数理解析研究所講究録
1151巻 2000年 1-71

The interplay between topological dynamics and theory of C*-algebras, Part 2 (after the Seoul lecture note 1992)

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Contents

1 Preliminaries

Let $\Sigma = (X, \sigma)$ be a topologicall dynamical system on an arbitrary compact Hausdorff space X with a homeomorphism σ. Let $C(X)$ be the algebra of all continuous functions on X. We consider then an automorphism α on $C(X)$ defined as $\alpha(f)(x) = f(\sigma^{-1}x)$ and the associated C*-algebra $A(\Sigma)$, which is the C*-crossed product of $C(X)$ with respect to the above automorphism considered as the action of the integer group Z. Call this algebra a homeomorphism C*-algebra associated to the dynamical system Σ.

This article contains further developments of our project to construct a broad bridge between topological dynamics and C*-theory after the author's Seoul lecture note [38]. The project is based on the following three principles.

(1) All key results should be formulated in equivalent forms for both sides,

(2) Allow periodic points in basic principle,

(3) Preferably without the assumption of metrizability for the space X.

The principle (3) means that in relation with operator algebras we have to treat sometimes dynamical systems in a big compact space such as a hyperstonean space. Therefore, unlesss we specify the space, X stands an arbitrary compact Hausdorff space without any countability assumption.

In this note we shall discuss mainly the author's results as well as joint works with his colleagues after the author's Seoul lecture note [38].

Main results included in this note are in two directions;

1) We clarify the C*-algebraic meanings of those elementary sets of topological dynamical systems such as the set of recurrent points, $c(\sigma)$ together with its closure, Birkhoff center, the difference $c(\sigma)\backslash Per(\sigma)$, and the nonwandering set, $\Omega(\sigma)$.

Together with the result in [29] for the set of chain recurrent points, $R(\sigma)$, and also with other author's results for the sets $Per(\sigma)$ and $Aper(\sigma)$ this means now we can understand the structure of all kinds of elementary sets in topological dynamical systems in terms of C*-algebras.

2). We have succeeded the analysis of the continuous full groups in connection with normalizers of $C(X)$ in $A(\Sigma)$ and also analized the structure of bounded topological orbit equivalences. As a consequence, we have solved the problem of the restricted isomorphism problem, by which we mean the problem to say under what relations of dynamical systems two homeomorphism C*-algebras are isomorphic each other by the isomorphism keeping their subalgebras of continuous functions.

The author regrets to say however that we have been unable to make substantial progress towards the general isomorphism problem. The final goal in this direction seems to be still far beyond our scope, even for dynamical systems in tori, or in the unit circle. .

There are of course many other important subjects to be done in our project such as the analysis of extensions of dynamical systems and entropies etc, etc. In this sense our present theory remains still immature at the stage to be able to make contributions towards topological dynamics from the side of C*-algebras. We should however notice here recent deep contributions by K.Matsumoto to the presentation of subshifts from the spirit of *-algebras in his series of works (notably [25]: see its references for his another papers). Furthermore, the project should be extended to the case of continuous mappings, and the interplay between the theory of flows of dynamical systems and C*-theory will also be wating for us (dynamical systems of diffeomorphisms also come to another problems, but to handle this class means that we have to be concerned not only with C*-algebras but with thier canonical dense C^∞- subalgebras).

Throughout this note, for published results we shall only present our results without proofs or with outlines of proofs for some hard results, whereas for other ones we give sometimes detailed proofs.

The above homeomorphism C*-algebra contains $C(X)$ as a subalgebra and generated by $C(X)$ and a special unitary element δ implementing the automorphism α. It follows that the algebra is just a closed linear span of generalized polynomials of $\{\delta^n\}$ over $C(X)$. Moreover, it is basically characterized as the universal C*-algebra having the following properties:

(a)

$$\| \sum_{-n}^{n} f_i\delta^i \| \geq \| f_0 \| \quad \text{for functions} \quad f_i \in C(X).$$

(b) $A(\Sigma)$ has the universal property for covariant representations of $\{C(X), \alpha, Z\}$.

Note that the condition (a) implies the assertion; $\{\delta^n\}$ is independent over $C(X)$. Namely,

$$\sum_{i=-n}^{n} f_i\delta^i = 0 \quad \text{implies } f_i = 0 \text{ for all } i.$$

Here a covariant representation of the above system means a pair of $\{\pi, u\}$, a representation of $C(X)$ on a Hilbert space H and a unitary operator u on H, such that $\pi(\alpha(f)) = u\pi(f)u^*$. Every representation of $A(\Sigma)$ arises from a covariant representation $\{\pi, u\}$. In this aspect we write a representation of $A(\Sigma)$ by $\tilde{\pi} = \pi \times u$. Moreover, the condition (b) implies the existence of the canonical projection of norm one E from $A(\Sigma)$ to $C(X)$ which becomes faithful in the sense that $E(a) = 0$ for $a \geq 0$ implies $a = 0$. For an element a of $A(\Sigma)$ we define the n-th generalized Fourier coefficient $a(n)$ as $E(a\delta^{*n})$.

We write $Per(\sigma)$ and $Aper(\sigma)$ the sets of periodic and aperiodic points respectively. The set $Per_n(\sigma)$ means the set of all n-periodic points, whereas we write

$$Per^n(\sigma) = \{ \quad x \quad |\sigma^n(x) = x\}.$$

We write $O_\sigma(x)$ the orbit of x by the homeomorphism σ or $O(x)$ if no confusion occurs. We recall classes of dynamical systems treated in [38].

Definition 1.1 *(1) Σ is said to be minimal if every orbit is dense in X:*

(2) Topologically transitive if for any pair of open sets: U, V there exists an integer n such that $\sigma^n U \cap V \neq \phi$:

(3) Topologically free if the set $Aper(\sigma)$ is dense in X:

(4) Free if there is no periodic points.

The third class covers almost all dynamical systems because dynamical systems in manifolds have often at most countable periodic points. This class however does not appear in usual literature of dynamical systems since it may be too broad to handle with standard arguments. This class is however quite important not only from the theory of C*-algebras but in topological dynamics. We have seen many evidences in the Seoul lecture note as well as from the results in §10 and §11. We note that a topologically transitive dynamical system in an infinite space becomes necessarily topologically free.

For the classes of C*-algebras we shall explain their structures when needed.

As is well known, when X is a metric space topological transitivity is equivalent to the existence of a dense orbit, but this is not the case in general.In fact, all topological dynamical systems in the spectrum of L^∞ space of the Lebesgue space coming from nonsingular ergodic transformations provide examples of such differences. We shall discuss in §6 these kinds of homeomorphisms.

2 Results from the Seoul lecture note

In this section we confirm several facts in [38] which will be often used in our coming discussions. Let $\tilde{\pi} = \pi \times u$ be a representation of $A(\Sigma)$ on a Hilbert space H.Write I the kernel of π, which is a closed ideal of $C(X)$. Hence it is written as the kernel of an invariant closed subset X_π,$k(X_\pi)$. The image $\pi(C(X))$ is then naturally isomorphic to the quotient algebra $C(X)/I$,which can be identified with the algebra $C(X_\pi)$. Thus we have the associated dynamical system $\Sigma_\pi = (X_\pi, \sigma_\pi)$ where σ_π is the restriction of σ to the invariant subset X_π, $\sigma|X_\pi$. On the other hand , the image is isomorphic (Gelfand representation) to the algebra of all continuous functions on a compact space X'_π. Since the automorphism $Adu : a \to uau*$ induces an automorphism α_π on $\pi(C(X))$ it gives rise a homeomorphism σ'_π on this space, that is, the dynamical system $\Sigma'_\pi = (X'_\pi, \sigma'_\pi)$.

One may then easily verify that two dynamical systems $\Sigma_\pi = (X_\pi, \sigma_\pi)$ and $\Sigma'_\pi = (X'_\pi, \sigma'_\pi)$ are topologically conjugate each other through the isomorphisms

$$C(X_\pi) \simeq C(X)/I \simeq \pi(C(X)) \simeq C(X'_\pi).$$

Henceforth, we identify these dynamical systems and call the system $\Sigma_\pi = (X_\pi, \sigma_\pi)$ the dynamical system induced by the representation $\tilde{\pi} = \pi \times u$.

We first recall that structure of those basic (irreducible) representations of $A(\Sigma)$ coming from the points of the space X. Namely, take a point x of X and denote by μ_x the point evalution. Let φ be a state extension of μ_x to $A(\Sigma)$. We write the GNS- representation by φ as $\{H_\varphi, \tilde{\pi}_\varphi, \xi_\varphi\}$. This kind of representations is equivalent to the class of induced covariant representations discussed in a broad C*-algebraic context. In our simple setting, however, we need not use such big machines.

Lemma 2.1 *For an element a and a function f of $C(X)$, we have*

$$\varphi(af) = \varphi(fa) = f(x)\varphi(a).$$

Keeping the notations as above we define the subspace H_n of H_φ by

$$H_n = \{\xi \in H_\varphi | \quad \tilde{\pi}_\varphi(f)\xi = f(\sigma^n x)\xi \quad \text{for every } f \in C(X)\}.$$

Write $u = \tilde{\pi}_\varphi(\delta)$, that is, $\tilde{\pi}_\varphi = \pi_\varphi \times u$.

Theorem 2.2 *([38, Proposition 4.2])*

(1) $\xi_\varphi \in H_0$ and $H_n = u^n H_0$. Two subspaces H_m and H_n are orthogonal if $m \neq n$ ($m \neq n$ mod p if x is a p-periodic point),

(2) If x is aperiodic,

$$H_\varphi = \sum_{n \in Z} \oplus H_n \quad \textit{(orthogonal sum).}$$

If x is p-periodic,

$$H_\varphi = H_0 \oplus H_1 \oplus \ldots \oplus H_{p-1} \quad \textit{(orthogonal sum).}$$

(3) The state φ is a pure state,i.e, $\tilde{\pi}_\varphi$ is irreducible if and only if H_0 is one dimensional.

As a corollary we have the following conclusion.

Corollary 2.3 *(1) For an aperiodic point x, the state extension of μ_x is unique and has the form, $\varphi_x = \mu_x \circ E$.*

(2) If x is a periodic point, the pure state extensions of μ_x is parametrized by the torus T, written as $\varphi_{x,\lambda}$. This parameter λ appears as $u^p = \lambda$ on the space H_φ.

Henceforth we denote those irreducible representations by $\tilde{\pi}_x$ for an aperiodic point x and by $\tilde{\pi}_{y,\lambda}$ for a periodic point y. The unitary equivalence of these induced representations is determined in the following way.

Proposition 2.4 *([37, Theorem 4.1.3])*

Take two points x and y, then

(1) $\tilde{\pi}_x$ and $\tilde{\pi}_y$ are unitarily equivalent if and only if $O(x) = O(y)$ when x and y are aperiodic,

(2) $\tilde{\pi}_{x,\lambda}$ and $\tilde{\pi}_{y,\mu}$ are unitarily equivalent if and only if $O(x) = O(y)$ and $\lambda = \mu$ when x and y are periodic points.

In the following, we denote by $P(\overline{x})$ the kernel of an irreducible representation $\tilde{\pi}_x$ for an aperiodic point x and by $P(\overline{y}, \lambda)$ the kernel of an irreducible representation for an irreducible representation, $\tilde{\pi}_{y,\lambda}$. Here we mean by $\overline{x}$ and $\overline{y}$ that those primitive ideals depend only their orbits, that is, classes of x and y respectively. We also denote by $Q(\overline{y})$ the intersection of the above primitive ideals for all parameters.

Now if all irreducible representations of $A(\Sigma)$ came up from the points of X representation theory of this algebra would become quite understandable. This is of course not the case in general. For instance, let $\Sigma_\theta = (\sigma_\theta, T)$ be the rotation on the circle by an irrational number θ. The algebra $A(\Sigma_\theta)$ (usually written as A_θ) has then the irreducible representation on the space $L^2(d\mu)$ for the Lebesgue measure $d\mu$ arised from the covariant representation $\{m, u_\theta\}$ where m is the representation of $C(T)$ as multiplication operators and u_θ is the translation unitary operator by θ. This representation can not be arised from the points of T. In fact. we can not find such common eigen subspaces of $L^2(d\mu)$ as described in the theorem. We shall discuss later the real obstruction of this phenomena.

Thus it will be quite meaningful that we still have the following result.

Proposition 2.5 *Every finite dimensional irreducible representation of $A(\Sigma)$ is unitarily equivalent to the GNS representation associated to the one induced from a periodic point. The dimension necessarily coincides with the poriod of that point.*

This is of course well known in the theory of covariant representations of transformation group C*-algebras in a much broader context, but here we are taking the way of pedestrian mathematics but still obtaining substancial results. In this sense, a key of the proposition lies in the following point. In fact, let $\tilde{\pi} = \pi \times u$ be a n-dimensional irreducible representation on a Hilbert space H, then the space X_π consists of a single periodic orbit $O(x) = \{x, \sigma(x), \ldots, \sigma^{k-1}(x)\}$. Let p_i be the characteristic function of the set $\{\sigma^i(x)\}$, then the automorphism Adu brings p_i to p_{i+1} with $modk$. Moreover $u^{\ell k}$ commmute with the algebra $C(X_\pi)$ for every ℓ. It follows that the C*-algebra $p_0\tilde{\pi}(A(\Sigma))p_0$ becomes a commutative C*-subalgebra acting irreducibly on the space p_0H. Hence it has to be one dimensional and $k = n$. Therefore, $\tilde{\pi}$ is naturally unitarily equivalent to the representation induced by the periodic point x with an appropriate parameter λ, $\lambda = u^n$.

Remark.(a) Actually we can say more; namely suppose that the center of the image $\tilde{\pi}(A(\Sigma))$ is trivial (such as the case of a factor representation), then the image is finite dimensional if and only if the space X_π is written as the orbit $O(x)$ for a periodic point x in X. In fact, $\tilde{\pi}(A(\Sigma))$ is isomorphic to the matrix algebra M_n if $per(x) = n$.

(b) Sometimes we have to be careful about the difference between a finite dimensional representastion and a representation with finite dimensional image. Note that except irreducible representations they are diffrent, and we have the following fact;

"The image of a representation of $\tilde{\pi} = \pi \times u$ is finite dimensional if and only if the center of the image is finite dimensional and X_π is a finite set".

Besides these results we add the following results which are not mentioned in the lecture note.

Proposition 2.6 *The map*

$$\Phi_n : Per_n(\sigma) \times T \longrightarrow \{\varphi(x, \lambda) | x \in Per_n(\sigma), \lambda \in T\}$$

is a homeomorphism with respect to the w-topology in the pure state space.*

On the other hand, the map

$$\Phi_\infty : x \in Aper(\sigma) \longrightarrow \varphi_x$$

is a homeomorphism into the pure state space of $A(\Sigma)$.

Proof. Suppose a net $\{(y_\alpha, \lambda_\alpha)\}$ converges to a point (y_0, λ_0). Since each $\varphi(y_\alpha, \lambda_\alpha)$ is a pure state extension of the point evaluation μ_{y_α}, $\varphi(y_\alpha, \lambda_\alpha)(f) = f(y_\alpha)$ converges to $f(y_0) = \varphi(y_0, \lambda_0)$ for every continuous function f. On the other hand, we have , by the definition of the parameter for pure state extensions ,that

$$\varphi(y_\alpha, \lambda_\alpha)(\delta^{nk}) = \lambda_\alpha^k \longrightarrow \lambda_0^k = \varphi(y_0, \lambda_0)(\delta^{nk}).$$

Moreover, the values of pure states of other powers of the unitary δ are all zero by Theorem 2.4. Now since

$$\varphi(y, \lambda)(f\delta^n) = f(y)\varphi(y, \lambda)(\delta^n)$$

by Lemma 2.1, we see that the net $\{\varphi(y_\alpha, \lambda_\alpha)\}$ converges to $\varphi(y_0, \lambda_0)$ in the w*-topology.

The converse continuity may be easily seen from the above argumens. The assertion for Φ_∞ is obvious because of the form of the extension φ_x.

When we do not fix the period, we can not expect this kind of result.

Denote by X/Z the orbit space of the dynamical system Σ. Then the above lemma easily implies a simple proof of [23, Theorem A]. Namely

Proposition 2.7 *(1) The space* $\widehat{A(\Sigma)}_n$, *equivalence classes of n-dimensional irreducible representations of* $A(\Sigma)$ *is homeomorphic to the product space* $(Per_n(\sigma)/Z) \times T$.

(2) The map Φ_∞ *induces a homeomorphism from* $Aper(\sigma)/Z$ *into the part of* $\widehat{A(\Sigma)}$ *induced from the aperiodic points.*

Proof. It suffices to notice that the canonical map from the pure state space of $A(\Sigma)$ to the space of primitive ideals is a continuous open map by [11, Theorem 3.4.11] and the latter is homeomorphisc to $\widehat{A(\Sigma)}_n$. Moreover the quotient map from $Per(\sigma)$ to X/Z is also continuous and open. This shows the assertion (1) and similarly the assertion (2) follows.

The next projection theorem and its consequences are the most important results in [38].

Theorem 2.8 *([38, Theorem 5.1]) For the representation $\tilde{\pi} = \pi \times u$, suppose that the induced dynamical system Σ_π is topologically free. Then there exists a faithful projection of norm one, ε_π from $\tilde{\pi}(A(\Sigma))$ to $\pi(C(X))$ such that*

$$\varepsilon_\pi \circ \tilde{\pi}(a) = \pi \circ E(a) \quad \text{for } a \in A(\Sigma).$$

The representation $\tilde{\pi}$ becomes an isomorphism if and only if π is an isomorphism.

Immediate important consequences of this theorem are the following facts ([38, Corollary 5.1A,5.1B and Proposition 5.2]).

Corollary 2.9 *Keep the same notations as above, then*

(a) The image $\tilde{\pi}(A(\Sigma))$ is canonicaly isomorphic to the homeomorphism C^*-algebra $A(\Sigma_\pi)$.

(b) Any image of an infinite dimensional irreducible representation of $A(\Sigma)$ is canonically isomorphic to the homeomorphism C^*-algebra $A(\Sigma_\pi)$.

Because in this case the dynamical system Σ_π is topologically transitive and X_π is an infinite set,hence becomes topoplogically free.

Let P be the kernel of this representation $\tilde{\pi}$, then

(c) An element a of $A(\Sigma)$ belongs to P if and only if every Fourier coefficient of a vanishes on X_π:

(d) P coincides with the closed linear span of generalized polynomials of $\{\delta^n\}$ over the subalgebra $k(X_\pi)$ of $C(X)$.

Actually here the asertions (c) and (d) are equivalent. We shall discuss in §4 the situation surrounding this fact.It should be also noticed here that the above theorem refers no existence of nontrivial ergodic measures but still implies the assertion (a) of the Corollary.

In relation with the assertion (a) we emphasize here that if the system is free all images of representations of $A(\Sigma)$ have the crossed product structure.

We have to mention one more result.

Theorem 2.10 *([38, Theorem 5.4])*
For the homeomorphism C^*-algebra $A(\Sigma)$ the following assertions are equivalent:

(1) Σ is topologically free,

(2) For any ideal I of $A(\Sigma)$, $I \cap C(X) \neq \{0\}$ if and only if $I \neq \{0\}$,

(3) $C(X)$ is a maximal abelian C^*-subalgebra of $A(\Sigma)$.

We shall see later many applications of this result.

When we treat a C*-crossed product $A \times_\alpha Z$ we meet a serious trouble in analysis if there exists an ideal I for which $I \cap A = \{0\}$

(ideal hidden behind A). The above assertion (2) shows that we should not meet this difficulty in topologically free dynamical systems.

The following observation is sometimes useful.

Proposition 2.11 *If the dynamical system is topologically free, then the canonical projection E in $A(\Sigma)$ is a unique projection of norm one from $A(\Sigma)$ to $C(X)$.*

Proof. Suppose we have another projection E' to $C(X)$, and take an aperiodic point x of X. We have then by the unicity of state extensions of μ_x,

$$\mu_x \circ E(a) = \mu_x \circ E'(a) \quad \text{for every } a \in A(\Sigma).$$

Hence,

$$E(a)(x) = E'(a)(x) \quad \text{for every } a \in A(\Sigma) \text{ and } x \in Aper(\sigma).$$

Therefore, $E = E'$.

3 Universal C*-crossed products by the integer group Z and approximation

In this section we reflect the construction of C*-crossed products of the integer group Z from our point of view of the interplay. That is, we introduce the universal C*-crossed product by Z and consider the approximation of its elements by generalized polinomials in norm whose coefficient functions are specified by Fourier coefficients of given elements. Another motivation of the introduction of the universal crossed products is to obtain a perspective for the isomorphism problem between homeomorphism C*-algebras, which will be discussed in §11.

Let A be a unital C*-algebra acting on a Hilbert space H with an automorphism α. Let $A \times_\alpha Z$ be the C*-crossed product with respect to the automorphism α (regarding it as an action of Z) with the generating unitary δ and the canonical projection of norm one $E : A \times_\alpha Z \to A$. Denote by $\{a(n)\}$ the Fourier coefficients of an element a of $A \times_\alpha Z$. Then the norm convergent property of the expansion of a, $a = \sum_{n \in Z} a(n)\delta^n$, is somewhat misleading (as is the case of the expansion of the elements of a von Neumann crossed product with respect to the strong topology),and this certainly does not hold. We have however the result stating that the generalized Cesàro mean $\sigma_n(a)$ converges to a in norm ([12, Theorem VIII.2.2]).

Since this result is quite useful we shall present this type of approximation theorem in a more general form including the case of Cesàro mean. Moreover, in connection with our problem of isomorphisms among homeomorphism C*-algebras we consider the approximation as results in the universal C*-crossed product by Z formulated in the following way.

Let

$$K = \ell_2 \otimes H = \ell_2(Z, H),$$

and consider the unitary representation v_t of the torus T where for each point t of the torus T the unitary operator v_t on K is defined as

$$v_t \xi(n) = e^{2\pi int} \xi(n).$$

Denote by λ the shift unitary operator on K, that is, $(\lambda\xi)(n) = \xi(n-1)$. Then through the covariant representation $\{\pi_\alpha, \lambda\}$ of $\{C(X), \alpha\}$ where

$$(\pi_\alpha(a)\xi)(n) = \alpha^{-n}(a)\xi(n)$$

we can identify the crossed product $A \times_\alpha Z$ with the C*-algebra generated by $\pi_\alpha(C(X))$ and λ. Hence we may assume $A \times_\alpha Z$ is the C*-algebra on the Hilbert space K. We write then the one parameter automorphism groups of $B(K)$ induced by Adv_t by $\hat{\omega}_t$. As is well known, the restriction of this action to each C*-crossed product $A \times_\alpha Z$ is called the dual action of α,usually written as $\hat{\alpha}_t$.

Now let $B(Z)$ be the C*-algebra in $B(K)$ consisting of all elements on which the action $\hat{\omega}_t(a)$ is norm continuous. This is a quite big irreducible C*-algebra on K absorbing all C*-crossed products of a single automorphism (if the space H is big enough). Let $B(\hat{\omega})$ be the fixed point algebra of the action $\hat{\omega}$. We define the projection of norm one E_Z from $B(Z)$ to $B(\hat{\omega})$ by

$$E_Z(a) = \int_0^1 \hat{\omega}_t(a)dt$$

At this stage we know the faithfulness of this projection. In fact, take a state φ on $B(Z)$ then

$$\varphi(E_Z(a)) = \int_0^1 \varphi(\hat{\omega}_t(a))dt.$$

Hence if $E_Z(a) = 0$ for a nonnegative element a the continuous function in the integral becomes zero for every state φ , and $a = 0$. We then define the generalized n-th Fourier coefficient of an element a in $B(Z)$ as $a(n) = E_Z(a\lambda^{*n})$. Note that

$$\hat{\omega}_t(\lambda) = e^{2\pi it}\lambda \quad \text{and} \quad E_Z(\lambda^n) = 0 \quad \forall n \neq 0.$$

Henceforth we regard this algebra as the universal C*-crossed product by the integer group Z.

Next recall that a sequence of real valued continuous functions $\{k_n(t)\}$ on the torus T is called a summability kernel if they satisfy the following three conditions:

(a)

$$\int_T k_n(t)dt = 1,$$

(b)

$$\int_T | k_n(t) | dt \leqslant C(\text{constant})$$

(c) For every $0 < \delta < 1$,

$$\lim_{n\to\infty} \int_\delta^{1-\delta} | k_n(t) | dt = 0.$$

Well known summability kernels are Fejér kernel,

$$K_n(t) = \sum_{-n}^{n} (1 - \frac{| j |}{n+1}) e^{2\pi ijt}$$

de la Vallée Poussin kernel,

$$V_n(t) = 2K_{2n-1}(t) - K_{n-1}(t),$$

and Jackson kernel etc, which are trigonometric polynomials. On the other hand, parameters of summability kernels need not be natural numbers in general.Whenever families of continuous functions satisfy the above three conditions with respects to the appropriate parameters, we can apply the same arguments. Therefore, we can regard the Poisson kernel $P(r,t)$ as a summability kernel with continuous parameter r . In this case $P_r(t)$ satisfies the condition (c) as $r \to 1$. This kernel is however not consisting of trigonometrical polynomials. The Dirichlet kernel $\{D_n(t)\}$ is not a summabilibty kernel because it does not satisfy the third condition, and this shows why we can not obtain the norm convergence of the sum $\sum_{-\infty}^{\infty} a(n)\delta^n$.

Let B be a Banach space and consider the space of all B-valued continuous functions on T,$C(T,B)$. We define the convolution $k_n \star F$ in $C(T,B)$ by

$$k_n \star F(t) = \int_T k_n(s)F(t-s)ds.$$

One then easily sees that the convolution is also a B-valued continuous function. We assert here the Banach space version of the following classical approximation theorem in Fourier analysis.

Proposition 3.1 *For any summability kernel $\{k_n\}$ and a continuous function $F(t)$ in $C(T, B)$, the convolution $k_n \star F(t)$ converges uniformly to $F(t)$ in B.*

The proof of this result is just a linear modification of the one given in the classical Fourier analysis, and we leave the readers its verification.

Define the n-th Fourier coefficient $\hat{F}(n)$ of F by

$$\hat{F}(n) = \int_T F(t)e^{-2\pi int}dt.$$

Then if $k_n(t)$ is a polynomial of a form,

$$k_n(t) = \sum_{-\ell_n}^{\ell_n} c_j e^{2\pi ijt},$$

we have

$$k_n \star F(t) = \sum_{-\ell_n}^{\ell_n} c_j \hat{F}(j)e^{2\pi ijt}.$$

Hence the above result says that the function $F(t)$ is uniformly approximated in norm by the above trigonometric polynomials.

We now apply this result to the algebra $B(Z)$ taking this algebra as the above Banach space B with the continous function $\hat{\omega}_t(a)$ for an element a of $B(Z)$.Write this function as $\tilde{a}(t)$. We have then

$$a(n)\lambda^n = \int_T \hat{\omega}_t(a\lambda^{*n})dt\lambda^n = \int_T \hat{\omega}_t(a)e^{-2\pi int}dt = \hat{\tilde{a}}(n).$$

Therefore we obtain the following approximation theorem in $B(Z)$.

Theorem 3.2 *Let $\{k_n(t)\}$ be a summability kernel on the torus T. Then an element a in $B(Z)$ is approximated in norm by the sequence $k_n \star \tilde{a}(0)$. In particular if the kernel consists of trigonometric polynomials of the form*

$$k_n(t) = \sum_{-\ell_n}^{\ell_n} c_j e^{2\pi ijt},$$

a is approximated by the generalized polynomials of λ with the form

$$k_n \star \tilde{a}(0) = \sum_{-\ell_n}^{\ell_n} c_j a(j)\lambda^j.$$

Hence $B(Z)$ is linearly spanned by $\{\lambda^n\}$ in norm over the fixed point algebra $B(\hat{\omega})$.

Thus, though we do not assume at first any crossed product structure for $B(Z)$ we are able to deduce the fact that it is linearly spanned by generalized polynomials of λ whose coefficients are specifically defined from the Fourier coefficients of the elements to which they converge.

Now take a crossed product $A\times_\alpha Z$ regarded as a C*-subalgebra of $B(Z)$. It is then obvious that the canonical projection E in $A\times_\alpha Z$ is just the restriction of E_Z and $A = B(\hat{\omega})\cap A\times_\alpha Z$. Hence the Fourier coefficients of an element a in $A\times_\alpha Z$ is nothing but those coefficients defined as an element of $B(Z)$.

Therefore from the above theorem we can derive usual conclusions on the unicity of the generalized Fourier coefficients etc. in a quite C*-algeraic manner.

We emphasize again that the advantage of the above approxiamation theorem lies in the fact that for the approximation of a fixed element we can refer to its Fourier coefficients for those approximation polynomials, even in various ways depending on which summability kernels we use. Among them the Cesàro mean for Fejér kernel,

$$\sigma_n(a) = K_n \star \tilde{a}(0) = \sum_{-n}^{n}(1 - \frac{|j|}{n+1})a(j)\delta^j,$$

is most elementary.

4 Noncommutative hulls and kernels; Classification of ideals of homeomorphism C*-algebras

Materials of this section stem from the article [43] together with additional results.

Here we recall the definitions of the ideals, $P(\overline{x})$, $P(\overline{y},\lambda)$ and $Q(\overline{y})$. Note first that the family $\{P(\overline{y},\lambda)|y \in Per(\sigma), \lambda \in T\}$ exhausts all primitive ideals of $A(\Sigma)$ which are kernels of finite dimensional irreducible representations. On the other hand, as far as the infinite dimensional irreducible representations are concerned a primitive ideal need not be the kernel of an irreducible representation induced by a point, that is , in our setting the family $\{P(\overline{x})|x \in Aper(\sigma)\}$ does not exhaust the primitive ideals of kernels of infinite dimensional irreducible representations of $A(\Sigma)$ unless X is metrizable. In fact if X is metrizable, for an infinite dimensional irreducible representation, $\tilde{\pi} = \pi \times u$, there exists a point x_0 in X_π with dense orbit because the induced dynamical system Σ_π is topologically transitive. Therefore, in this case the kernel of $\tilde{\pi}$ coincides with that of the irreducible representation arising from x_0 by (c) of Corollary 2.9.

In a broad context of transformation C*-algebras this problem had been greatly discussed as Effros-Hahn conjecture. In nonseparable case, that is, for a dynamical system in an arbitrary compact space the conjecture does not necessarily hold even in our simplest setting.

A counter example: Let σ_θ be an irrational rotation on the torus T with the Lebesgue measure μ. Denote by Γ the spectrum of $L^\infty(T,\mu)$, that is, $L^\infty(T,\mu) \simeq C(\Gamma)$. We have then a dynamical system $\tilde{\Sigma} = (\Gamma, \tilde{\sigma}_\theta)$, where $\tilde{\sigma}_\theta$ is the homeomorphism induced from the automorphism α of $L^\infty(T,\mu)$. Here the homeomorphism C*-algebra $A(\tilde{\Sigma})$ is the C*-crossed product of $L^\infty(T,\mu)$ with respect to α and the ergodicity of σ_θ implies the topological transitivity of $\tilde{\sigma}_\theta$. Now consider the irreducible representation of this homeomorphism C*-algebra through the standard covariant representation using multiplication of $L^\infty(T,\mu)$ and the translation unitary u on $L^2(T,\mu)$. Then by the projection Theorem 2.8 this is an isomorphism. Since in this dynamical system the closure of every orbit becomes a null set for μ , the trivial primitive ideal can not be realized as the one induced from a point of Γ. We refer §6 for those results used here.

We , however, still have the following

Proposition 4.1 *Every ideal of $A(\Sigma)$ is the intersection of those primitive ideals of $P(\overline{x}_\alpha)$ and $P(\overline{y}_\beta, \lambda)$ where x_α , y_β and λ are ranging over some sets of aperiodic points , periodic points and parameters from the torus, respectively.*

This fact has been mentioned already in [1](and [40, Proposition 4.5] without proof) . Since we do not impose any countability condition on the space X, the result is not so trivial and depends heavily on a particular structure of the images of infinite dimensional irreducible representations of $A(\Sigma)$ (crossed product structure) explained before together with the fact that any finite dimensional irreducible representation of $A(\Sigma)$ comes from a periodic point in X. Though the reference [1] is not easily available, the proof is found in [43].

As an immeadiate consequence, we have

Corollary 4.2 *Any maximal ideal of the homeomorphism C^*-algebra $A(\Sigma)$ has the form of primitive ideal induced by a point of X.*

Henceforth we mean by an ideal of $A(\Sigma)$ a closed ideal. Now we consider the classification of the ideals of $A(\Sigma)$. Let I be an ideal of $A(\Sigma)$, then the image $E(I)$ becomes an ideal of $C(X)$ (not necessarily closed) because of the module properties of the projection E. Hence, either it remains to be a proper ideal of $C(X)$ or coincides with $C(X)$. Thus, ideals of $A(\Sigma)$ are divided into the following three classes.

Definition 4.3 *Let I be an ideal of $A(\Sigma)$.*

(a) We call I well behaving if $E(I) \subset I$,

(b) Call I badly behaving if $E(I) = C(X)$,

(c) Call I a plain ideal if $E(I)$ is a proper ideal of $C(X)$ but not contained in I.

Note that in case of a well behaving ideal I the image $E(I)$ becomes necessarily a closed invariant ideal of $C(X)$.

As we mentioned in §2 , kernels of representations for which induced dynamical systems are topologically free (hence in particular the ideal $P(\overline{x})$ for an aperiodic point x) are typical examples of well behaving ideals. Actually, they become the kernel of the following elementary operation in general. Namely take an invariant closed set S. Then the map $\rho_S : f \to f|S$ and the automorphism α_S on $C(S)$ defined as $\alpha_S(f)(x) = f(\sigma^{-1}x)$ give rise to the representation $\tilde{\rho}_S = \rho \times u$ of $A(\Sigma)$, whose kernel becomes obviously a well behaving ideal. As a result, the ideal $Q(\overline{y})$ for a periodic point y is well behaving.

It is to be noticed here that if we restrict the system Σ to the orbit $O(y)$ then the twisted part of the crossed product disappears and the homeomorphism algebra on $O(y)$ is isomorphic to the algebra of all continuous functions on T taking the value in M_n , where n is the period of y (cf.[40, Proposition 3.5]).

On the other hand, an ideal $P(\overline{y}, \lambda)$ is an example of a badly behaving ideal. In fact, writing the period of y as n the element $\lambda - \delta^n$ belongs to the ideal, hence the constant function λ belongs to $E(P(\overline{y}, \lambda))$, which coincides with $C(X)$.

A plain ideal I in $A(\Sigma)$ is a mixture of these two kinds of ideals. Hence a simple example of a plain ideal is the intersection I of $P(\overline{x})$ and $P(\overline{y}, \lambda))$ where the orbit $O(y)$ of a periodic point y is not included in $\overline{O(x)}$. For in this case take a continuous function f which vanishes on $\overline{O(x)}$ and $f|O(y) = 1$ and an element b in $P(\overline{y}, \lambda)$ such that $E(b) = 1$. Then the element fb belongs to I ,but $E(fb) = f$ does not belong to $P(\overline{y}, \lambda)$.

Now while in the algebra $C(X)$ we have used usual notations of the hull of an ideal J and the kernel of a subset S of X as $h(J)$ and $k(S)$, we shall consider its noncommutative versions. Namely,

Definition 4.4 *(a) Let S be a closed invariant subset of X, then define the (noncommutative) kernel of S in $A(\Sigma)$ as*

$$Ker(S) = \{a \in A(\Sigma) | \quad a(n)|S = 0 \quad \text{for every } n\},$$

(b) Let I be an ideal of $A(\Sigma)$, then define the hull of I as

$$Hull(I) = \{x \in X| \quad a(n)(x) = 0 \quad \text{for all } a \text{ in } I \text{ and } n\}.$$

We have then,

Proposition 4.5 *(a) $Ker(S)$ is a well behaving ideal of $A(\Sigma)$ and it is a closed linear span of generalized polynomials over the functions of $k(S)$ written as $J(k(S))$. Hence, we have that $S = Hull(Ker(S))$.*

(b) $Hull(I)$ is a closed invariant subset of X, but $Ker(Hull(I))$ does not coincide with I in general.

Proof. In order to show the property of ideal for $Ker(S)$. take an element a of $Ker(S)$ and an arbitrary element b of $A(\Sigma)$. Then the generalized Cesàro mean of a, $\sigma_n(a)$ (clearly contained in $Ker(S)$), converges to a in norm, hence $b\sigma_n(a)$ and $\sigma_n(a)b$ converge to ba and ab, respectively. On the other hand, we see by definition that $E(Ker(S)) = k(S)$, which is apparantly contained in $Ker(S)$. Now since

$$b\sigma_n(a) = \sum_{-n}^{n}(1 - \frac{|j|}{n+1})ba(j)\delta^j \quad \text{where } a(j) \in k(S),$$

we have for any integer k

$$(b\sigma_n(a))(k) = E(b\sigma_n(a)\delta^{*k}) = \sum_{-n}^{n}(1 - \frac{|j|}{n+1})b(k-j)\alpha^{k-j}(a(j)).$$

Hence $b\sigma_n(a)$ belongs to $Ker(S)$, and similarly $\sigma_n(a)b$, too. Thus, both ba and ab belong to $Ker(S)$.

For the assertion (b), we first note that $a(n)(\sigma^{-1}x) = \alpha(a)(n)(x)$ and $\alpha(a)$ belongs to I as well as a. As for an example we take the primitive ideal $P(\overline{y}, \lambda)$, then $Ker(Hull(P(\overline{y}, \lambda))) = A(\Sigma)$.

Now we shall characterize a well behaving ideal in the following way. In the theorem the assertion (3) is suggested by A.Kishimoto.

Theorem 4.6 *The following assertions are equivalent for an ideal I of $A(\Sigma)$:*

(1) I is a well behaving ideal,

(2)

$$I = Ker(Hull(I)) = J(k(Hull(I)))$$

,that is, I is an intersection of all $P(\overline{x})$ and $Q(\overline{y})$ for x and y in $Hull(I)$,

(3) I is invariant by the dual action $\hat{\alpha}$,

(4) The quotient algebra $A(\Sigma)/I$ is canonically isomorphic to the C^*-crossed product $q(C(X)) \times_{\alpha_I} Z$ with respect to the induced automorphism α_I of $q(C(X))$ in such a way that

$$q \circ E(a) = E_I \circ q(a)$$

where q and E_I are the quotient homomorphism and the canonical projection in $q(C(X)) \times_{\alpha_I} Z$, respectively.

In particular, when the dynamical system is free, there is a one to one correspondence between the set of closed ideals of $A(\Sigma)$ and the set of closed invariant subsets of X.

Proof. Assume the assertion (1). Then one sided inclusion is clear for (2) and the other inclusion is obtained by using Cesàro mean. (One may of course refer here the old Zeller-Meier's result [45, Proposition 5.10] but we want to emphasize the important aspect of the crossed products by Z discussed in §3)).

The assertion (2) clearly implies (3) by the properties of dual actions, and the assertion (3) leads to (1) by the definition of the projection E.

The assertion (1)⇒(4).Define the map ε_I by $\varepsilon_I(q(a)) = q(E(a))$. Then by the assumption, this map is well defined and one may easily verify that it is a projection of norm one from $A(\Sigma)/I$ to $q(C(X))$ satisfying the relation

$$\varepsilon_I \circ q = q \circ E.$$

. Now since the quotient algebra $A(\Sigma)/I$ is generated by $q(C(X))$ and $q(\delta)$, there exists a homomorphism Φ from the crossed product $q(C(X)) \times_{\alpha_I} Z$ to $A(\Sigma)/I$ such that $\Phi(\delta_I) = q(\delta)$ where δ_I stands for the generating unitary of the crossed product. Moreover, the above property of the projection ε_I implies the relation,

$$\varepsilon_I \circ \Phi = \Phi \circ E_I.$$

. Here E_I is the faithful canonical projection of the crossed product $q(C(X)) \times_{\alpha_I} Z$ and Φ is naturally faithful on $q(C(X))$. It follows that Φ is an isomorphism.

The assertion (4)⇒(2) is easily seen once we refer the elementary homomorphism ρ from $A(\Sigma)$ to $A(\Sigma_{S_I})$ mentioned above , denoting the kernel of q on $C(X)$ by $k(S_I)$ for an invariant closed subset S_I of X.

The statement of the second half is clear because in this case , by Theorem 2.6, every ideal of $A(\Sigma)$ becomes well behaving.

This completes all proofs.

We notice that this gives another background for the classical equivalence between simplicity of $A(\Sigma)$ and minimality of the dynamical system Σ.

Remark. Actually all the equivalences except the assertion (2) are valid for an arbitrary crossed product $A \times_\alpha Z$, but we are interested in those properties only from the point of view of their relationships to the dynamical system Σ.

Recall that a unital C*-algebra A always contains the largest ideal K of type 1 for which the quotient algebra A/K has no type 1 portion (called an antiliminal C*-algebra). C*-algebras of type 1 ,or postliminal C*-algebras are the most tractable class among C*-algebras. In C*-theory we are used to regard commutative C*-algebras as the starting class having only one dimensional irreducible representations. Then comes the class of n-homogeneous C*-algebras defined as the ones having only irreducible representations of the fixed dimension n as in the case of the matrix algebra M_n. Roughly speaking, an algebra of type 1 is an infinite piling of n-homogeneous C*-algebras passing through liminal C*-algebaras (a liminal C*-algebra is defined as an algebra every image of whose irreducible representation consists of compact operators).

It is also known that A contains the largest liminal ideal L. It is defined as the ideal for which for any irreducible representation of A images of all elements are compact operators. Write these ideals of $A(\Sigma)$ by $K(\sigma)$ and $L(\sigma)$. Note that condition (3) of the theorem implies both ideals $K(\sigma)$ and $L(\sigma)$ are good examples of well behaving ideals. We shall give later their characterizations in terms of elementary sets of Σ.

Now as described above, all troubles of ideals stem from the presence of periodic points. Thus take an ideal I which is the intersection of a family of primitive ideals $\{P(\overline{y_\alpha}, \lambda_\beta)\}$. In this case we may write I as an intersection of the family $\{P_\alpha\}$, where P_{α_1} and P_{α_2} are associated to different periodic points y_1 and y_2. We have then the following

Lemma 4.7 *If the above ideal I becomes a well behaving ideal, then the intersection of $\{P_\alpha\}$ coincides with the intersection of the family, $\{Q(\overline{y_\alpha})\}$.*

Proof. Let $S = Hull(I)$, then $I = Ker(S)$. Now suppose there exist an orbit $O(y_{\alpha_0})$ which is not contained in S, that is , disjoint from S. We have then a function f vanishing on S and having the value 1 on $O(y_{\alpha_0})$. This is however a contradiction. Hence every orbit $O(y_\alpha)$ is contained in S and I is contained in $Q(y_\alpha)$ for every α.

Thus we have the conclusion.

On the other hand, if I is badly behaving every ideal P_α does not coincide with $Q(\overline{y}_\alpha)$.

The reason that we might not obtain an exclusive description of a plain ideal however seems to stem from the following situation. Namely, we have still an example of a topologically free dynamical system in which there exists a countable set $\{y_n\}$ of periodic points without isolated points and ideals $\{P_n\}$ with $P_n \supsetneqq Q(\bar{y}_n)$ but never-the-less we have

$$\bigcap_{n=1}^{\infty} P_n = \bigcap_{n=1}^{\infty} Q(\overline{y_n}).$$

([43, p.12]). For such an example one may use a rational rotation making use of Proposition 2.6, but this dynamical system seems too restrictive to use for our example. Thus in [43] we have used the dynamical system on the two dimensional tori T^2 for the toral automorphism defined by the matrix $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Since the system arises from the group C*-algebra of 3-dimensional discrete Heisenberg group (cf.[38, §6]), we denote this system as $\Sigma_H = (T^2, \sigma_H)$. By definition, this dynamical system is a mixture of rational and irrational rotations according to the first axis. Hence this is topologically free but not topologically transitive. We note also that the set of periodic points is also dense in T^2.

Now contrary to the above case we have

Proposition 4.8 *Let $\{P_1, P_2, \ldots, P_n\}$ be the ideals associated with the set of periodic points,$\{y_1, y_2, \ldots, y_n\}$ whose orbits are disjoint each other.*

Suppose that $P_i \supsetneqq Q(\overline{y_i})$ for every i, then the intersection P of those P_i's is a badly behaving ideal.

Proof. We assert first that each P_i is badly behaving. For if $E(P_i)$ were a proper ideal of $C(X)$ we could write $\overline{E(P_i)} = k(S)$ for some nonempty invariant closed set S. Then $S \subset O(y_i)$, and $S = O(y_i)$, contradicting to the strict inclusion for a pair $\{P_i, Q(\overline{y_i})\}$. Next assume $E(P)$ were a proper ideal of $C(X)$, and write $\overline{E(P)} = k(S)$. Then similarly as above S is contained in the union of orbits $O(y_i)$.

Hence it could contain an orbit $O(y_{i_0})$. Take a continuous function f such that $f|O(y_{i_0}) = 1$ and vanishes on other orbits. Moreover, choose an element b of P_{i_0} with $E(b) = 1$. The element fb belongs then to P but

$$E(fb)|O(y_{i_0}) = f|O(y_{i_0}) = 1,$$

a contradiction. Thus $E(P) = C(X)$ and P is a badly behaving ideal.

Next let I be a plain ideal and write $\overline{E(I)} = k(S_0)$ for an invariant (nonempty) closed set S_0 of X. Let $I_0 = Ker(S_0)$, then we can write as

$$I = I_0 \cap \left(\bigcap_{\alpha} P_\alpha \right),$$

where $\{P_\alpha\}$ are associated to a set of periodic points $\{y_\alpha\}$ as stated before. For this family we may assume that every orbit $O(y_\alpha)$ is disjoint from S_0 and P_α contains strictly $Q(\overline{y_\alpha})$. Write I_1 their intersection.

We say this expression $I = I_0 \cap I_1$ a standard decomposition for a plain ideal I. In this decomposition it would be most desirable if we could conclude that I_1 is a badly behaving ideal. In some cases, this is actually true as in the following result. Let S_1 be the closure of the union of all periodic orbits related to those ideals P_α.

Proposition 4.9 *Keep the above notations. Then if S_0 is disjoint from S_1. the ideal I_1 becomes a badly behaving ideal.*

The converse does not necessarily hold.

Proof. Suppose $E(I_1)$ is a proper ideal of $C(X)$. We can write then as $\overline{E(I_1)} = k(S)$ for some nonempty closed invariant set S. It follows that $S \subset S_1$. On the other hand, since $I \subset I_1$ we have the relation, $E(I) \subset E(I_1)$, and S is also contained in S_0, a contradiction.

As a counter example for the converse we consider the situation in $Per_n(\sigma)$ where a sequence $\{y_n\}$ of periodic points converges to a point y_0 with the ideals $\{P(\overline{y_n}, \lambda)\}$. Then the ideal

$$I = Q(\overline{y_0}) \cap I_1 \quad \text{for} \quad I_1 = \textstyle\bigcap_n P(\overline{y_n}, \lambda)$$

is a plain ideal and I_1 is a badly behaving ideal because by assumption the element $\lambda - \delta^n$ belongs to every ideal $P(\overline{y_n}, \lambda)$. But the intersection of S_0 and S_1 is $O(y_0)$.

Unfortunately, more pathological phenomena may happen for the above type of ideal $I_1 = \bigcap_\alpha P_\alpha$. Namely there is a case where I_1 becomes again a plain ideal. For a further example, consider the same situation as above in which $y_0 \neq y_n$ for any n. Put

$$P_n = \bigcap_{n,\lambda} P(\overline{y_n}, \lambda) \quad \text{where } \lambda \in [0, 1 - 1/n].$$

We have then the strict inclusion, $Q(\overline{y_n}) \subsetneqq P_n$. Now by Proposition 2.6 for every parameter μ the pure state $\varphi(y_0, \mu)$ is approximated in the w*-topology by the family

$$\{\varphi(y_n, \lambda) | \quad \lambda \in [0, 1 - 1/n]\}.$$

We assert here the ideal $I_1 = \bigcap_n P_n$ is a plain ideal. Indeed, from the above argument we see that $P(\overline{y_0}, \mu) \supset I_1$ and then $Q(\overline{y_0}) \supset I_1$. Thus

$$E(I_1) \subset E(Q(\overline{y_0})) = k(O(y_0)) \subsetneq C(X),$$

whereas $E(I_1)$ is not included in I_1. In fact, take a function f vanishing on all orbits $O(y_n)$ except $n \neq n_0$ for an integer n_0 and $f|O(y_{n_0}) = 1$. Choose an element b_0 in P_{n_0} such that $E(b_0) = 1$(cf.Proposition 4.8). Then the element fb_0 belongs to I_1 but $E(fb_0) = f$ does not belong to P_{n_0} nor to I_1.

Now consider two homeomorphism C*-algebras $A(\Sigma_1)$ and $A(\Sigma_2)$ which are isomorphic each other. We then usually assume that their structures of ideals are the same , but the above discussions for ideals of $A(\Sigma)$ suggest us that we should not follow this way of thinking once we are concerned with the structure of $A(\Sigma)$ in connection with dynamical systems. In fact, for an isomorphism between those algbras we are not assured that it preserves the class of those ideals discussed above. It preserves naturally kernels of infinite dimensional irreducible representations, but there is a case where an isomorphism does not keep the ideals of the form $Q(\overline{y})$ for periodic points.

We shall come back to this problem in §12.

5 Representations of dynamical systems and homeomorphism C*-algebras

When we treat a dynamical system Σ we are used to specify that dynamical system into a class, such as topologically transitive, minimal etc. The corresponding C*-algebra $A(\Sigma)$ has however its representation theory, and to consider a representation with the kernel I does not simply mean only the quotient algebra $A(\Sigma)/I$ but means the whole structure of the representation, the image, its action on the represented Hilbert space etc.

Thus,for a repesentation $\tilde{\pi} = \pi \times u$, we regard the dynamical system $\Sigma_\pi = (X_\pi, \sigma_\pi)$ a repesentation of the dynamical system Σ embedded into $\tilde{\pi}(A(\Sigma))$. This means to consider the system Σ_π not only as a restricted corner of the system Σ but as a system associated with the C*-algebra $\tilde{\pi}(A(\Sigma)$ together with the representation space H.

Based on the article [42] we shall discuss in this section this stand point of view: a subshift is a representation of the full shift.

It is to be noticed here that in case of measurable dynamical systems we need not think over this kind of problem.

In the following, we say a representation Σ_π finite or infinite if the space X_π is a finite set or an infinite set respectively.

We also use the terminologies of minimal, topologically transitive and topologically free representations etc if the dynamical system $\Sigma_\pi = (X_\pi, \sigma_\pi)$ is minimal, topologically transitive and topologically free etc. respectively.

Furthermore, we say a representation $\tilde{\pi} = \pi \times u$ of the C*-algebra $A(\Sigma)$ simple, prime and primitive if the image $\tilde{\pi}(A(\Sigma))$ is simple, prime and primitive respectively.

We shall discuss relations between two categories of representations. This means that we have to rereinforce our previous results along this line. A main diference between this kind of arguments and previous ones is at the point that we may not assume apriori the represented C*-algebras to have crossed product structure.

We start first to consider a finite representation of the system $\Sigma = (X, \sigma)$. The relation will be seen from (b) of the remark after Proposition 2.4. Namely we have the following observation.

Proposition 5.1 *The representation Σ_π of Σ is finite if and only if the center of the image $\tilde{\pi}(A(\Sigma))$ is finite dimensional.*

As we already noticed, a finite representation need not imply the finite dimensionality of the image of $A(\Sigma)$.

Next we discuss a considerably wide class of representations of dynamical systems, topologically free representations. The most basic result in this direction is Theorem 2.6.

Let x be a periodic point of x with the point evaluation μ_x. Then the pure state extensions of μ_x to $A(\Sigma)$ can be parametrized by the torus. Let $\varphi(x, \lambda)$ be such an extension. We say that the image $\tilde{\pi}(A(\Sigma))$ absorbes the extension $\varphi(x, \lambda)$ if x is a periodic point of X_π and there exists a pure state extension φ' of μ_x to $\tilde{\pi}(A(\Sigma))$ such that

$$\varphi(x, \lambda) = \varphi' \circ \tilde{\pi}.$$

For instance, in case of the finite dimensional irreducible representation $\tilde{\pi}(x, \lambda_0)$, the image absorbes only the pure state extension $\varphi(x, \lambda_0)$.

The following theorem is a reformulation of Theorem 2.8 in the representatioins of dynamical systems.

Theorem 5.2 *If Σ_π is a topologically free representation of Σ, then the C^*-algebra $\tilde{\pi}(A(\Sigma))$ satisfies the following two conditions;*

(a) Any closed ideal I of $\tilde{\pi}(A(\Sigma))$ is non-zero if and only if the intersection $I \cap \pi(C(X))$ is non-zero,

(b) The algebra $\pi(C(X))$ is a maximal abelian C^*-subalgebra.

The converse holds if $\tilde{\pi}(A(\Sigma))$ satisfies one of the above two conditions and moreover if it absorbs every pure state extension of any periodic point of X_π.

As we can see from finite dimensional irreducible representations, the assumption for the converse assertions can not be dropped.

Next are the relations between simple and prime representations with minimal and topologically transitive representations, respectively. We note again that here we may not assume the image of $A(\Sigma)$ to have the crossed product form.

Theorem 5.3 *(a) If $\tilde{\pi}$ is a simple representation, Σ_π is a minimal representation of Σ. Conversely, an infinite minimal representation of Σ is a simple representation of $A(\Sigma)$.*

(b) If $\tilde{\pi}$ is a prime representation, then Σ_π becomes a topologically transitive representation. Conversely, if Σ_π is an infinite topologically transitive representation, $\tilde{\pi}$ is a prime representation.

. As of now, no characterization of primitive representations are obtained. One might claim here that since primitivity of a C*-algebra is not algebraic but concerns with the action out of the C*-algebra we could not find the characterization of the primitive representation of a topological dynamical system. When we talk about representations of dynamical systems, however, we do not mean merely the restriction Σ_π of the original dynamical system Σ but we also consider its embedding into the C*-frame $\tilde{\pi}(A(\Sigma))$. For instance, assume $X = O(x)$ for a periodic point x. It is then minimal and has no subdynamical system. Although we have not given the precise definition of the equivalency for representations of dynamical system yet it has however embeddings(representations) of continuously many different aspects as sitting inside in irreducible representations of $A(\Sigma)$ associated to the point x. Moreover, if we ask the crossed product structure for such embedding the resulting C*-algebra is even not necessarily prime.

On the other hand, if a C*-algebra A is separable an old result of Dixmier asserts that A is primitive if and only if it is prime. This result exactly corresponds to the fact that in case of dynamical systems in a metrizable compact space the system is topologically transitive if and only if it has a dense orbit. For, in this case the irreducible representation induced by the point with dense orbit is faithful and the algebra becomes primitive.

The non-separable version of Dixmier's result has been remained open, whereas we know that for dynamical systems two notions of topologiacal transitivity and dense orbit property are different in general as we have already mentioned in §4. Thus, if we could have the difference between primeness and primitivity of C*-algebras in general the difference should bring some reflection to representations of dynamical systems.

6 Topological realizations of measurable dynamical systems

Let T be a non-singular ergodic isomorphism in a σ-finite measure space (X, ν). When the map T is measure preserving there is a topological realization theorem known as the Jewett-Krieger theorem. It says for a probability measure space that any such system has a uniquely ergodic topological realization (Y, σ_T, μ). Namely, σ_T is a μ-invariant ergodic homeomorphism in a compact metric space Y and the topological dynamical system (Y, σ_T, μ) is measurably conjugate to the given system (X, T, ν). This result has been remarkably improved recently by N.Ormes [28] by means of minimal homeomorphisms on the Cantor set so as to generalize both Dye's theorem and the Jewett-Krieger theorem.

In this section however we propose a completely different way of topological realization of measurable dynamical systems for non-singular measurable isomorphisms. Some people might feel that our method is too different nature from their standard way of thinking so that it could not bring any substancial contribution. The author believes however that the way may lead us another fruitful aspects in the investigation of the interplay between measurable dynamics and topological dynamics, although we can include here only some introductry results because of the author's circumstances.

Let (X, T, μ) be a measurable dynamical system where T is a (not necessarily measure preserving) non-singular measurable isomorphism on the σ-finite measure space (X, μ). Denote by Γ the spectrum of the algebra of bounded measurable functions, $L^\infty(X, \mu)$, which turns out to be a hyperstonean space.

We recall here the definition of a hyperstonean space. A compact space X is called a stonean space if its algebra of real continuous functions forms a complete lattice. A probability measure μ on a stonean space is then said to be normal if $\mu(sup f_\alpha) = sup \mu(f_\alpha)$ for every increasing net $\{f_\alpha\}$ of real continuous functions. A stonean space is then called a hyperstonean space if the union of all supports of normal measures is dense in the space. In the case of the space $L^\infty(X, \mu)$ the measure μ transfered from X to Γ becomes a faithful normal measure. For standard results about stonean and hyperstonean spaces we refer the readers [36, section 3.1].

Now the map T induces an automorphism (write also as α) of the algebra $C(\Gamma)$, therefore we have a homeomorphism σ_T in Γ corresponding to α. Conversely if we have a topological dynamical system (Γ, σ), it gives rise to an automorphism α in $L^\infty(X, T, \mu)$ and by von Neumann's theorem we can find a non-singular transformation associated to α provided that the measure space is reasonable enough such as the Lebesgue space. Since our theory does not primarily assume metrizability of the underlying space X, we can apply most of our results to this kind of topological dynamical systems.

When the starting system arises from a topological dynamical system $\Sigma = (X, \sigma)$ and σ becomes a non-singular transformation with respect to a suitabale measure μ on X, we denote the induced dynamical system as $\tilde{\Sigma} = (\Gamma, \tilde{\sigma})$.

The simplest example of this kind of system is the one coming from the shift s on the integer group Z. We see then $\Gamma = \beta Z$(the Stone-Čech compactification of Z) and the extended homeomorphism $\tilde{s}$ on βZ is known to be free(cf.[37, p.85]).

The following is a dictionary for this transplant of measurable dynamical systems, but before the theorem we recall some definitions.

At first, we call a point ω in Γ a nonwandering point for σ_T if for any neighborhood U of ω there exists a positive integer n such that $\sigma_T^n(U) \cap U \neq \varphi$. The set of all nonwandering points of σ_T is denoted as $\Omega(\sigma_T)$. In case of a usual topological dynamical system, $\Sigma = (X, \sigma)$ we are discussing, we use the same notation $\Omega(\sigma)$ for the set of nonwandering points in X.

On the other hand, a non-singular transformation T in a σ-finite measure space (X, μ) is said to be dissipative if there exists a wandering set W for T such that $X = \bigcup_{n \in Z} T^n W$. T is said to be recurrent if there exist no wandering sets of positive measure.

It is said to be aperiodic if the set of periodic points is a null set. The map T is said to be free if there exist no absolutely invariant sets of positive measure, or if the corresponding automorphism α in $L^\infty(X, \mu)$ is free, that is, the relation

$$ba = a\alpha(b) \quad \text{for every } b \text{ in } L^\infty(X, \mu)$$

implies $a = 0$. It is then known that T is aperiodic if and only if it is free.

Theorem 6.1 *Let (Γ, σ_T) be the topological dynamical system corresponding the measurable dynamical system (X, T, μ).*

(a) T is dissipative if and only if the set of wandering points of σ_T is dense in Γ'

(b) T is recurrent if and only if $\Omega(\sigma_T) = \Gamma$,

(c) T is aperiodic if and only if σ_T is topologically free,

(d) T is ergodic if and only if σ_T is topologically transitive.

Moreover, if (X, μ) is the Lebesgue space the closure of every orbit of σ_T in Γ becomes a null set for the measure μ.

Thus when T is an ergodic transformation in the Lebesgue space the topological dynamical system (Γ, σ_T) turns out to be a topologically transitive system in which there is no point with dense orbit as mentioned before in §5. This system is far from being uniquely ergodic because there appear so many singular invariant measures. Moreover the aspects here are so different from standard aspects in ergodic theory. However,if we once admit periodic points for a given topological dynamical system it means that we can no more look for the unicity of invariant measures because we may have many atomic invariant measures arising from periodic points

Proof. Proofs of the assertions (c) and (d) as well as the last assertion are found in [40, Proposition 1.2]. Hence we shall give here the proofs of (a) and (b).

For a measurable set E in X with positive measure, we denote by $\tilde{E}$ the corresponding clopen set in Γ through the characteristic function χ_E of E. Now the only if part of the assertion (a) is trivial, and we shall show the converse. Take a wandering point ω in Γ, then there exists a neighborhood U of ω such that $\sigma_T^n(U) \cap U = \phi$ for all positive integers. Therefore we have the same relation for all negative integers. It follows that the family $\{\sigma_T^n(U) | \quad n \in Z\}$ is a disjoint family. Now consider the set $\mathcal{M}$ of disjoint such families, which is apparantry inductive by natural order of inclusions. Let $\mathcal{M}_0$ be a maximal set in $\mathcal{M}$ and choose one open set from from each family in $\mathcal{M}_0$. Write the union of those representatiove open sets as $\tilde{W}$. Then by definition $\{\sigma_T^n(\tilde{W})\}$ is a disjoint family. Now if the union of these sets is not dense in Γ there exists a wandering point ω_0 in the compliment and we can find another family $\{\sigma_T^n(V)\}$ of disjoint sets for a open neighboirhood V of ω_0 such that their union is again in the compliments. This contradicts however the maximality of $\mathcal{M}_0$. Therefore we have

$$\Gamma = \overline{\bigcup_{n \in Z} \sigma_T^n(\tilde{W})},$$

and $X = \bigcup_{n \in Z} T^n(W)$ where W is a wandering set corresponding to $\tilde{W}$.

Next assume T is not a recurrent map,then we have a wandering set W of positive measure. It follows that the set of wandering points in Γ has non-empty interior and $\Omega(\sigma_T)$ is strictly contained in Γ.

Therefore if the nonwandering set of σ_T exhausts Γ, T must be a recurrent map. Conversely, if $\Omega(\sigma_T) \subsetneqq \Gamma$ then as we mentioned above we can find a wandering set of positive measure in X. Hence T is not a recurrent map. This completes the proof.

Now consider the decomposition,

$$\Gamma = \Omega(\sigma_T) \cup (\Gamma \setminus \Omega(\sigma_T)) .$$

The closure $\overline{\Gamma \setminus \Omega(\sigma_T)}$ becomes then a clopen set as the closure of an open set. Let p be the characteristic function of this set, which turns out to be a projection in $L^\infty(X, \mu)$. The projection $1 - p$ is the characteristic function of the interior of $\Omega(\sigma_T)$ (also becomes a clopen set of Γ). Let ΠE_1 and E_2 be the corresponding measurable sets of X. The decomposition ,

$$X = E_1 \cup E_2$$

is then nothing but the standard decomposition of the map T on X such that $T|E_1$ is dissipative and $T|E_2$ is recurrent.

Note that since in this case topological transitivity of the system (Γ, σ_T) implies the identity, $\Omega(\sigma_T) = \Gamma$, we see another way to show the assertion:

"An ergodic transformation is necessarily recurrent".

Let $\Sigma = (T, \sigma_\theta)$ be the dynamical system of irrational rotation θ and let $\tilde{\Sigma} = (\Gamma, \tilde{\sigma_\theta})$ be the corresponding dynamical system. Then since the C*-algebra $A(\tilde{\Sigma})$ contains the simple algebra $A(\Sigma) = A_\theta$, so-called an irrational rotation C*-algebra, it can not have finite dimensional irreducible representations. Therefore the dynamical system $\tilde{\Sigma}$ is free. We shall see in the next section that this algebra is antiliminal,i.e. $K(\tilde{\sigma}) = 0$.

In order to illustrate our method we shall show the proof of Rohlin's tower theorem from our context in the following way.

Theorem 6.2 *Let T be an aperiodic transformation in the Lebesgue probability measure space (X, μ). Then for any positive integer p and ε there exists a mesurable set E for which $\{E, TE, \ldots, T^{p-1}E\}$ are disjoint each other and*

$$\mu(X \setminus \bigcup_{j=0}^{p-1} T^j E) < \varepsilon.$$

Proof. From the above theorem, the dynamical system (Γ, σ_T) becomes topologically free. Hence the set $Per(\sigma_T)$ is a union of closed sets $Per^n(\sigma_T)$ without interior points, and it becomes a null set ([40, Proposition 1.11]). Take a clopen set F contaning $Per(\sigma_T)$ such that $\mu(F) < \varepsilon$. We can choose an clopen set U in $\Gamma \setminus F$ such that the p-tuple

$$\{U, \sigma_T(U), \ldots, \sigma_T^{p-1}(U)\}$$

consists of disjoint sets. Let $\mathcal{M}$ be the disjoint family of all such sets, which turns out to be inductive by natural ordering of inclusions. Hence take a maximal family $\mathcal{M}_0$ and consider the union E' of those clopen sets choosing one set from each family of sets in $\mathcal{M}_0$. Let $\tilde{E}$ be the closure of E', then as the closure of an open set it becomes a clopen set and $\mu(\tilde{E}) = \mu(E')$. Moreover, since in Γ any nonempty clopen set should have positive measure for μ those sets $\{\tilde{E}, \sigma_T(\tilde{E}), \ldots, \sigma_T^{p-1}\}$ are also mutually disjoint each other. Therefore, by the maximality of $\mathcal{M}_0$ we have

$$\Gamma \setminus F = \tilde{E} \cup \sigma_T(\tilde{E}) \cup \ldots \cup \sigma_T^{p-1}(\tilde{E}).$$

Thus, transplanting this situation to X we obtain the set E having the required property and

$$\mu(X \setminus \bigcup_{j=0}^{p-1} T^j(E)) < \varepsilon.$$

This completes the proof.

We remark that in the above proof we simply use aperiodicity of points and need not use the return map as in the case of the standard proof in Ergodic theory. Thus, we need not consider the difference between dissipative maps and recurrent maps.

7 The set of recurrent points and type 1 portions of homeomorphism C*-algebras

For an arbitrary compact Hausdorff space X, we call a point x recurrent if there exists a (nontrivial) net $\{\sigma^{n_\alpha}\}$ in $O(x)$ (subsequence if X is metrizable) converging to x. Denote the set of all (generalized) recurrent points by $c(\sigma)$, whose closure is known to be as the Birkhoff center or simply called as the center. When X is metrizable it is known that the set $c(\sigma)$ is always nonempty. That this is true for any compact space is proved in the following algebraic way.

Proposition 7.1 *For any dynamical system (X, σ) where X is a compact Hausdorff space, the set $c(\sigma)$ is not empty.*

Proof. Take a maximal ideal M of $A(\Sigma)$, then the quotient algebra $A(\Sigma)/M$ is simple. By theorem 5.3 the associated dynamical system becomes minimal. Therefore whatever the quotient algebra is finite dimensional or not we have periodic points or nontrivial recurrent points in Σ.

Henceforth by a nontrivial recurrent point we mean a recurrent point which is not periodic.

Now there are many literatures to show when the algebra $A(\Sigma)$ becomes an algebra of type 1 in the broad context of transformation group C*-algebras. Those results are however formulated towards the theory of operator algebras and not for dynamical systems themselves. Thus even for our simplest dynamical systems (single homeomorphism on a compact space) it is often hard to see whether or not a given dynamical system yields the algebra $A(\Sigma)$ of type 1.

In this section, particularly when the space X is metrizable, we shall exactly determine the size of the largest ideal of type 1 in $A(\Sigma)$, $K(\sigma)$, from which we can easily see when the algebra is of type one. The author does not know whether or not the same result holds for dynamical systems in an arbitrary compact space.

Let $\tilde{\pi} = \pi \times u$ be an ireducible representation of $A(\Sigma)$ on a Hilbert space H. In this case the dynamical system $\Sigma_\pi = (X_\pi, \sigma_\pi)$ induced by this representation becomes topologically transitive. We shall determine the case when the image contains the algebra of compact operators, $C(H)$.

Proposition 7.2 *The image $\tilde{\pi}(A(\Sigma))$ contains the algebra of compact operators, $C(H)$, if and only if there exists a point x not belonging to the set $c(\sigma) \setminus Per(\sigma)$ with dense orbit in the space X_π.*

This point x becomes necessarily an isolated point of X_π.

The proof is a modification of [4, Proposition 1], where X is assumed to be metrizable.

Proof. We first note that in case of an irreducible representation of $A(\Sigma)$ it is finite dimensional if and only if X_π is a finite set, though the if part is somewhat nontrivial.. Thus for the proof we may assume that X_π is an infinite set.

($\Rightarrow$). From the above assumption, the dynamical system Σ_π is topologically free and by Corollary 2.9 (c) and Theorem 2.10 the intersection of $\pi(C(X))$ and $C(H)$ contains a nonzero selfadjoint element $\pi(f)$. Take a spectral projection p of $\pi(f)$ which is naturally contained in $\tilde{\pi}(A(\Sigma))$. It follows by Theorem 2.10 (3) together with the property of spectral projections that p belongs to $\pi(C(X))$. Let S be the support of p in X_π, then it is an open and closed set .

Moreover as p is finite dimensional S must consist of isolated points of finite number. Now take a point x in S, then the orbit $O(x)$ is an invariant open (infinite) set. Hence it has to be dense because Σ_π is topologically transitive and since x is an isolated point it can not be a recurrent point.

($\Leftarrow$).Let x be the point in X_π with dense orbit which is not a recurrent point. We assert that the set $X_\pi \setminus O(x)$ is a closed set. In fact, let $\{y_\alpha\}$ be a net in the above set converging to y. If y belong to $O(x)$,say $y = \sigma^k(x)$ it becomes a recurrent point because each y_α is in the closure of $O(x)$, a contradiction. Hence y must also belong to $X_\pi \setminus O(x)$. It follows that there exists a continuous function f such that

$$f(x) = 1 \quad \text{and} \quad f|X_\pi \setminus O(x) = 0.$$

Let

$$S = \{y | f(y) > 1/2\}$$

, whose closure is obviously contained in $O(x)$. Now if S is an infinite set it must have an accumulation point in $O(x)$ and x become a recurrent point, a contradiction. Hence S has to be a finite set and then it becomes an open and closed set consisting of isolated points. Let p be the characteristic function of $\{x\}$. We have then for any a in $\tilde{\pi}(A(\Sigma))$ and g in $\pi(C(X))$,

$$papg = g(x)pap = gpap.$$

Therefore, pap belongs to $\pi(C(X))$ and

$$pap = \lambda p \quad \text{for some scalar } \lambda.$$

This shows p is a minimal projection of $\tilde{\pi}(A(\Sigma))$. Now since this image is strongly dence in $B(H)$ it becomes a minimal projection of $B(H)$, and it is one dimensional. It follows that the image of $A(\Sigma)$ contains $C(H)$ because of the irreducibility of $\tilde{\pi}$. This completes the proof.

In the above proof, a main trouble for non-metrizable case is at the point that for an irreducible representation $\tilde{\pi}$ we can not assume apriori the existence of a point with dense orbit (whereas in case X being metrizable it is the consequence of the equivalency between topological transitivity and dense orbit property). Thus the author does not know if the equality in the next theorem always holds or not.

Recall that the ideal $P(\bar{x})$ is the kernel of the irreducible representation arised from an aperiodic point x. We see then from Theorem 2.2 that $X_\pi = \overline{O(x)}$. Therefore, if x is a nontrivial recurrent point (i.e not isolated point in X_π) the image $\tilde{\pi}_x(A(\Sigma))$ can not contain the algebra of compact operators by the above Proposition. Thus, we have the following

Theorem 7.3

$$K(\sigma) \subseteq \bigcap P(\bar{x}) \quad x \in c(\sigma) \backslash Per(\sigma).$$

The equality holds when X is metrizable.

In fact, when X is metrizable we can show that the ideal, $\bigcap P(\bar{x})$, is of type 1.

The result exactly shows how $A(\Sigma)$ differs from being of type 1 in terms of dynamical systems. Thus we naturally obtain the following immediate consequences, which show the meaning of the difference $c(\sigma)\backslash Per(\sigma)$ in C^*-algebra theory.

Theorem 7.4 *Let Σ be a topological dynamical system in a compact Hausdorff space X.*

(a) If $A(\Sigma)$ is of type 1 then $c(\sigma) = Per(\sigma)$.

(b) If the set $c(\sigma)\backslash Per(\sigma)$ is dense in X, $A(\Sigma)$ becomes an antiliminal C^*-algebra.

The converse of both assertions hold also if X is metrizable.

. The so-called Morse-Smale dynamical systems are examples of the systems of type 1 . In fact in this case the set of non-wandering points, $\Omega(\sigma)$, is a finite set and naturally consists of only periodic points. It is known that most differential dynamical systems on the circle fall into this class. Another interesting example is the case of a Möbius homeomorphism on the extended compact plain $\overline{C}$, that is,

$$\sigma(z) = \frac{a + cz}{b + dz} \quad \text{where } ad - bc \neq 0 \text{ and } d \neq 0.$$

It is then known that most of the cases this dynamical system $\{\overline{C}, \sigma\}$ satisfies the condition, $c(\sigma) = Per(\sigma)$ (see [34] for applications of this observation).

On the other hand, a topologically transitive system on a metric compact space has always a point with dense orbit, whence if it is non-trivial (if the point is not an isolated point) the set of recurrent points is dense in X. Thus we have abundence examples of the above case (b) such as the case of Bernoulli shifts.

Now as we mentioned before, many dificulties about the structure of $A(\Sigma)$ appear when there exist infinite dimensional irreducible representations not arising from aperiodic points. In this aspect the problem had been considerably discussed in leterature such as Effros [14]. The results might have been satisfactory ones to the side of C*-algebras, but not so much from the side of dynamical systems.Thus we have to modify the old arguments in [14] to show exactly the obstruction in our dynamical systems. Discussions from this point of view are also found in [26, Chap.9] but under the assumption of free action.

Proposition 7.5 *Let X be a compact metric space, then for any point x in $c(\sigma)\backslash Per(\sigma)$ there exists a non-atomic quasi-invariant ergodic measure μ_x supported on the set $\overline{O(x)}$. The irreducible representation of $A(\Sigma)$ on the space $L^2(X, \mu_x)$ induced by μ_x is not unitarily equivalent to the ones induced from the points of X.*

Note that since the above μ_x is nonatomic it is not concentrated on the orbit $O(x)$.

Proof. We may assume the orbit $O(x)$ is dense in X. Moreover, it suffices to show the existence of a nonatomic ergodic measure μ because one may easily obtain a quasi-invariant measure with the same properties from μ.

For a nonnegative integer n, consider an n-tuple $(i_1, i_2, \ldots, i_n)$ where i_k's are 0 or 1. We shall show that for each n there exists a family of open sets $\{P(i_1, i_2, \ldots, i_n)\}$ and a power of σ, $g(n) = \sigma^{k_n}$ satisfying the following conditions;

$(a)_n$ $x \in P(0.0.\ldots, 0_n)$,

$(b)_n$ if

$$(i_1, i_2, \ldots, i_n) \neq (j_1, j_2, \ldots), \text{then} \quad P(i_1, i_2, \ldots, i_n) \cap P(j_1, j_2, \ldots, j_n) = \phi,$$

$(c)_n$ the closure of $P(i_1, i_2, \ldots, i_n)$ is included in $P(i_1, i_2, \ldots, i_{n-1})$ where $n \geq 1$,

$(d)_n$ diameter of $P(i_1, i_2, \ldots, i_n)$ is less than $1/n$,

$(e)_n$

$$g(k)P(0, \ldots, 0_k, i_{k+1}, \ldots, i_n) = P(0, \ldots, 0_{k-1}, 1_k, i_{k+1}, \ldots, i_n) \quad (n \geq 1).$$

We first put,

$$g(0) = \sigma^0 = id, \quad \text{and} \quad P(\phi) = X,$$

then they satisfy the conditions, $(a)_0 - (e)_0$. Now suppose we have defined the sets, $g(k), P(i_1, i_2, \ldots, i_n)$ satisfying those conditions, $(a)_k - (e)_k$ up to n. We shall construct the next sets.

Let $\{U_n\}$ be a decreasing countable open base for the point x, then none of their intersection with $P(0, 0, \ldots, 0_n)$ can be dense in that set. For if there were dense, by Baire category theorem the intersection of those sets became also dense in $P(0,0,\ldots,0_n)$, and x became an isolated point. Thus, there exists a set U_m whose closure can not contain the set $P(0,0,\ldots,0_n)$ and we can find an iteration of $\sigma, g(n+1)$, such that

$$g(n+1)x \in P(0,0,\ldots,0_n) \setminus \overline{U_m}.$$

There exists then an open neighborhood $P(0,0,\ldots,0_{n+1})$ satisfying the conditions;

$$\overline{P(0,0,\ldots,0_{n+1})} \subset P(0,0,\ldots,0_n) \cap U_m,$$

$$g(n+1)\overline{P(0,0,\ldots,0_{n+1})} \subset P(0,0,\ldots,0_n) \setminus \overline{U_m},$$

and for integers, $1 \leq k_i < k_2 < \ldots < k_r \leq n+1$

$$diameter(g(k_1)\ldots g(k_r)P(0,0,\ldots,0_{n+1})) < \frac{1}{n+1}.$$

Write

$$P(i_1,i_2,\ldots,i_{n+1}) = g(k_1)g(k_2)\ldots g(k_r)P(0,0,\ldots,0_{n+1})$$

where the coordinates at $\{k_1,k_2,\ldots,k_r\}$ are specified to be 1.

From the above definition we easily see that the assertions $(a)_{n+1}$, $(d)_{n+1}$ and $(e)_{n+1}$ are satisfied. As for the assertion $(c)_{n+1}$, let $\{k_1,k_2,\ldots,k_s\}$ be the positions in $\{i_1,i_2,\ldots,i_n\}$ whose coordinates are 1. We have then,

$$\begin{aligned} P(i_1,i_2,\ldots,i_n) &= g(k_1)g(k_2)\ldots g(k_s)P(0,0,\ldots,0_n) \\ &\supseteq g(k_1)g(k_2)\ldots g(k_s)\overline{P(0,0,\ldots,0_{n+1})} \\ &= \overline{P(i_1,i_2,\ldots,i_n,0)}. \end{aligned}$$

Moreover,

$$\begin{aligned} P(i_1,i_2,\ldots,i_n) &\subseteq g(k_1)g(k_2)\ldots g(k_s)g(k_{n+1})\overline{P(0,0,\ldots,0-n+1)} \\ &= \overline{P(i_1,i_2,\ldots,i_n,1)}. \end{aligned}$$

The assertion $(c)_{n+1}$ follows.

Next, take two different tuples,$(i_1,i_2,\ldots,i_{n+1})$ and $(j_1,j_2,\ldots,j_{n+1})$. If $i_k \neq j_k$ for some $k < n+1$, they are disjoint because of the previous condition $(b)_n$. If both coordinates are the same up to n and $i_{n+1} = 0, j_{n+1} = 1$, consider the positions $k_1 < k_2 < \ldots < k_s$ where the coordinates there are to be 1 in $\{i_1,i_2,\ldots,i_n\}$. We see then,

$$P(i_1,i_2,\ldots,i_n,0) = g(k_1)\ldots g(k_s)P(0,0,\ldots,0_{n+1})$$

and

$$P(i_1, i_2, \ldots, i_n, 1) = g(k_1)\ldots g(k_s)g(n+1)P(0, 0, \ldots, 0_{n+1}).$$

Hence by definition of $P(0, 0, \ldots, 0_{n+1})$, the above two sets are disjoint.

Thus we have constructed the sets assigned.

Let K be the Cantor set expressed as the infinite product space of the set $\{0, 1\}$ over the natural numbers. Consider the open base $\{K(i_1, i_2, \ldots, i_n)\}$ of K fixing the first n-coordinates as $(i_1, i_2, \ldots, i_n)$. Take a point $z = (i_1, i_2, \ldots)$ of K, then by $(c)_n$ and $(d)_n$ the intersection of all $P(i_1, i_2, \ldots, i_n)$ over the positive numbers is reduced to one point, say $\theta(z)$. Therefore we can define an injective map θ from K into X. Furthermore, since

$$\theta(K(i_1, i_2, \ldots, i_n)) = \theta(K) \cap P(i_1, i_2, \ldots, i_n),$$

we see that the map is a homeomorphism. Thus we identify K with $\theta(K)$.

Now note that there exists a unique non-atomic measure μ on K with the property;

$$\mu(K(i_1, i_2, \ldots, i_n)) = \frac{1}{2^n}.$$

Write as the same μ its natural extension to X as a Borel measure. We assert μ is an ergodic measure. Define the map $\delta(k)$ on K as

$$\delta(k)(i_1, i_2, \ldots, i_k, i_{k+1}, \ldots) = (i_1, i_2, \ldots, 1 - i_k, i_{k+1}, \ldots).$$

It is then measure preserving and by $(e)_n$

$$\delta(k)|K(0, 0, \ldots, 0_k) = g(k)|K(0, 0, \ldots, 0_k).$$

Hence, $g(k)$ is μ-preserving on the above set. (*)

Let T be an invariant measureable set in X with nonzero μ-measure, and take a positive ϵ less than 1. As K is a compact metrizable space, μ is a regular measure. Therefore we can choose an open set Q in K containing $K \cap T$ such that

$$\mu(T) \geq (1 - \epsilon)\mu(Q).$$

On the other hand, since $\{K(i_1, i_2, \ldots, i_n)\}$ is the base of K consisting of clopen sets there exists a disjoint family $\{K_j\}$ among them such that

$$Q = \bigcup_{j=1}^{\infty} K_j.$$

It follows that

$$\sum_{j=1}^{\infty} \mu(T \cap K_j) = \mu(T) \geq (1-\epsilon) \sum_{j=1}^{\infty} \mu(K_j).$$

Thus there exists an intejer j with

$$\mu(T \cap K_j) \geq (1-\epsilon)\mu(K_j) \quad K_j = K(i_1, i_2, \ldots, i_n).$$

Now the set K is expressed as a disjoint sum of $K(j_1, j_2, \ldots, j_n)$ ranging over n-tuples of dyadics, whence

$$\mu(T) = \sum_{j_1, j_2, \ldots, j_n = 0,1} \mu(T \cap K(j_1, j_2, \ldots, j_n)).$$

Let $k_1 < k_2 < \ldots < k_r$ be as before for $(j_1, j_2, \ldots, j_n)$. We have by the property (*) of μ,

$$\begin{aligned} \mu(T \cap K(j_1, j_2, \ldots, j_n)) &= \mu(T \cap g(k_1) \ldots g(k_r) K(0, \ldots, 0_n)) \\ &= \mu(T \cap K(0, 0, \ldots, 0_n)) \end{aligned}$$

Therefore for an arbitrary n-tuple $(j_1, j_2, \ldots, j_n)$,

$$\begin{aligned} \mu(T \cap K(j_1, j_2, \ldots, j_n)) &= \mu(T \cap K(0, 0, \ldots, 0_n) \\ &= \mu(T \cap K(i_1, i_2, \ldots, i_n)) \\ &= (1-\epsilon)2^{-n}. \end{aligned}$$

Thus $\mu(T)$ is not less than $1-\epsilon$ for any ϵ, and

$$\mu(T) = 1 = \mu(X).$$

which shows the ergodicity of μ. This completes the first half part of the proof.

We shall show the second part. Let μ be the quasi-invariant ergodic measure constructed from the nonatomic measure as above. Consider the measure μ_σ defined as $\mu_\sigma(E) = \mu(\sigma^{-1}E)$. Let π_0 be the representation of $C(X)$ on the Hilbert space $H = L^2(X, \mu)$ as multiplication operators. We define the unitary operator u in a standard way, namely it is defined using the Radon-Nikodym derivatives,

$$uf(x) = \left(\frac{d\mu_\sigma}{d\mu}\right)^{\frac{1}{2}} (x) f(\sigma^{-1}x).$$

Then as usual $\{u^n\}$ gives a unitary representation of the group Z. Since

$$u^* f(x) = \left(\frac{d\mu \circ \sigma}{d\mu}\right)^{\frac{1}{2}} (x) f(\sigma x),$$

$\{\pi_0, u\}$ becomes a covariant representation of $\{C(X), \alpha\}$. Moreover, as μ is an ergodic measure, the representation $\tilde{\pi} = \pi_0 \times u$ turns out to be irreducible.

Now suppose this representation were unitarily equivalent to an irreducible representation induced by a point x_0 of X, then there would exist a subspace H_0 such that

$$\tilde{\pi}(f)\xi = f(x_0)\xi \quad \forall \xi \in H_0, \forall f \in C(X).$$

It follows that for a non-zero function g in H_0 we have

$$(f(x) - f(x_0))g(x) = 0 \quad a.e \quad \text{for every } f \in C(X).$$

Hence we can take the measurable set E of positive measure on which any continuous function f on X takes the constant $f(x_0)$, but this contradicts to the nonatomic property of μ.

Thus, the above irreducible representation is not induced from the points of X. This completes all proofs.

As an immeadiate consequence we obtain,

Corollary 7.6 *Let X be a compact metric space. If the set $c(\sigma) \setminus Per(\sigma)$ is not empty, then the algebra $A(\Sigma)$ has irreducible representations which are not unitarily equivalent to those irreducible representations induced from the points of X.*

Suming up all previous results and picking useful equivalent conditions from [14] , we now obtain the following theorem.

Theorem 7.7 *Suppose that X is metrizable, then the following assertions are equivalent:*

(1) $A(\Sigma)$ is of type one,

(2) All irreducible representations of $A(\Sigma)$ are induced from the points of X,

(3) Two irreducible representations of $A(\Sigma)$, $\tilde{\pi}_1$ and $\tilde{\pi}_2$ are unitarily equivalent if and only if their have the same kernel,

(4) The set $c(\sigma)$ coincides with $Per(\sigma)$,

(5) The orbit space of Σ is a T_0-space,

(6) The quotient map from X to the orbit space X/Z has a Borel cross section.

In the theorem, except for the condition (4) other equivalencies are old known results. We believe however that, considering the condition (4) which is a purely dynamical version as well as showing Proposition 7.5, arguments surrounding the dynamical systems of type 1 become more transparent. Since there are many impoetant dynamical systems of type 1 such as Morse-Smale systems this point seems to be quite meaningful in spite of the present role of the class of type 1 in the theory of C*-algebras.

The equivalence of (1) and (3) is a famous theorem by J.Glimm, and holds for an arbitrary separable C*-algebra of type 1, but here we need no serious arguments of Glimm to include the condition (3). Thus we give a direct proof of the theorem in our context.

Proof of theorem. Suppose $A(\Sigma)$ is of type 1 and let $\tilde{\pi}_1$ and $\tilde{\pi}_2$ be irreducible representations whose kernels coincide. Since the image of $\tilde{\pi}_1$ contains the algebra of compact operators, by [11, Corollary 4.1.10] $\tilde{\pi}_1$ is unitarily equivalent to $\tilde{\pi}_2$.

The implication (3)$\Rightarrow$ (2). As we already mentioned before, when X is metrizable for any irreducible representation $\tilde{\pi}$ we can find the irreducible representation arising from a point x_0 of X_π (with $\overline{O(x_0)} = X_\pi$) whose kernel coincides with that of $\tilde{\pi}$ (cf. Corollary 2.9 (c)). Hence $\tilde{\pi}$ is unitarily equivalent to $\tilde{\pi}_{x_0}$.

The assertion (2)$\Rightarrow$ (4) is established by Proposition 7.5 and (4)$\Rightarrow$ (1) is proved in Theorem 7.4. Moreover, an elementary (purely topological) proof of the equivalency, (4) and (5), is found in [40, p.757-758].

Next suppose there exists a Borel section from the orbit space X/Z to X and yet $c(\sigma) \neq Per(\sigma)$. Take a nonatomic quasi-invariant probability measure μ constructed in Proposition 7.5. From the assumption we have a Borel set B in X such that

$$X = \bigcup_{n \in Z} \sigma^n(B) \quad \text{(disjoint union)}.$$

Then B has positive measure, and as μ is nonatomic we can split B into the disjoint union of Borel sets B_1 and B_2 of positive measures. It follows that the union $\bigcup_{n \in Z} \sigma^n(B_1)$ becomes a non-trivial invariant set for μ. This is a contradiction, showing the implication (6)$\Rightarrow$(4).

The assertion, (4)$\Rightarrow$ (6). We recall first that a Polish space means a topological space which is homeomorphisc to a separable complete metric space. Let q be the quotient map from X to X/Z and consider its restriction to the set $Per(\sigma)$. Then the inverse image of each point is apparently closed in $Per(\sigma)$. Moreover, as a Borel subset of X, there is a Polish space P and a continuous surjective injection $\theta : P \rightarrow Per(\sigma)$ (cf.[6, Theorem 3.2.1]). Thus applying the cross section theorem for the Polish space ([6, Theorem 3.4.1]) for the map $q|Per(\sigma) \circ \theta$ we get a Borel cross section from $Per(\sigma)/Z$ to P, hence to $Per(\sigma)$.

Next for each positive integer k, let

$$F_k = \bigcap_{n \in Z} \{x \in Aper(\sigma) | \quad d(x, \sigma^n x) \geq 1/k\},$$

which turns out to be a closed set in $Aper(\sigma)$. By the separability of $Aper(\sigma)$ we can find a family of countable closed sets (in $Aper(\sigma)$), $\{W_{k,j}\}$ ($j \geq 1$) such that $F_k = \bigcup_j W_{k.j}$ and their diameters are less than $1/2k$. Then each set $W_{k,j}$ becomes a wandering set. For if

$$W_{k.j} \cap \sigma^n(W_{k,j}) \neq \phi \quad \text{for a non-zero integer } n,$$

there exists an element y in $W_{k.j}$ such that $\sigma^n(y) \in W_{k.j}$. But this contradicts to the definition of F_k and the choice of $W_{k,j}$.

Since $Aper(\sigma) = \bigcup_k F_k$ by the assumption (4), we can collect up a suitable countable family from $\{W_{k,j}\}$ so that its union forms a Borel transversal subset of $Aper(\sigma)$. Thus combining the result for $Per(\sigma)$ and $Aper(\sigma)$ we see that there exists a Borel cross section from the orbit space X/Z to X.

This completes all proofs.

If one fully make use of the situation of C*algebras of type 1 for the implication :(4) $\rightarrow$ (6) one may use of Dixmier's cross section theorem from $\widehat{A(\Sigma)}$ to the space of irreducible representations of $A(\Sigma)$ and then use the realization of $\widehat{A(\Sigma)}$ by the space, $Aper(\sigma)/Z \bigcup Per(\sigma)/Z \times T$, but we prefer the above topological way of the proof.

8 Shrinking steps of nonwandering sets and composition series in homeomorphism c*-algebras

Throughout this section we assume that X is a compact metric space with the metric $d(x, y)$. Results in this section are base on the author's recent preprint [44].

Recall that a point x of X is called a nonwandering point if for any neighborhood U of x there is a natural number n such that $\sigma^n(U) \cap U \neq \phi$. The set of all nonwandering points for Σ, $\Omega(\sigma)$, is an invariant closed subset of X.

In order to handle with nonwandering sets, we must observe at first an exceptional behavior of the set $\Omega(\sigma)$ from other elementary sets. Namely other sets are not changed when we restrict the domain of the homeomorphism σ ; mostly by their definitions except the set of chain recurrent points that will be discussed in the next section.

The nonwandering set of the restriction of σ to $\Omega(\sigma)$ is however not the same set in general. It shrinks and if we consider further the nonwandering set of this restricted domain the set would become more smaller one. It is then known that this shrinking step stops at the center ([2, Proposition 17]). Thus, starting from the sets $\Omega_0(\sigma) = X$ and $\Omega_1(\sigma) = \Omega(\sigma)$,we obtain a decreasing series of closed invariant sets $\{\Omega_\alpha(\sigma)\}$ indexed by countable ordinal number α $(0 \leq \alpha \leq \gamma\ )$ such that

$$\Omega_{\alpha+1}(\sigma) = \Omega(\sigma \mid \Omega_\alpha(\sigma)),$$

and if α is a limit ordinal number

$$\Omega_\alpha(\sigma) = \bigcap_{\lambda<\alpha} \Omega_\lambda(\sigma).$$

Moreover, steps end as $\Omega_{\gamma+1}(\sigma) = \Omega_\gamma(\sigma) = \overline{c(\sigma)}$. The minimal such γ is called the depth of the center of the homeomorphism σ and is denoted by $d(\sigma)$. It is then to be noticed that, although the fact that the shrinking steps end at the center was known in old days of G.D. Birkhoff, as we have noticed before, realizations of these steps are only recently confirmed by Kato [21]. Namely, for any given countable ordinal number γ there exists a homeomorphism σ in some compact metric space X for which $d(\sigma) = \gamma$.

Now as we have shown before, $Hull(K(\sigma))$ becomes the closure of the set $c(\sigma)\backslash Per(\sigma)$. Let $F(\sigma)$ be the intersection of all kernels of finite dimensional irreducible representations of $A(\Sigma)$. We write $J(\sigma)$ the ideal defined as the intersection of $K(\sigma)$ and $F(\sigma)$. This becomes then the well behaving ideal of type 1 in $K(\sigma)$ whose Hull coincides with the center. Hence we see that $J(\sigma)$ becomes the Kernel ideal of $\overline{c(\sigma)}$.

By definition, the ideal $J(\sigma)$ does not have non-trivial finite dimensional irreducible representations and then every ideal I of $J(\sigma)$, as an ideal of $A(\Sigma)$, becomes a well behaving ideal. It follows by Theorem 5.7 that there exists a one-to-one correspondence between the family of closed invariant sets $\{S_\lambda\}$ containing the center and the family of ideals $\{I_\lambda\}$ of $J(\sigma)$ given as

$$\begin{aligned} I_\lambda &= \bigcap_{x \in S_\lambda \backslash \overline{c(\sigma)}} P(\tilde{x}) \bigcap J(\sigma) \\ &= Ker(S_\lambda), \end{aligned}$$

and

$$\begin{aligned} S_\lambda &= \{x \notin \overline{c(\sigma)} \mid \tilde{\pi}_x(I_\lambda) = 0\} \cup \overline{c(\sigma)} \\ &= Hull(I_\lambda). \end{aligned}$$

In this situation, the following lemma is easily verified.

Lemma 8.1 *Let S be the intersection of the above kind of closed invariant sets $\{S_\lambda\}$ then the associated ideal I of $J(\sigma)$ is the closure of the union of all associated ideals I_λ's.*

Let A be a C*-algebra of type 1, then the algebra admits an ascending series of ideals $\{I_\alpha\}$ indexed as $0 \leq \alpha \leq \gamma$ by ordinal numbers,called a composition series, such that $I_0 = 0, I_\gamma = A$ and if α is a limit ordinal,$I_\alpha = \overline{\bigcup_{\lambda<\alpha} I_\lambda}$. For this composition series we can specify each quotient algebra $I_{\alpha+1}/I_\alpha$ to be a liminal C*-algebra. Let $\hat{A}$ be the dual of a C*-algebra A. In case of a C*-algebra of type 1, $\hat{A}$ is identified with the space $Prim(A)$ of all primitive ideals of A equipped with the hull-kernel topology (cf. Theorem 7.7(3)). We then call A a C*-algebra with continuous trace if $\hat{A}$ is a Hausdorff space and for each point π_0 of $\hat{A}$ there exist an element a of A and a neighborhood W of π_0 such that $\pi(a)$ is a projection of rank one for every irreducible representation π in W ([11, Proposition 4.5.4]). This algebra becomes necessarily a liminal C*-algebra, and it is known ([11, Theorem 4.5.5]) that the composition series $\{I_\alpha\}$ of the C*-algebra A of type 1 can be furthermore specified as all quotient algebras are C*-algebras with continuous trace. This type of composition series of A may well be not unique in general.

Now let S_1 and S_2 be closed invariant sets of X containing the center with $S_1 \supset S_2$ and let I_1 and I_2 be the corresponding ideals of $J(\sigma)$.

The following two facts are key results of our discussion. But before going into their discussions it would be better to give the detailed structure of the dual of the quotient algebra I_2/I_1 for those readers who are not so much familier with C*-theory. We leave details of proofs to our preprint [44].

We note that every irreducible representation $\tilde{\pi}$ of I_2/I_1 arises from a point of $S_1 \setminus S_2$, so that the dual of I_2/I_1 is identified with the orbit space $(S_1 \setminus S_2)/Z$. In fact, the representation $\tilde{\pi}$ is regarded as a representation of I_2 vanishing on I_1 and moreover it can be considered as the restriction of the irreducible representation of $A(\Sigma)$ (denoted as $\tilde{\pi} = \pi \times u$). Let $\Sigma_\pi = (X_\pi, \sigma_\pi)$ be the dynamical system induced by this representation $\pi \times u$. Then it is topologically transitive and since X is a compact metric space there exists a point x in X_π with dense orbit. Moreover, since $\tilde{\pi}$ does not vanish on I_2 this point x does not belong to the center (nor to S_2), hence by Proposition 4.1 the image $\tilde{\pi}(A(\Sigma))$ contains the algebra of compact operators. It follows by [11, Proposition 4.1.10] that $\tilde{\pi}$ is unitarily equivalent to the irreducible representation $\tilde{\pi}_x$ induced by x because they have the same kernel. Thus, the dual $\widehat{I_2/I_1}$ is identified with the orbit space $(S_1 \setminus S_2)/Z$ as sets and, since the algebra is of type 1, identified as topological spaces, too. This last assertion holds however only because the algebra I_2/I_1 does not have finite dimensional irreducible representations (see Proposition 2.4 (1)).

With this identification, henceforth we denote the points of $\widehat{I_2/I_1}$ by $\bar{x}$ taking a point x in $S_1 \setminus S_2$. We write also the quotient map from $S_1 \setminus S_2$ to its orbit space by q.

Lemma 8.2 *The quotient algebra I_2/I_1 becomes liminal if and only if the boundary set $\partial O(x) = \overline{O(x)} \setminus O(x)$ is contained in S_2 for every point x in $S_1 \setminus S_2$.*

Proposition 8.3 *Keep the notations as above. Then the algebra I_2/I_1 becomes a C^*-algebra with continuous trace if and only if the following conditions hold for the space $S_1 \setminus S_2$;*

(1) All points of $S_1 \setminus S_2$ are wandering points with respect to the homeomorphism $\sigma|S_1$,

(2) For any neighborhood U of x in $S_1 \setminus S_2$, the set $\overline{\bigcup_{n \in Z} \sigma^n(U)} \setminus \bigcup \sigma^n(U)$ is contained in S_2.

It is to be noticed that in general the above second condition need not hold for gaps of nonwandering sets hence the notion of C*-algebras of continuous trace seems to be too strong to describe those gaps. By similar reason we can not use another class of generalized C*-algebras with continuous trace introduced by Dixmier [10] either. Thus we reach the following

Theorem 8.4 *Keep the notations as above. Then the set S_2 becomes the nonwandering set with respect to the homeomorphism σ on S_1 if and only if I_2 is a largest ideal of $J(\sigma)$ containing the ideal I_1 which satisfies the following conditions;*

(1) the quotient algebra I_2/I_1 is liminal,

(2) for each point $\bar{x}_0$ of the dual $\widehat{I_2/I_1}$ there exist a continuous function f belonging to I_2 and a neighborhood W of $\bar{x}_0$ such that $\pi_x(f)$ is a projection of rank one for every irreducible representation $\tilde{\pi}_x$ with $\bar{x} \in W$.

Note that in the theorem we lose the assumption, Hausdorff property, but have to ask instead more stronger condition for an element f to be in $C(X) \cap I_2$.

In the theorem,the author does not know whether he may replace f by an element a in I_2. If this would be the case, we could say that shrinking steps of nonwandering sets are also algebraic invariant as in the case of other elementary sets. Algebraic invariants of topological dynamical systems will be discussed in §12.

We finally discuss C*-versions of the cases where the depth of the center is zero and one, that is, $X = \Omega(\sigma)$ and $\Omega(\sigma|\Omega(\sigma)) = \Omega(\sigma)$. As we mentioned before these amount to the cases where $X = \overline{c(\sigma)}$ and $\Omega(\sigma) = \overline{c(\sigma)}$.

Proposition 8.5 *(1) Depth of the center is zero if an only if the largest ideal of type 1,$K(\sigma)$, in $A(\Sigma)$ becomes a residually finite dimensional C*-algebra or 0,*

(2) Depth is one if and only if the ideal $J(\sigma)$ is a liminal ideal of $A(\Sigma)$ and satisfies the second condition of Theorem 8.4.

Here a C*-algebra is said to be residually finite dimensional if there exist sufficiently many finite dimensional irreducible representations.

We meet the first case, for instance, when the dynamical system is topologically transitive (although in this case $K(\sigma) = 0$ if X is infinite). For the case of depth one, perhaps a good class of examples is a dynamical system satisfying the Pseudo-orbit tracing property, POTP.

Recall that for a positive δ a sequence $\{x_i\}$ is called a δ-pseudo orbit for σ if

$$d(\sigma(x_i), x_{i+1}) < \delta \quad \text{for every i.}$$

For a positive ε, it is then said to be ε - traced if there exists a point $x \in X$ such that

$$d(\sigma^i(x), x_i) < \varepsilon \quad \text{for every i.}$$

We say that a dynamical system $\Sigma = (X, \sigma)$ satisfies POTP if for every ε there exists $\delta = \delta(\varepsilon) > 0$ such that every δ-pseudo-orbit is ε-traced. It is then known ([3, Theorem 7.20]) that the restriction of σ to $\Omega(\sigma)$ has also POTP. Since in this case the set of chain recurrent points coincides with the nonwandering set, we have

$$\Omega(\sigma|\Omega(\sigma)) = R(\sigma|R(\sigma)) = R(\sigma) = \Omega(\sigma).$$

Hence, by [2, Proposition 17], $\Omega(\sigma) = \overline{c(\sigma)}$, that is, depth of the center is one.

Next we shall further treat problems related to our preceeding main results. From the above arguments, one may recognize that the C*-algebra $A(\Sigma)$ may hardly become a C*-algebra with continuous trace. That this is actually the case is assured in the next result.

Proposition 8.6 *The algebra $A(\Sigma)$ becomes a C^*-algebra with continuous trace if and only if the dynamical system Σ consists of periodic points with finite numbers periodicities,$\{n_1, n_2, \ldots, n_k\}$, and each set $Per_{n_j}(\sigma)$ is open and closed.*

Proof. Suppose $A(\Sigma)$ be a C*-algebra with continuous trace. It is then a liminal C*-algebra and since $A(\Sigma)$ is unital every irreducible representation becomes finite dimensional. Hence by [38, Theorem 4.6] X consists of periodic points and as we already mentioned before the dual of $A(\Sigma)$ is regarded as the product set (not as the product topological space) $X/Z \times T$ with the correspondence $[\tilde{\pi}_{x,\lambda}] \leftrightarrow (\bar{x}, \lambda)$. Now take an arbitrary point $(\bar{x}_0, \lambda_0)$ in $\widehat{A(\Sigma)}$ where $x_0 \in Per_p(\sigma)$. By the assumption, there exist an element a of $A(\Sigma)$ and a neighborhood U of $(\bar{x}_0, \lambda_0)$ such that $\tilde{\pi_{x,\lambda}}(a)$ is a projection of rank one for every point $(\bar{x}, \lambda)$ in U. Let $\{e_{ij}\}$ be a matrix units of $\tilde{\pi}_{x_0,\lambda_0}(A(\Sigma))$ ($= M_p$) with $e_{11} = \tilde{\pi}_{x_0,\lambda_0}(a)$. We can then find another neighborhood V in U for which there exists a set $\{a_{ij}\}$ in $A(\Sigma)$ such that $a_{11} = a$ and $\{\tilde{\pi}_{x.\lambda}(a_{ij})\}$ keep all relations of matrix units for every $(\bar{x}, \lambda)$ ([19, Lemma 10 and 11]) , inducing the system $\{e_{ij}\}$ at $(\bar{x}_0, \lambda_0)$. Here the element $1_{x,\lambda} - \sum_{i=1}^{p} \tilde{\pi}_{x,\lambda}(a_{ij})$ is a projection at each point $(\bar{x}, \lambda)$ in V . Since the norm functions are continuous by the Hausdorff property of the dual ([11, Corollary 3.3.4]) there exists a neighborhood W in V on which the above projection becomes zero. Thus the phenomena of so-called "dimension drops" do not occur in our case and the p-dimensional component $\widehat{A(\Sigma)}_p$ becomes open in $\widehat{A(\Sigma)}$.

We assert that the set $Per_p(\sigma)$ is open in X. Let G be the image of W in the space $Per_p(\sigma)/Z$ by the projection map on $Per_p(\sigma)/Z \times T$. We shall show that G is an open neighborhood of $\bar{x}_0$ in $Per(\sigma)/Z$ so that its inverse image becomes an open set of X contained in $Per_p(\sigma)$. In fact, let G^c be the complement of G. It follows then the product space $G^c \times T$ is included in the complement W^c, which is a closed subset in the dual space. Take a point $\bar{y}_0$ in G then we find a number μ_0 in T such that $(\bar{y}_0, \mu_0)$ belongs to W. Since the topology of $\widehat{A(\Sigma)}$ is regarded as the hull-kernel topology, the ideal $P(\bar{y}_0, \mu_0)$ does not include the ideal of intersection;

$$\bigcap_{G^c \times T} P(\bar{x}, \lambda) = \bigcap_{G^c} Q(\bar{x}).$$

Now suppose there exist a sequence $\{\bar{x}_n\}$ in G^c converging to $\bar{y}_0$ in X/Z. We have then,

$$O(y_0) \subseteq \overline{\bigcup_n O(x_n)}.$$

It follows that

$$\bigcap_{G^c} Q(\bar{x}) \subseteq \bigcap_n Q(\bar{x}_n) \subseteq Q(\bar{y}_0) \subseteq P(\bar{y}_0.\mu_0),$$

a contradiction. After all, the inverse of G in $Per(\sigma)_p$ becomes an open subset of X/Z, hence $Per_p(\sigma)$ is an open set of X. Since X consists of periodic points, it must be also a closed set. Thus, there appear only finite numbers of periodicities in Σ.

Conversely suppose we have such a dynamical system Σ and let $\{q_1, q_2, \ldots, q_k\}$ be characteristic functions of the sets $\{Per_{p_i}(\sigma)\}$. Since these sets are invariant one may easily see that those functions (regarded as projections) are invariant central projections of $A(\Sigma)$ and $A(\Sigma)$ is written as a direct sum of $A(\Sigma)q_i$. Here each such direct summand becomes a p_i-dimensional homogeneous C*-algebra, by which we mean its irreducible representation is always p_i-dimensional. It is then well known that a homogeneous C*-algebra is a C*-algebra with continuous trace. Therefore, as a direct sum of finite number of such algebras, $A(\Sigma)$ becomes a C*-algebra with continuous trace.

Now we consider the case where $\Omega(\sigma)$ is a finite set. We meet often this kind of dynamical systems such as the case of Morse-Smale dynamical systems.

In this case, the set $\Omega(\sigma)$ naturally consists of only periodic points and $A(\Sigma)$ is of type 1.

Proposition 8.7 *The set $\Omega(\sigma)$ is a finite set if and only if $J(\sigma)$ is the ideal of $A(\Sigma)$ having the properties (1) and (2) in Theorem 1 (as $I_1 = 0$) such that its quotient algebra is a direct sum of finite numbers of algebras of all matrix valued continuous functions on the torus.*

Proof. Suppose $\Omega(\sigma)$ is finite. Then the ideal $J(\sigma)$ coincides with the ideal J, and it has the properties in Theorem 8.4. Since the quotient algebra is naturally considered as the homeomoprhism C*-algebra on the set $\Omega(\sigma)$, it is a direct sum of finite numbers of matrix valued continuous functions on the torus T by [40, Proposition 3.5]. Here the central projection for each direct summand appears as the quotient image of a continuous function having the value 1 on an assigned periodic orbit and vanishing on other periodic orbits.

Conversely if $A(\Sigma)$ has that structure it is of type one. Moreover, it implies that the nonwandering set consists of finite numbers of periodic points by Theorem 8.4.

9 The set of chain recurrent (pseudo nonwandering) points and quasidiagonality of quotients and ideals of homeomorphism C*-algebaras

Throughout this section we assume X is a compact metric space. For a positive δ, a sequence $\{x_i\}$ is called a δ-orbit if

$$d(\sigma(x_i), x_{i+1}) < \delta \quad \text{for every } i.$$

A point x is then called a chain recurrent (or pseudo nonwandering) point if for every δ there exists a (nontrivial) cyclic δ-orbit for x. We denote by $R(\sigma)$ the set of all chain recurrent point in X. It is an invariant closed set and naturally contains the set $\Omega(\sigma)$. We first cite here basic known facts for the chain recurrent set in a theoem altogether from the article [3, Remark 7.2, Theorem 7.4 and 7.14].

Theorem 9.1 *(a) $R(\sigma) = R(\sigma^k)$ for any integer $k \neq 0$.*
(b)$R(\sigma|R(\sigma)) = R(\sigma)$.
(c) For a point x in X the following assertions are equivalent:
(1) x does not belong to $R(\sigma)$,
(2) There exists an open neighborhood U of x such that $\sigma(\overline{U}) \subset U$ but $x \notin \sigma(\overline{U})$,
(3)There exists an open set U such that $x \notin U$, $\sigma(\overline{U}) \subset U$ and $\sigma^k(x) \in U$ for some $k > 0$,
(4) There is an attractor K such that $x \notin K$ and $\omega(x) \subset K$.

Here a set K is called an attractor if there is an open set U containing K with $K = \omega(U)$. The set $\omega(U)$ is defined as

$$\omega(U) = \bigcap_{n=0}^{\infty} \{\overline{\bigcup_{k \geq n} \sigma^k(U)}\},$$

hence $\omega(\{x\}) = \omega(x)$ is the set of all accumulation points of positive orbit of x. Similarly we define the set $\alpha(\{x\}) = \alpha(x)$ as the set of all accumulation points of negative orbit of x.

We start with the following result by Pimsner [29].

Theorem 9.2 *Let X be a compact metric space, then the following assertions are equivalent;*
(1) $R(\sigma) = X$,
(2) $A(\Sigma)$ is embedded into an AF-algebra,
(3) $A(\Sigma)$ is a quasidiagonal C^*-algebra,

An AF(approximately finite)-algebra is a C*-algebra defined as the inductive limit of (presumably countable) finite dimensional C*-algebras, which turns out to appear in various interesting classes of C*-algebras beyond the class of type 1 C*-algebras . C*-algebras of this class are often quite computable, therefore besides of its own interesting structure they are sometimes fitted for the first steps of further investigations of structural problems.

Now recall the definition of a quasidiagonal algebra. A separable C*-algebra A is said to be quasidiagonal if it has a faithful representation ρ for which there exists an increasing sequence $\{P_n\}$ of finite rank projections tending strongly to the identity such that

$$\|P_n\rho(a) - \rho(a)P_n\| \xrightarrow{n} 0 \quad \text{for every } a \in A \ .$$

This is also an interesting and important class of C*-algebras. By definition, all subalgebras of a quasidiagonal algebra are also quasidiagonal, but a quotient algebra need not be a quasidiagonal algebra. Actually every unital C*-algebra can be a quotient image of a quasidiagonal algebra through the cone CA. Moreover, there comes another trouble because a C*-algebra of type 1 is not necessarily quasidiagonal(Toeplitz algebra as an example). Therefore, although interesting is this class obstructions to be quasidiagonal are still somewhat mysterious.

Thus in this section we shall discuss quasidiagonality of quotient algebras (and ideals) of $A(\Sigma)$ as a good model to discuss about the structure of this class of C*-algebras.

Write the dynamical system $(R(\sigma), \sigma|R(\sigma))$ by $\Sigma_{R(\sigma)}$. Then as an immeadiate consequence of this theorem combining the above Theorem 7.1 (b), we have

Corollary 9.3 *The homeomorphism algebra $A(\Sigma_{R(\sigma)})$ is quasidiagonal.*

Typical cases where the condition of the theorem is satisfied are the cases where X coincides with the center,$\overline{c(\sigma)}$, $Per(\sigma)$ or the closure of $Per(\sigma)$ etc.

In the proof of Theorem 9.2 the most hard part is the implication (1) $\Rightarrow$ (2). For the arguments Pimsner has employed the following alternative definition of chain recurrent points, which seems to be fitted to the point of view of operator algebras.

Namely, consider an open covering of X, $\mathcal{V} = \{V_i\}_{i \in I}$. We call a subset $\{V_{\omega(n)}\}$ with respect to the index set $\{\omega = (\omega(n))_{n \in Z}| \ \omega(n) \in I\}$ a V-seudo-orbit if

$$V_{\omega(n)} \cap \sigma^{-1}(V_{\omega(n+1)}) \neq \phi \quad \text{for all } n.$$

On this occasion, if there exists a positive integer p such that $\omega(n+p)=\omega(n)$ for any n we say this V-pseudo-orbit is periodic, and write its period as $p(\omega)$. Then x becomes a chain recurrent point if and only if for any open covering $\mathcal{V}=\{V_i\}$ for which $x \in V_{i_0}$ there exists a periodic V-pseudo-orbit starting from V_{i_0} (i.e $\omega(0)=i_0$). The advantage of this definition is that it refers no metrics in X.

For the implication (3) $\Rightarrow$ (1) Pimsner uses the property (2) of (c) in Theorem 9.1. When an open set U has the property, $\sigma(\overline{U}) \subsetneq U$, N.Brown calls this situation that σ compresses the open set U. By (1) of Theorem 9.2 we may then add the following equivalent condition as the fourth assertion;

(4) σ compresses no open set in X.

It would be quite interesting to reconsider Pimsner's arguments (1) $\Rightarrow$ (4) from the whole aspects in the interplay between elementary sets of dynamical systems and C*-algebras. Namely, the embedding given in the above paper is quite rough and it would be interesting to investigate the structure of embedding according to the various size of elementary sets. Actually in case of an irrational rotation C*-algebra (usually denoted by A_θ) proved before by Pimsner-Voiculescu, we have the most simplest situation;

$$c(\sigma)=\Omega(\sigma)=R(\sigma)=X.$$

In this case, the embedded AF-algebra is closely enough to $A(\Sigma)$ having the same K_0-groups, and the result leads us to determine under what condition of dynamical systems two such algebras A_θ and $A_{\theta'}$ are isomorphic each other ($\theta=\theta'$ or $1-\theta'$).

On the other hand, it is to be noticed here that $A(\Sigma)$ itself never becomes an AF-algebra. Indeed, if it were an AF-algebra its K_1-group should be trivial, and we can derive a contradiction by using six exact sequences about K_*-groups of related C*-algebras by Pimsner-Voiculescue (cf.[37, p.152]).

Now the simplest example of a dynamical system satisfying the condition, $R(\sigma)=X$, is the case of the one point compactfication of the integer group Z with the shift, fixing the infinite point. In this case the set of nonwandering points just consists of the single infinite point. On the contrary, let X be the two point compactification of Z for positive and negative infinite points with the shift as well. Then we have

$$R(\sigma)=\Omega(\sigma)=\{-\infty,+\infty\}.$$

Let Σ be a dynamical system for which $X \neq R(\sigma)$. Then, as we have Corollary 9.3 the algebra $A(\Sigma_{R(\sigma)})$ is quasidiagonal. On the other hand, the ideal $Ker(\Omega(\sigma))$ is a liminal C*-algebra by Lemma

8.2 hence the $Ker(R(\sigma))$, too. Therefore by [32, Corollary of Theorem 3.6] it is also a quasidiagonal algebra. This seems to provide a good example in which the extension for quasidiagonal algebras fails to be quasidiaagonal.

We remark that the condition (4) mentioned above can not be tranfered directly to the case of locally compact spaces. In fact, take the group Z with shift σ on it and consider the one point compactification X with the extended homeomorphism $\tilde{\sigma}$ fixing the infinite point. We see then, as mentined before, that $R(\tilde{\sigma}) = X$ hence $A(\Sigma)$ (also $C_0(Z) \times Z$ as well) becomes quasidiagonal but there are many open sets in Z compressed by the shift.

Let I be a well behaving ideal of $A(\Sigma)$,then it is written as $Ker(S)$ for an invariant closed set $S = Hull(I)$.

Proposition 9.4 *(a) Let I be a well behaving ideal written as $Ker(S)$, then the quotient algebra becomes quasidiagonal if and only if $R(\sigma|S) = S$. In particular if the system $(S, \sigma|S)$ is topologically transitive and x_0 is a point of S with dense orbit, then the quotient algebra is quasidiagonal except the case where x_0 is an isolated point for which $\alpha(x_0) \cap \omega(x_0) = \phi$.*

(b) If J is a badly behaving ideal , then the quotient algebra is quasidiagonal.

Proof. For a well behaving ideal $Ker(S)$, let Σ_S be the dynamical system for the restriction map. We know by Theorem 4.6(4) that the quotient algebra is canonically isomorphic to the homeomorphism algebra $A(\Sigma_S)$. Hence the conclusion follows by Theorem 9.2.

The second assertion holds because $R(\sigma|S) = S$ if x_0 is not an isolated points. Moreover even if x_0 is an isolated point if we have the condition $\alpha(x_0) \cap \omega(x_0) \neq \phi$ we can see that every point in $O(x_0)$ becomes a chain recurrent point as in the case of one point compactification of the shift dynamical system on Z. Since $R(\sigma|S)$ is closed, it coincides with S.

Next let J be a badly behaving ideal, then the quotient algebra can not have any infinite dimensional irreducible representation because by definition $E(J) = C(X)$. Thus it becomes a liminal C*-algebra and by [32, Corollary of Theorem 3.6] it is quasidiagonal.

Unfortunately, a useful general criterion which tells us when the chain recurrent set with respect to the restricted homeomorphism exhausts the set S is not known. In case of the image of an irreducible representation, however, its associated representation of the dynamical system is topologically transitive. and we can apply the second assertion of (a). Hence the image becomes mostly quasidiagonal.

We are planning to describe the condition when a quotient algebra of $A(\Sigma)$ becomes quasidiagonal, but for a plain ideal we have not obtained a satisfactory answer because of its troublesome structure. As of now we have only the following result.

Let I be a plain ideal of $A(\Sigma)$ and let $I = I_0 \cap I_1$ be a standard decomposition described before Proposition 4.9 with corresponding closed invariant sets S_0 and S_1.

Theorem 9.5 *For the ideal I suppose that $S_0 \cap S_1 = \phi$. Then the quotient algebra $A(\Sigma)/I$ becomes quasidiagonal if and only if $R(\sigma|S_0) = S_0$.*

Proof. Note first that I_1 is a badly behaving ideal by Proposition 4.9. We assert there is no non-trivial ideal containing the sum of I_0 and I_1. In fact, as I_1 is badly behaving no primitive ideals of the type $P(\bar{x})$ contain the sum. On the other hand , if the primitive ideal $P(\bar{y},\lambda)$ contains I_0+I_1 the associated finite dimensional irreducible representation $\tilde{\pi}_{y,\lambda}$ is factored through both homeomorphism algebras $A(\Sigma_{S_0})$ and $A(\Sigma_{S_1})$ for the dynamical systems $(S_0, \sigma|S_0)$ and $(S_1, \sigma|S_1)$. It follows from Proposition 2.5 that the orbit $O(y)$ is contained in both S_0 and S_1, a contradiction. Hence the sum $I_0 + I_1$ is dense in $A(\Sigma)$, and

$$I_0 + I_1 = A(\Sigma).$$

Therefore, the quotient C*-algebras I_0/I and I_1/I are isomorphic to $A(\Sigma)/I_1$ and $A(\Sigma)/I_0$, respectively. Moreover, we see that $A(\Sigma)/I$ is the sum of those quotient algebras I_0/I and I_1/I. Now as I_1 is a badly behaving ideal, its quotient algebra is quasidiagonal by the assertion (a) of the above proposition. The quotient algebra by I_0 is quasidiagonal if and only if $R(\sigma|S_0) = S_0$ by (b) of the same proposition.Thus, we have the conclusion.

The next observation is due to N.Brown.

Proposition 9.6 *Let Y be a locally compact Hausdorff space with the second countability axiom and σ be a hemeomorphism in Y.*

Let $\Sigma = (X, \tilde{\sigma})$ be the dynamical system cosisting of the one point compactification of Y, X, and the extended homeomorphism $\tilde{\sigma}$ fixing the infinite point.Then the C-crossed product $C_0(Y) \times Z$ is quasidiagonal if and only if $A(\Sigma)$ becomes also quasidiagonal (hence $R(\tilde{\sigma}) = X$).*

Proof. Note first that the first crossed product is regarded as the ideal, $Ker(\infty)$ in $A(\Sigma)$ and we have ashort exact sequence where the part $C(T)$ appears because the infinite point is fixed by $\tilde{\sigma}$ (cf. [40, Proposition 3.5]).

$$0 \longrightarrow Ker(\infty) \longrightarrow A(\Sigma) \longrightarrow C(T) \longrightarrow 0.$$

Here the generating unitary in the group $K_1(C(T))$ can be lifted to the unitary of $A(\Sigma)$ (just as δ), hence the connected map from $K_1(C(T))$ to $K_0(Ker(\infty))$ is easily seen to be zero (cf.[27, p.146]). It follows by recent result due to N.Brown that $A(\Sigma)$ is quasidiagonal. The other implication is trivial.

As for ideals of $A(\Sigma)$, so far $X = X(\sigma)$ all ideals are quasidiagonal as subalgebras of a quasidiagonal algebra $A(\Sigma)$. In general, at least, for an invariant closed set S containning the nonwandering set $\Omega(\sigma)$ we can say that the ideal $Ker(S)$ becomes quasidiagonal because $Ker(\Omega(\sigma))$ is quasidiagonal as we mentined before.

We finally notice that in order to see when a well behavind ideal $Ker(S)$ is quasidiagonal the above proposition would be sometimes useful. Indeed, its quasidiagonality is equivalent to the quasidiagonality of the homeomorphism algebras of one point compactification Y of $X \setminus S$, and we could check the behavior of σ through the condition (4) for the extension $\tilde{\sigma}$ to Y, that is, whether the homeomorphism $\tilde{\sigma}$ compresses no open sets of Y or not.

10 Full groups of Σ and normalizers of $C(X)$ in $A(\Sigma)$

For a homeomorphism σ we define its full group $[\sigma]$ as

Definition 10.1

$$[\sigma] = \{\tau \in Homeo(X) | \quad \tau(O_\sigma(x)) \subseteq O_\sigma(x) \quad \forall x\}$$

For such a homeomorphism τ there exists an integer valued function $n(x)$ (called the jump function) such that

$$\tau(x) = \sigma^{n(x)}(x).$$

Note that the values of $n(x)$ are not defined uniquely on the set $Per(\sigma)$ and this situation causes main troubles in our coming discussions. Now the jump function $n(x)$ brings the following cocycle function $f(k,x)$ on $Z \times X$,

$$f(k,x) = \begin{cases} n(x) + n(\tau x) + \ldots + n(\tau^{k-1}x), & \text{for } k > 0 \\ 0, & \text{for } k = 0 \\ -[n(\tau^{-1}x) + n(\tau^{-2}x) + \ldots + n(\tau^{k}x)], & \text{for } k < 0 \end{cases}$$

Then by definition $f(k,x)$ satisfies the relation,

$$\tau^k(x) = \sigma^{f(k,x)}(x)$$

and the cocycle condition,

$$f(k+l,x) = f(k,\tau^l x) + f(l,x).$$

In the case of mesurable dynamical systems, problems for the jump function $n(x)$ is only mesurability but in topologicaly dynamical systems we have to see various aspects of $n(x)$. Thus we consider further subgroups of $[\sigma]$.

Definition 10.2 *(a) The bounded full group $[\sigma]_b$ for σ is a subgroup of $[\sigma]$ consisting of those homeomorphisms for which we can choose $n(x)$ as bounded functions,*

(b) The continuous or topological full group $[\sigma]_c$ for σ is a subgroup of $[\sigma]_b$ consisting of those homeomorphisms for which we can choose $n(x)$ as continuous functions.

We also define normalizers of those groups in a standard way. For instance the normalizers of $[\sigma]$ is;

$$N[\sigma] = \{\tau \in Homeo(X) | \quad \tau\rho\tau^{-1} \in [\sigma] \quad for\ \rho \in [\sigma]\}.$$

In general it is hard to see how large are the full groups. But in case of Cantor minimal systems discussed by Giordano-Putnam-Skau [17], the continuous full group for such a dynamical system (Γ, φ) becomes sufficiently large. In fact,for any pair of points (x,y) there exists a positive integer n such that $\varphi^n(x) \neq y$. We can then choose a clopen neighborhood U of x such that

$$U \cap \varphi^n(U) = \phi, \quad y \notin U \cup \varphi^n(U).$$

Hence if we consider the map τ defined as φ^n on U, φ^{-n} on $\varphi^n(U)$ and the identity on the compliment of $U \cup \varphi^n(U)$ it becomes a homeomorphism in $[\varphi]_c$,and separates x and y.

Recenly they have proved in [18] that the Dye's program for full groups of ergodic transformations can be also carried out for the Cantor minimal ssystem. Namely, group isomorphisms between those full groups,$[\varphi]$ and $[\varphi]_c$ determine the relations of dynamical systems up to topological orbit eqiuivalence and flip conjugate respectively.

The following is a negative result.

Proposition 10.3 *If X is connected and σ is free, then*

$$[\sigma] = \{\sigma^n | \quad n \in Z\}.$$

Proof. Write
$$X_n = \{x| \quad \tau(x) = \sigma^n(x)\}.$$
Then the set $\{X_n\}$ forms a disjoint partitiion of closed sets because the system is free. Hence by Sierpinski's theorem only one component remains and $\tau = \sigma^n$ for some integer n.

The above case is applied to irrational rotations on the torus as well as to Denjoy homeomorphisms discussed in [31]. Further examples of free dynamical systems on the n-dimensional tori, T^n, are the family of Furstenberg transformations. On T^2 this transformation is written as
$$(s,t) \Rightarrow (s+\theta, t+f(s))$$
for an irrational number θ and a continuous function $f(s)$.

When a dynamical system Σ admits periodic points, as we mentioned above it is hard to see how large its full groups are even for a dynamical system on the tori. We shall discuss later full groups of the dynamical system $\Sigma_H = (T^2, \sigma_H)$. The difficulty to determine full groups of a dynamical system stems from the situation that we have to go beyond the concept of homeomorphism C*-algebras. As we shall prove, so far the continuous full groups are concerned we can remain inside our homeomorphism C*-algebras, but once we consider bounded full groups we have to consider C*-crossed products of L^∞ algebras (if there exist suitable measures for given homeomorphisms). Moreover in order to catch up elements of general full groups we further need to consider von Neumann crossed products of L^∞-algebras.

Proposition 10.4

$$N[\sigma] = \{\tau \in Homeo(X)| \quad \tau(O_\sigma(x)) = O_\sigma(\tau x)\}.$$

Proof. If $\tau \in N[\sigma]$, then for any point x and an integer n, the homeomorphisms $\tau\sigma^n\tau^{-1}$ and $\tau^{-1}\sigma^n\tau$ belong to $[\sigma]$. Hence,
$$\tau(\sigma^n(x)) = \tau\sigma^n\tau^{-1}(\tau(x)) \in O_\sigma(\tau(x)),$$
and
$$\tau^{-1}(\sigma^n(\tau(x)) = \tau^{-1}\sigma^n\tau(x) \in O_\sigma(x).$$
Therefore,
$$\tau\left(O_\sigma(x)\right) \subset O_\sigma(\tau(x)) \quad \text{and} \quad O_\sigma(\tau(x)) \subset \tau(O_\sigma(x)),$$
which shows that $\tau(O_\sigma(x)) = O_\sigma(\tau(x))$.

Conversely, take a homeomorphism τ such that $\tau\left(O_\sigma(x)\right) = O_\sigma(\tau(x))$ for all $x \in X$. For any element ρ in $[\sigma]$, we have then

$$\tau\rho\tau^{-1}(x) = \tau\rho(\tau^{-1}(x)) = \tau\sigma^m(\tau^{-1}(x)) = \sigma^n(x)$$

for some integers m and n. Hence by definition τ belongs to $[\sigma]$.

Let $U(A(\Sigma))$ and $U(C(X))$ be the unitary groups of those algebras respectively. We denote the normalizers of $C(X)$ in $A(\Sigma)$ by

$$N(C(X), A(\Sigma)) = \{v \in U(A(\Sigma)) | \quad vC(X)v* = C(X)\}.$$

Similarly we write the groups of automorphisms of $A(\Sigma)$ and that of inner sutomorphisms of $A(\Sigma)$ keeping the subalgebra $C(X)$ as $Aut(C(X), A(\Sigma))$ and $Inn(C(X), A(\Sigma))$ respectively.

Next, define the homomorphism ι from $U(C(X))$ to $Aut(C(X), A(\Sigma))$ as

$$\iota(g)f = f \quad \forall f \in C(X), \quad \text{and } \iota(g)\delta = g\delta.$$

Wright

$$U_\sigma = \{f \in U(C(X))| \quad f = g\alpha(\bar{g}) \quad \text{for some } g\}$$

We shall first give the C*-version of the results known for mesurable full groups and normalizers in corresponding factors for a fairly wide class of topological dynamical systems, topologically free dynamical systems.

Let v be a normalizer of $C(X)$ in $A(\Sigma)$ and let τ be the homeomorphism of X determined by v. In order to show that τ belongs to the topological full group $[\sigma]_c$ we need a series of lemmas.

Lemma 10.5

$$\tau(O_\sigma(x)) = O_\sigma(x) \quad \forall x \in X.$$

Though the statement of this lemma is a topological one, we make use of Theorem 2.2 (representations of $A(\Sigma)$) for the homeomorphisms σ and τ. Note that with this lemma the jump function $n(x)$ for σ is uniquely determined on the set $Aper(\sigma)$. A difficult part of the next arguments is to find a continuous extension of $n(x)$ to the whole space X.

Let x be an aperiodic point and let φ be a unique pure state extension of the point evaluation μ_x to $A(\Sigma)$. Write by η the map from $A(\Sigma)$ to the Hilbert space H_φ associated to the GNS representation of φ.

Lemma 10.6 *The following assertions are equivalent;*

(1) $\tau(x) = \sigma^n(x)$,

(2) $\eta(v) = \lambda\eta(\delta^n)$ *for some* λ *with modulus one,*

(3) $\mid v^*(-n)(x) \mid = 1$ *(and consequently* $v^*(-m)(x) = 0$ *for any* $m \neq n$*)* .

Lemma 10.7 *Let* $\{F_n\}$ *be a closed covering of* X*, then* X *coincides with the closure of the union* $\cup_{n \in Z} F_n^\circ$*, where* F° *means the interior of* F*.*

Now we give a sketch of the final stage of the proof. Put

$$F_n = \{x| \quad \tau(x) = \sigma^n(x)\}.$$

Then X is the union of these closed sets hence by the above lemma X coincides with the closure of the union of open sets $\{U_{n_k}\}$, which are nonempty interiors of F_{n_k}. Then using Lemma 10.6 (3), we can show that each closure $\overline{U_{n_k}}$ is separated from the closure of the union of all other sets. Thus it turns out to be a clopen set, and moreover we can further verify that the family $\{\overline{U_{n_k}}\}$ becomes of finite numbers. But once we know this fact, the function $n(x)$ defined as n_k on $\overline{U_{n_k}}$ should be the continuous extension of the function $n(x)$ on $Aper(\sigma)$ to the whole space X.

Comparing the above arguments, the other side is relatively easy. In fact, take a homeomorphism τ in $[\sigma]_c$. It provides then a partition of clopen sets $\{X(\tau, n_k)\}$ of finite number according to the range of $n(x)$. Let $p(\tau, k)$ be the characteristic function of $X(\tau, n_k)$. Put

$$v = \sum_{k=1}^{k(\tau)} \delta^{n_k} p(\tau, n_k).$$

We can show then v is a normalizer of $C(X)$ and $Adv = \alpha_\tau$.

Thus we have the assertion (a) of the following

Theorem 10.8 *Suppose that the dynamical system* Σ *is topologaically free. Then we have the following short exact sequences,and they all split.*

(a)

$$1 \longrightarrow U(C(X)) \longrightarrow N(C(X), A(\Sigma)) \longrightarrow [\sigma]_c \longrightarrow 1,$$

(b)

$$1 \longrightarrow U(C(X)) \xrightarrow{\iota} Aut(C(X), A(\Sigma)) \xrightarrow{\kappa} N[\sigma]_c \longrightarrow 1,$$

(c)

$$1 \longrightarrow U_\sigma \xrightarrow{\iota} Inn(C(X), A(\Sigma)) \xrightarrow{\kappa} [\sigma]_c \longrightarrow 1.$$

Proofs of (b) and (c) are included in [42], but for readers convenience we give here their proofs.

Proof. For the assertion (b), take an automorphism β in $Aut(C(X), A(\Sigma))$ and let τ_β be the induced homeomorphism in X. Then for a function f and a homeomorphism τ of $[\sigma]_c$ we have

$$\begin{aligned} f \cdot (\tau_\beta \cdot \tau \cdot \tau_\beta^{-1})^{-1} &= f \cdot \tau_\beta \cdot \tau^{-1} \cdot \tau_\beta^{-1} \\ &= \beta \cdot Adv_\tau \cdot \beta^{-1}(f) = Ad(\beta(v_\tau))(f), \end{aligned}$$

where v_τ is a normalizer of $C(X)$ implementing τ by the split sequence (a). This means that the homeomorphism $\tau_\beta\tau\tau_\beta^{-1}$ is induced by the normalizer $\beta(v_\tau)$ and it belongs to $[\sigma]_c$. Hence τ_β belongs to the normalizer group $N[\sigma]_c$.

Next, take a homeomorphism τ in $N[\sigma]_c$. The homeomorphism $\tau\sigma\tau^{-1}$ belongs to $[\sigma]_c$, hence by (a) there exists a normalizer v which implements the above homeomorphism. Let γ be the automorphism of $C(X)$ induced by τ. We have then

$$\begin{aligned} Adv \cdot \gamma(f)(x) &= \gamma(f)(\tau\sigma^{-1}\tau^{-1}(x)) \\ &= f(\sigma^{-1}\tau^{-1}(x)) = \gamma\alpha(f)(x). \end{aligned}$$

It follows that the automorphism $\gamma\alpha$ coincides with $Adv \cdot \gamma$ as an automorphisms of $C(X)$. Therefore, considering the covariant representation $\{C(X), \gamma\}$ we obtain the automorphism $\hat{\gamma}$ of $A(\Sigma)$ such that

$$\hat{\gamma}(f) = \gamma(f) \quad \text{and} \quad \hat{\gamma}(\delta) = v,$$

that is, $\kappa(\hat{\gamma}) = \tau$. The sequence splits.

On the other hand, if the automorphism β belongs to the kernel of κ,

$$\beta(\delta)f\beta(\delta)^* = \beta(\delta f \delta^*) = \alpha(f) = \delta f \delta^*.$$

Since $C(X)$ is a maximal abelian C*-subalgebra of $A(\Sigma)$ by Theorem 2.10 $\delta * \beta(\delta)$ belongs to $U(C(X))$. Thus if we write it as f_β we see that $\iota(f_\beta) = \beta$, and the short exact sequence (b) is finished.

Finally if the adjoint of a normalizer v of $C(X)$ belongs to the kernel of κ v turns out to be a unitary function in $C(X)$, say g. In this case the corresponding function f to the automorphism Adv has the form,

$$f = \beta(\delta)\delta * = g\delta\bar{g}\delta * = g\alpha(\bar{g}).$$

This completes the sequence (c).

11 Bounded and continuous topological orbit equivalence and full groups

Given two dynamical systems,$\Sigma_1 = (X, \sigma)$ and $\Sigma_2 = (Y, \tau)$ we say that they are topologically orbit equivalent if there exists a homeomorphism h from X to Y satisfying,

$$h(O_\sigma(x)) = O_\tau(h(x)) \quad \text{for every point } x.$$

For two topologically orbit equivalent systems Σ_1 and Σ_2 we consider jump functions $n(x)$, $m(x)$ defined as,

$$\tau h(x) = h\sigma^{n(x)}(x),$$

$$h\sigma(x) = \tau^{m(x)}(h(x)).$$

Note that both functions $n(x)$ and $m(x)$ are not defined uniquely on the set $Per(\sigma)$. Moreover, in case of measurable dynamical systems the only problem for these jump functions is the measurability, whereas in the topological setting we have to consider the following cases as in the case of full groups;

(a) Continuous orbit equivalence $\leftrightarrow$ $n(x)$ is continuous.

(b) Bounded orbit equivalence $\leftrightarrow$ $n(x)$ is bounded,

(c) General orbit equivalence where no conditon is assigned on $n(x)$.

Note that when the system is free two equivalences (a) and (b) coincide. Moreover, though it is quite reasonable we should remark here a serious gap between measursable orbit equivalence and topological orbit equivalence. For instance, on the torus T all irrational rotations are known to be orbait equivalent as measurable isomorphisms. On the contrary, two irrational rotations σ_{θ_1} and σ_{θ_2} are topologically orbait equivalent if and only if they are flip conjugate (hence conjugate) each other. For if σ_{θ_1} is topologically orbit equivalent to σ_{θ_2}, by Proposition 10.3 σ_{θ_1} is conjugate to $\sigma_{\theta_2}^n$ for some n and similarly σ_{θ_2} is conjugate to $\sigma_{\theta_1}^m$ for some m. Hence they are flip conjugate.

In this secton we shall clarify the structure of bounded orbit equivalences based on the author's joint paper with Boyle [8].

We have however no systematic informations for the general topologicaal orbit equivalences except for the case of Cantor minimal systems.

Now in order to go our analysis for bounded topological orbit equivalence it may be enough to assume that

(*) X = Y and σ,τ have the same orbits.

Thus, throughout our discussions we keep this assumption. Our results are however easily reformulated into the situation where σ is a homeomorphism of X and τ is a homeomorphism of another space Y.

Recall the cocycle function $f(k,x)$ in relation with σ and τ. The following Proposition is the key result of our discussion.

Proposition 11.1 *(Bijection of coordinates)*

Suppose that the homeomorphism σ is topologically free and the jump function $n(x)$ is bounded. For a point x_0, if $n(x)$ is continuous on the orbit of x_0 then the map ; $k \to f(k, x_0)$ is a bijection on Z.

We emphasize here that when the point x_0 is aperiodic the proposition is trivial, and dificulty only appears for periodic points.

Theorem 11.2 *(Bounded orbit equivalence theorem)*

Assume that the homeomorphism σ is topologically free and the function $n(x)$ is baounded by N. Let P be the union of all orbits which contain some discontinuous points of $n(x)$. We have then,

(a) P is a closed nowhere dense subset of X consisting of periodic points with periods bounded by $2N$.

(b) The compliments $X \backslash P$ is decomposed into two invariant open sets A and B and there exists a integer valued continuous function $a(x)$ on $X \backslash P$ such that

$$n(x) = \begin{cases} 1 + a(x) - a(\tau x), & \text{on } A \\ -1 + a(x) - a(\tau x), & \text{on } B \end{cases}$$

If we define the homeomorphism h by

$$h(x) = \sigma^{a(x)}(x)$$

then by this homeomorphism h we have

$$\sigma \mid A \cong \tau \mid A, \quad \sigma \mid B \cong \tau^{-1} \mid B.$$

In the proof the property of P is easily shown. In fact, if we put $A_i = \{x : n(x) = i\}$, the space X is the union of those A_i for which $\mid i \mid \leq N$. The function $n(x)$ is continuous in the interior of each A_i and discontinuous elsewhere. For $-N \leq i < j \leq N$, let D_{ij} be the intersection of the boundaries of A_i and A_j. Then $n(x)$ is discontinuous at x if and only if x is in some D_{ij}, in which case

$$\sigma^i(x) = \tau(x) = \sigma^j(x),$$

and $\sigma^{i-j}(x) = x$ with $| i - j | \leq 2N$.

Therefore, each D_{ij} is a closed set of periodic points with periods at most $2N$, and P is a union of finitely many sets of the form $\sigma^k D_{ij}$. Since the system is topologically free, this shows the property of the set P.

Thus, the difficult part of the proof is to find the function $a(x)$. We can prove this in a series of lemmas but the most technical difficulty lies in the proof of the above proposition.

In the following we denote by $[m,n]$ the interval consisting of integers between m and n.

Lemma 11.3 *For a positive integer M and a point x_0, there exist a neighborhood U and a positive integer $\overline{M}$ such that*

$$[-M, M] \subseteq \{f(k,y); \quad k \in [-\overline{M}, \overline{M}] \quad \textit{for every } y \in U\}.$$

Now for a positive integer m we define,

$$A_m = \{x \in X \backslash P; \quad f(n,x) > 0 \quad \text{and} \quad f(-n,x) < 0 \quad \text{for any } n \geq m\},$$

$$B_m = \{x \in X \backslash P; \quad f(n,x) < 0 \quad \text{and} \quad f(-n,x) > 0 \quad \text{for any } n \geq m\}.$$

By definition, A_m and B_ℓ are disjoint for any m and ℓ.

Lemma 11.4 *Let A and B be the union of all sets A_m and B_m for all positive integers, then they are invariant open sets whose union is $X \backslash P$.*

Next, for a positive integer M and a point of A, define the following functions;

$$c_M(x) = \sharp\{(-M, \infty) \cap (f(i,x); i \leq 0)\}.$$

$$a_M(x) = c_M(x) - M.$$

The functions c_M and a_M are constant on A because any point in A has a neighborhood U on which c_M is constant for a sufficiently large M. Then we can get the function $a(x)$ as the limit of the above function $a_M(x)$ when M goes to infinity (exists) ,and the final lemma comes in.

Lemma 11.5

$$n(x) = 1 + a(x) - a(\tau x) \quad \forall x \in A$$

The arguments for the set B may be done in a similar way.

Now for completeness we include a sketch of the final step of the proof of our theorem.

The argument of the above lemma was applied to the triple (σ, τ, n) to define the function $a(x)$ on A. For x in B, we have $\tau x = \left(\sigma^{-1}\right)^{-n(x)}(x)$, and we apply the argument of the above lemma to the triple $(\sigma^{-1}, \tau, -n)$ to produce a continuous function $b(x)$ on B such that for x in B,

$$-n(x) = 1 + b(x) - b(\tau x).$$

Defining then $a = -b$ on B, we get

$$n(x) = -1 + a(x) - a(\tau x), \quad x \in B.$$

This proves the cocycle relations claimed for n and a. We can then finally show that the map $h(x) = \sigma^{a(x)}(x)$ becomes a homeomorphism satisfing the required conditions.

Corollary 11.6 *If σ is topologically transitive, then σ and τ are flip conjugate on the set $X \backslash P$.*

Because in this case one of A or B disappears.

Next specializing the theorem to the case of a continuous cocycle we have the following

Theorem 11.7 *With the same conditions for σ and τ as above, suppose that $n(x)$ is continuous. Then X is decomposed into invariant open sets A and B, on each of which σ is conjugate to τ and to τ^{-1}, respectively.*

This means that so far the continuous orbit equivalence we have completely clarified the structure under fairly general condition, topological freeness. As an immediate consequence we have

Corollary 11.8 *Keeping the assumption, if σ is topologically transitive or X is connected then σ and τ are continuously orbit equivalent if and only if they are flip conjugate each other by a homeomorphism $h(x) = \sigma^{a(x)}(x)$, where the transfer function is continuous.*

Remark. The above result is a topological analogue of Belinskaya's theorem ([7]) in ergodic theory. Belinskaya proved that if S and T are ergodic automorphisms of Lebesgue probability space with the same orbits by an integrable cocycle $n(x)$, then T is flip conjugate (in the measurable category) to S by a measure preserving transformation $g(x) = S^{a(x)}(x)$, where the transfer function a is measurable.

Proposition 11.9 *The continuous orbit equivalence is a symmetric relation, that is , if the jump function $n(x)$ is continuous we can find a continuous jump function $m(x)$.*

In the case of bounded orbit equivalence, we can not expect this kind of result with the assumption (*), but once we allow a perturbed homeomorphism h we do not know the conclusion.

Recall the dynamical system $\Sigma_H = (T^2, \sigma_H)$ coming from the 3-dimensional discrete Heisenberg group mentined in §4. We shall determine the structure of homeomorphisms of $[\sigma_H]$. Here the concinuous full group is trivial,i.e.,$\{\sigma_H^n\}$, but its bounded full group $[\sigma_H]_b$ is rather big.

Let τ be an element of $[\sigma_H]$ and put

$$X_n(\tau) = \{x = (s,t) | \quad \tau(x) = \sigma_H^n(x)\}.$$

This is a closed set and $T^2 = \bigcup_{n\in Z} X_n(\tau)$. Hence at any s-level, we have

$$(s,T) = \bigcup_{n\in Z} ((s,T) \cap X_n(\tau)).$$

When s is an irrational number, this is a union of disjoint closed set. It follows by Sierpinski's theorem that there exists an integer n_0 such that

$$(s,T) \subset X_{n_0} \quad (n_0 \text{ may become zero}) .$$

On the other hand, since the union of slits of irrational levels is dense in T^2 each closed set $X_n(\tau)$ should consist of closed strips and slits along t-axis. Moreover, the (nonempty) intersection $X_i(\tau) \cap X_j(\tau)$ consists of lines at rational levels. Therefore, all of these sets are invariant under σ_H.

With this picture we can find many homeomorphisms of $[\sigma_H]$ based on piecewise transforms of the s-axis. For instance, consider the homeomorphism τ_0 defined as

$$\tau_0 = \begin{cases} \sigma, & 0 \le s \le 1/2 \\ \sigma^{-1}, & 1/2 \le s \le 1 \end{cases}$$

Then τ_0 belongs to $[\sigma_H]_b$ and the jump function $n(x)$ for this map is 1 on $0 \le s \le 1/2$ and -1 on $1/2 \le s \le 1$.The set P in Theorem 11.2 is the slit $(1/2, T)$ for discontinuous points of $n(x)$.

Now let τ be an arbitrary homeomorphism in $[\sigma_H]_b$, then the above observation tells us that the set P consists of slits of finite number at rational levels according to the range of the jump function $n(x)$. Conversely if any tuple of slits of finite numbers at rational levels is given we can easily contruct many homeomorphisms of $[\sigma_H]_b$ by making use of transforms of the s-axisa to piecewise linear curves. For such a curve, on each slit at level s any point of the orbit of (s, o)

for σ_H can be a connecting point of that curve. In this sense, the case τ_0 mentioned first is based on the transform of s-axis defined by

$$t = \begin{cases} s, & \text{on } 0 \le s \le 1/2 \\ 1-s, & \text{on } 1/2 \le s \le 1 \end{cases}$$

Next we consider the map σ_H with respect to the Lebesgue measure μ on T^2. It is then easily verified by Fubini theorem that σ_H is a measure preserving automorphism of T^2 and is aperiodic (though it is not ergodic).Therfore, we can consider the C*-crossed product $L^\infty(T^2,\mu) \rtimes_{\alpha_H} Z$ with respect to the automorphism α_H of $L^\infty(T^2,\mu)$ induced by σ_H. Let p be the characteristic function on the domain $[0,1/2] \times T$, then it turns out to be a central projection of the L^∞-crossed product. Put $\delta_1 = p\delta + (1-p)\delta^*$, which is a unitary element of $L^\infty(T^2,\mu) \rtimes_{\alpha_H} Z$. Then, although the homeomorphism τ_0 does not have a realization as a normalizer of $C(T^2)$ in $A(\Sigma_H)$, the automorphism $Ad\delta_1$ on $C(T^2)$ coincides with the automorphism α_{τ_0} induced by τ_0.

Thus in this way together with the structure of $[\sigma_H]_b$ mentioned above all homeomorphisms in $[\sigma_H]_b$ may be realized as normalizers of $L^\infty(T^2,\mu)$ in $L^\infty(T^2,\mu) \rtimes_{\alpha_H} Z$ keeping further the algebra $C(T^2)$ invariant. In fact, for a homeomrophism τ of $[\sigma_H]_b$ let $\{X_{n_1}, X_{n_2}, \ldots, X_{n_k}\}$ be the associated domains in T^2 for τ and let $\{p_1, p_2, \ldots, p_k\}$ be their characteristic functions. Then they are orthogonal projections because the intersections $\{X_{n_i} \cap X_{n_j}\}$ are all null sets (contained in $Per(\sigma_H)$).The unitary element u defined as

$$u = \sum_{i=1}^{k} p_i \delta^{n_i} \quad \text{in } L^\infty(T^2,\mu) \rtimes_{\alpha_H} Z$$

implements the automorphism α_τ on $C(T^2)$ induced by τ.

Thus the bounded full group for σ_H is relatively big to be able to separate s-coordinates. However, because of the form of each component $X_n(\tau)$ for τ in $[\sigma_H]$ the full group is not big enough to separate t-coordinates.

As for associated C*-algebras we can show that $A(\Sigma)$ and $A(\Sigma_{\tau_0})$ are not isomorphic each other because τ_0 is homotopic to the identity map whereas σ_H is not (cf[8, p.327]).

12 Algebraic invariants of topological dynamical system and isomorphism problem of homeomorphism C*-algebras

One of the most important problems in the interplay is to find the relations between Σ and Σ' when their associated homeomorphism C*-algebras are isomorphic each other. As is well known, this problem has been completely analized in case of measurable dynamical systems, that is, for ergodic transformations and their associated factors. In case of topological dynamical systems we believe that the C*-algebra $A(\Sigma)$ carries all basic informations of the dynamical system $\Sigma = (X, \sigma)$. Thus we put the following two basic questions.

Suppose we are given two dynamical systems $\Sigma = (X, \sigma)$ and $\Sigma' = (Y, \tau)$.

General isomromorphism problem: What is the relation between Σ and Σ' when those two C*-algebras $A(\Sigma)$ and $A(\Sigma')$ are isomorphic each other?

Restricted isomorphism problem: What is the relation between Σ and Σ' when $A(\Sigma)$ and $A(\Sigma')$ are isomorphic keeping their distinguished subalgebras of continuous functions $C(X)$ and $C(Y)$?

Until the time we find the final answer of the above general isomorphism problem, however, we do not know the exact evidence of our strategy.

In this section we discuss the situation towards the final answer of the general isomorphism problem , though it would be far beyond our present scope. We shall instead settle the restricted isomorphism problem (Theorem 12.2); the resulting relation is reduced to flip conjugate in limited cases.

So far we are working on the C*-algebra $A(\Sigma)$ this is the best thing we can hope for because the C*-crossed product with respect to an automorphism is isomorphic to the C*-crossed product with respect to its inverse. Actually if we want to make a distinction for these two C*-algebras restricting ourselves to treat the so-called analytic crossed product $A(\Sigma)_+$ (the subalgebra of $A(\Sigma)$ generated by $C(X)$ and non-negative powers of the generating unitary δ, not *-algebra) There is the following result by S.C.Power [30]. Namely,

Theorem. Two analytic crossed products $A(\Sigma)_+$ and $A(\Sigma')_+$ are isomorphic each other if and only if σ is conjugate to τ (no conditions on dynamical systems!).

This is quite nice result for the interplay. We feel however that the interplay within the rich structure of C*-context would bring more fruitful harvest for both sides as we have been illustrating. Thus we prefer to stay in the category of *-algebras, namely C*-algebras. Unfortunately, we came to know then even in the case of the Cantor minimal systems flip conjugacies are too restrictive to derive the conclusion from isomorphic relations among homeomorphism C*-algebras. In fact, the analysis by [17] tells us that we have to allow another extra condition, strong topological orbit equivalence. Namely, in this case two homeomorphism C*-algebras $A(\Sigma)$ and $A(\Sigma')$ are isomorphic each other if and only if Σ and Σ' are topologically orbit equivalent and their related jump function is continuous except at most one point. If the point of discontinuity disappears they are continuously topologically orbit equivalent, hence we have the conclusion of flip conjugate for minimal systems (Corollary 11.8). Thus we face an obstruction between general isomorphism problem and restricted isomorphism problem, contrary to the case of measurable dynamical systems. Here,the fact that an isomorphism between two factors associated to nonsingular ergodic transformations can be perturbed to a restricted isomorphism of them keeping their distinguished subalgebras of L^∞-functions (hence leading to the measurable orbit equivalence of those ergodic transformations) was a main crucial difficulty of the isomorphism problem between ergodic factors.

Now in a connected (at least path connected) space the discontinuity of jump function at only one point does not appear. Furthermore, there are some classes of homeomorphisms for which we have solutions of the general isomorphism problem. They are rotations and Denjoy homeomorphisms (the ones which are free and not topologically transitive on the torus, [30] for the latter), and certain class among Furstenberg homeomorphisms (by R.Ji [20] and [22]). In the results for all these kinds of homeomorphisms, we see however no obstrauction between general isomorphism problem and restricted isomorphism problem. For instance , besides for rotations and Denjoy homeomorphisms if we consider rather simple case of a Furstenberg homeomorphism $\sigma_{\theta,n}$ defined for an irrational number θ and an integer n ,

$$(s,t) \to (s+\theta, t+ns),$$

then two associated homeomorphism C*-algebras $A_{\theta,n}$ and $A_{\theta',m}$ are isomorphic if and only if

$$\theta = \theta' \quad \text{or } 1-\theta' \text{ and } \quad |n| = |m|,$$

in which these two homeomorphisms are even conjugate each other.

After all, in these arguments their associated C*-algebras are isomorphic each other if and only if those relevant homeomorphisms are flip conjugate (and simply conjugate if τ and τ^{-1} are conjugate).

Right now we do not have any categorical evidence for homeomorphisms on connected space (or even on tori), but we are inclined to believe that this might be the general conclusion for reasonable classes of homeomorphisms. That is, the algebra $A(\Sigma)$ carries almost all basic informations of the dynamical system Σ.

We should note here that if we put no conditions on dynamical systems we can not expect naturally the topological orbit equivalence of those dynamical systems. Actually,there exists a pair of compact connected manifolds (X, Y) which are not homeomorphic each other but their product spaces with torus are homeomorphic,that is, $X \times T \approx Y \times T$([9]). Hence if we consider the trivial dynamical systems in these compact manifolds, we have an isomorphism

$$A(\Sigma) = C(X \times T) \simeq C(Y \times T) = A(\Sigma').$$

Note that even in this situation the isomorphism brings the information that both dynamical systems are trivial.

Now we shall show the answer to the restricted isomorphism problem under a fairly general condition for dynamical systems, topological freeness. In order to state our restricted isomorphism theorem, we need however one more result.

Proposition 12.1 *Assume that the system $\Sigma = (X, \sigma)$ is topologically free and let v be a normalizer in $A(\Sigma)$. Then the C^*-algebra $A(v)$ generated by $C(X)$ and v coincides with $A(\Sigma)$ if and only if the orbit of τ_v coincides with the orbit of σ for every point of X.*

Thus combining this result with successive uses of Theorem 10.8, Theorem 11.7 and Proposition 11.9 we reach the following

Theorem 12.2 *(Restricted isomorphism theorem)*

Let $\Sigma = (X, \sigma)$ and $\Sigma' = (Y, \tau)$ be topologically free dynamical systems. Then there exists an isomorphism between homeomorphism C^*-algebras $A(\Sigma)$ and $A(\Sigma')$ keeping their subalgebras of continuous functions if and only if there are decompositions of X and Y into invariant open sets in such a way that

$$X = X_1 \cup X_2, \quad Y = Y_1 \cup Y_2$$

and

$$\sigma|X_1 \cong \tau|Y_1, \quad \sigma|X_2 \cong \tau^{-1}|Y_2.$$

Next we consider those properties which are carried over by isomorphisms of homeomorphism C*-algebras towards the general isomorphism problem. Thus we put the following definition.

Definition 12.3 *We say that a property of a topological dynamical system $\Sigma = (X, \sigma)$ is an algebraic invariant if the other dynamical system $\Sigma' = (Y, \tau)$ has the same property if $A(\Sigma')$ is isomorphic to $A(\Sigma)$.*

In this sense, being a trivial dynamical system is one of the simplest algebraic invariant.

We illustrate (nontrivial) algebraic invariants known as of now.

1) The property that $Per(\sigma)$ is dense or the whole space because these properties are characterized as the case where $A(\Sigma)$ is residually finite dimensional (that is, having sufficiently many finite dimensional irreducible representations) and as the case where $A(\Sigma)$ is liminal(cf.[38, Theorem 4.6]).

2).Topological freeness. We have already seen the importance of this concept as in Theorem 2.10. The results there are however not translated in the algebraic way without using the algebra of continuous functions. But we can see the algebraic invariance of topological freeness in the following way.

Proposition 12.4 *A dynamical system is topologically free if and only if its homeomorphism C^*-algebra has sufficiently many infinite dimensional irreducible representations.*

Proof. Let I be the intersection of all kernels of infinite dimensional irreducible representations. Then it is a well behaving ideal and by Corollary 2.9 (b) we have

$$Hull(I) = \overline{Aper(\sigma)}.$$

Therefore,

$$\overline{Aper(\sigma)} = X \quad \text{if and only if} \quad Ker(Hull(I)) = I = 0.$$

Moreover the property of freeness is easily senn to be algebraic invariant.

Next suppose that X is metrizable.

3). The properties that $c(\sigma)\backslash Per(\sigma)$ is dense in X and $c(\sigma) = Per(\sigma)$ are algebraic invariant as seen from Theorem 7.4.

4) The property $R(\sigma) = X$ due to Pimsner's result (Theorem 9.2).

5) The property $\Omega(\sigma) = X$(i.e depth of the center is 0) by Proposition 8.5 as well as the case where the depth of the center is one.

We have been however unable to conclude that the depth of the center (shrinking steps of the nonwandering set $\Omega(\sigma)$ down to the center $\overline{c(\sigma)}$) is an algebraic invariant or not.

In the context of C*-theory we are used to believe that two isomorphic C*-algebras should have the same structure of ideals. We have to be carefull however when we face the general isomorphism problem. In fact, when two homeomorphism C*-algebras $A(\Sigma)$ and $A(\Sigma')$ are isomorphic by an isomorphism Φ we can not say in general Φ keeps those classes of ideals discussed in §4. Actually, in the example of a particular pair of manifolds (X, Y) mentined above no isomorphisms between $A(\Sigma_X)$ and $A(\Sigma_Y)$ keep the types of ideals $Q(\bar{y})$ for all points of X because if there exist such an isomorphism it would induce a homeomorphism between X and Y.

Here an isomorphism brings certainly unitarily equivalent irreducible representations to equivalent pairs. But classes of finite dimensional irreducible representations have two parameters, namely orbits and numbers from the torus,and the trouble arises from the circumstances where we can not tell how these two kinds of parameters change according to each isomorphism.

Now since all these troubles come from the presence of periodic points if a dynamical system is free all isomorphisms seem to be well behaving.

Futhermore we have

Proposition 12.5 *In the dynamical system Σ if the set $Per(\sigma)$ is at most countable every isomorphism through $A(\Sigma)$ keeps the three types of ideals of $A(\Sigma)$.*

Proof. Suppose Φ be an isomorphism from $A(\Sigma)$ to another homeomorphism C*-algebra $A(\Sigma_1)$ for $\Sigma_1 = (Y, \tau)$. We shall show first that Φ keeps the type of the ideal $Q(\bar{y})$ for a periodic point y in Y with period p. We assert that the inverse image $I = \Phi^{-1}(Q(\bar{y}))$ has the same form. Note first that the quotient algebra $A(\Sigma_1)/Q(\bar{y})$ is regarded as the homeomorphism algebra on the orbit $O_\tau(y)$ and it is canonically isomorphic to the algebra of all M_p-valued continuous functions on the torus T([40, Proposition 3.5]). Hence, the dual of that algebra is homeomorphic to T, a compact connected space. On the other hand, dual of the corresponding quotient algebra, $A(\Sigma)/I$, is by [23, Theorem A] a compact subset Π of the product space $(Per_p(\sigma)/\sim) \times T$ as a part of the p-dimensional dual of $A(\Sigma)$. Now by the assumption for $Per(\sigma)$, Π is written as the sum of at most countable numbers of disjoint closed sets,

$$F_n = \{(\bar{x}_n, \lambda) \in \Pi\}.$$

Hence, by Sierpinski's theorem we have that

$$\Pi = F_{n_0} \quad \text{for some } \bar{x}_{n_0}.$$

Thus,

$$T' = \{\lambda \mid \quad (\bar{x}_{n_0}, \lambda) \in \Pi\}$$

becomes a compact subset of T, which is homeomorphic to T. It follows that $T' = T$ and I has the form $Q(\bar{x}_{n_0})$. Therefore, by Theorem 4.6 we see that Φ keeps well behaving ideals.

Next let I be a badly behaving ideal of $A(\Sigma)$, then $\Phi(I)$ must be also badly behaving. For if it is not we can find a well behaving ideal J containing $\Phi(I)$ and the well behaving ideal $\Phi^{-1}(J)$ contains I, a contradiction. Finally, if I is a plain ideal it is not well behaving but is contained in a well behaving ideal. Hence $\Phi(I)$ has to be a plain ideal, too.

This completes the proof.

We notice that most of examples of dynamical systems in manifolds satisfy the above condition (thus becoming topologically free dynamical systems). The author does not know whether or not an isomorphism between homeomorphism C*-algebras of topologically free dynamical systems satisfies always the above property..

So far, we come to know that for most of those reasonable dynamical systems all isomorphisms between associated C*-algebaras are relatively better behaving ones. Unfortunately this fact does not mean that we can perturb them towards extremely well-behaving isomorphisms, that is, restricted isomorphisms even in the Cantor minimal system.

For a topologically free dynamical system, we can give a characterization of a restricted isomorphism as the one which almost commutes with dual actions. This will be shown by the arguments based on the short exact sequence concerning the normalizer of the topological full group and the automorphism group of $A(\Sigma)$ preserving $C(X)$.

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