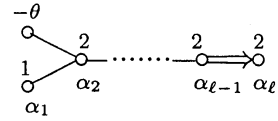
$\exists_1 \Delta_1 \subset \Delta$ s.t. $\mathfrak{s} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{s}_p \cong (X_\ell, \Delta_1)$

Classification of $\mathfrak{s} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{s}_p$ with $(*)$ is equivalent to

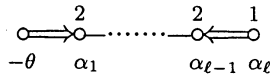
Classification of Parabolic subalgebras $\mathfrak{s}' = \bigoplus_{p \geq 0} \mathfrak{s}_p$



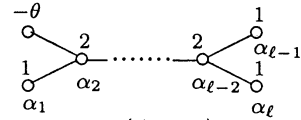
$A_\ell (\ell > 1)$



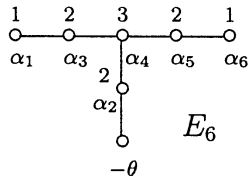
$B_\ell (\ell > 2)$



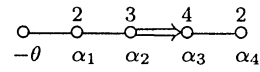
$C_\ell (\ell > 1)$



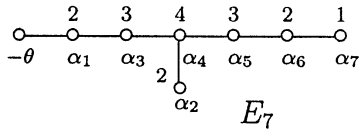
$D_\ell (\ell > 3)$



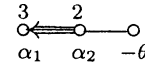
E_6



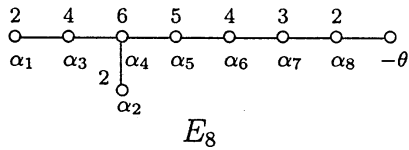
F_4



E_7



G_2



E_8

Extended Dynkin Diagrams
with the coefficient of the highest root

§8. Second Cohomology

We summarize here the results on the second cohomology.

$$H^2(\mathfrak{m}, \mathfrak{g})_{r,-1} \cong H^2(\mathfrak{l}_-, \mathfrak{b}_{-1})_r,$$

Proposition 8.1.

- (1) $H^2(\mathfrak{m}, \mathfrak{g})_{r,-1} = 0$ for all $r \geq 2$.
- (2) $H^2(\mathfrak{m}, \mathfrak{g})_{1,-1} \neq 0$ iff the sequence $(X_\ell, \Delta_1, \lambda)$ is one of the following
 $(A_\ell, \{\alpha_1\}, j\varpi_{\ell-1} + k\varpi_\ell)$ ($\ell \geq 2, j, k \geq 0, j+k \geq 1$), $(A_\ell, \{\alpha_2\}, k\varpi_\ell)$ ($\ell \geq 3, k \geq 1$),
 $(C_2, \{\alpha_2\}, k\varpi_1)$ ($k \geq 1$)

$$H^2(\mathfrak{m}, \mathfrak{g})_{r,0} \cong H^1(\mathfrak{l}_-, H^1(\mathfrak{b}, \mathfrak{g})_0)_r$$

Proposition 8.2.

- (1) $H^2(\mathfrak{m}, \mathfrak{g})_{r,0} = 0$ for $r \geq 2$ except for the case when $(X_\ell, \Delta_1) = (A_\ell, \{\alpha_1\})$ or $(A_\ell, \{\alpha_\ell\})$.
- (2) $H^2(\mathfrak{m}, \mathfrak{g})_{1,0} = 0$ if (X_ℓ, Δ_1) is one of $(A_\ell, \{\alpha_i\})$ ($\ell \geq 4, 1 < i \leq \lfloor \frac{\ell+1}{2} \rfloor$), $(C_\ell, \{\alpha_\ell\})$ ($\ell \geq 3$), $(D_\ell, \{\alpha_{\ell-1}\})$ ($\ell \geq 5$), $(E_6, \{\alpha_1\})$, $(E_7, \{\alpha_7\})$.
- (3) If $(X_\ell, \Delta_1) = (A_\ell, \{\alpha_1\})$, then $H^2(\mathfrak{m}, \mathfrak{g})_{r,0} = 0$ for $r \geq \min\{m_1, m_l\} + 2$.

$$H^2(\mathfrak{m}, \mathfrak{g})_{r,1} \cong H^0(\mathfrak{l}_-, H^2(\mathfrak{b}_-, \mathfrak{g})_1)_r \oplus H^1(\mathfrak{l}_-, \check{\mathfrak{b}}_1)_r,$$

$$H^2(\mathfrak{m}, \mathfrak{g})_{r,2} \cong H^0(\mathfrak{l}_-, H^2(\mathfrak{b}_-, \mathfrak{g})_2)_r$$

Proposition 8.3.

- (1) For $s = 1, 2$,

$$H^0(\mathfrak{l}_-, H^2(\mathfrak{b}_-, \mathfrak{g})_s)_r = 0 \quad \text{for } r \geq s(\mu - 1) + 1,$$

- (2) If $X_l = B_l, C_l$ or E_7 , then

$$H^0(\mathfrak{l}_-, H^2(\mathfrak{b}_-, \mathfrak{g})_s)_r = 0 \quad \text{for } r \geq \lfloor s(\mu + 1)/2 \rfloor + 1.$$

Department of Mathematics, Faculty of Science,
Hokkaido University, Sapporo 060-0810, Japan
E-mail:yamaguch@math.sci.hokudai.ac.jp

Department of Mathematics,
Hokkaido University of Education, Asahikawa Campus, Asahikawa 070-8261, Japan
E-mail:tomoaki@atson.asa.hokkyodai.ac.jp