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Stable rank for a pair of C*-algebras

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1 Introduction and Main Result

The (topological) stable rank of Rieffel[19] is noncommutative generalization of the dimension of a compact Hausdorff space. In fact, when X is a compact Hausdorff space, the stable rank of $C(X)$ is $\left[\frac{\dim X}{2}\right]+1$, where $\dim X$ is covering dimension of X. Recall that a unital C*-algebra A has stable rank n if for any element $a_1, a_2, \cdots, a_n$ and $\varepsilon > 0$ there exist $b_1, b_2, \cdots, b_n$ in A such that

$$(1) ||a_i - b_i|| < \varepsilon$$
$$(2) \sum_{i=1}^{n} b_i^* b_i > 0.$$

The condition (2) is equivalent to that there exist $c_1, c_2, \cdots, c_n$ in A such that $\sum_{i=1}^{n} c_i b_i = 1$. If A has no unital, we define stable rank of A as stable rank of the unitaization of A. Note that stable rank one condition is equivelent to that the set of invertible elemenets is dense in a given C*-algebra.

Many mathematicians tried to determine stable rank of interesting C*-algebras, in particular, simple unital C*-algebras ([5] [6] [8] [10] [11] [12] [13] [14] [15] [18] [20] [21] [22] etc). For examples, AF C*-algebras and non-commutative tori have stable rank one ([18]), Toeplitz algebra has stable rank two, and Cuntz algebra has an infinity ([19]).

It has been a problem of considerable interest to determine stable rank of a crossed product algebra $A \times_\alpha G$ of a unital C*-algebra A with stable rank one by a finite group G. Blackadar presented this problem in the case that A is an AF C*-algebra ([2]), and constructed a symmetry α on $A = C[0,1] \otimes UHF$ whose crossed product algebra $A \times_\alpha Z_2$ has stable rank two. So, to consider the above problem, we need the assumption of the simplicity on a given C*-algebra A.

In this direction Jeong and the author conclude ([10][11]) that a crossed product algebra $A \times_\alpha G$ has the cancellation property if A is simple with stable rank one and the SP-property. Recall that a C*-algebra A is said to have the SP-property if any non-zero hereditary subalgebra of A has non-zero projection. For example, an AF C*-algebra has the SP-property. Therefore, we could conclude by [1] that a crossed product algebra $A \times_\alpha G$ has stable rank one if we add real rank zero condition to this crossed product algebra, that is, the set of self-adjoint elements with finite spectra in $A \times_\alpha G$ is dense in the set of self-adjoint elements. As Elliott presented a crossed product algebra $UHF \times_\alpha Z_2$ with real rank one, however, we can not always hope that a given crossed product algebra has real rank zero.

In this talk we try to estimate stable rank of a given unital C*-algebra B by stable rank of a C*-subalgebra A with common unit. In case that B is a crossed product algebra of A by a finite group G, $sr(B) \leq sr(A) \times |G|$ ([11]). More generally, we have the following result:

Theorem 1 *Let $1 \in A \subset B$ be unital C*-algebras. Suppose that B is a finitely generated left A-module, that is, there are some n elements $v_1, v_2, \cdots, v_n$ in B such that $\sum_{i=1}^{n} Av_i = B$. Then, $sr(B) \leq sr(A) \times n$.*

2 Stable rank

We prove main theorem with using the technique of matrix algebras. To this end the following lemma is needed.

Lemma 2 (Spatial case of Rieffel[19]) *Let $n \in \mathbf{N}$.*

$$sr(M_n(A)) \leq sr(A).$$

Proof. We will give a sketch of the proof. Suppose that $sr(A) = m$. Take m elements $T_1, \cdots, T_m$ from $M_n(A)$. Set $S = (T_1, T_2, \cdots, T_m)^t$ in $M_{nm,m}(A)$. Let $(a_1, a_2, \cdots, a_{nm})$ be the first row in S. Since $sr(A) = m$, we may assume that there exist $c_2, \cdots, c_{m+1}$ such that

$$c_2a_2 + c_3a_3 + \cdots + c_{m+1}a_{m+1} = 1 - a_1.$$

Consider

$$\begin{pmatrix} 1 & c_2 & \cdots & c_{m+1} & 0 & \cdots & 0 \\ & 1 & & & & & \\ & & \ddots & & & & \\ & & & \ddots & & & \\ & & & & \ddots & & \\ & & & & & & 1 \end{pmatrix} S.$$

Then, the new first row is $(1, b_2, \cdots, b_{nm})^t$. Doing the iteration there is an invertible matrix $R \in M_{nm}(A)$ such that $RS = diag(1, S')$, $S' \in M_{nm-1,n-1}$. By induction there is $U \in M_{n-1,nm-1}$ such that $US' = I_{n-1}$. Note that $||R^{-1}diag(1, S') - S||$ is small.

Write

$$R^{-1}diag(1, S') = (S_1, \cdots, S_m)^t$$
$$diag(1, U)R = (U_1, \cdots, U_m),$$

where $S_1, \cdots, S_m, U_1, \cdots, U_m$ are in $M_n(A)$. Then, we have $||T_i - S_i||$ is small, and $\sum_{i=1}^{m} U_iS_i = I_n$. □

Definition 3 *Define*

$$Lg_n(A) = \{(a_1, a_2, \cdots, a_n) \in A^n | \sum_{i=1}^{n} Aa_i = A\}.$$

Then, $sr(A) \leq n$ if and only if $Lg_n(A)$ is dense in A^n.

Proof of Theorem 1.

We give only the proof of the case of $sr(A) = 1$ and $G = \mathbf{Z_2}$. That is, we will show that $sr(A \times_\alpha Z_2) \leq 2$ for a unital C*-algebra A. In general case we can guess it from the proof of Lemma 2.

Take $a_0 + a_1 u$, $b_0 + b_1 u$ in $A \times_\alpha Z_2$, where u is a unitary implementing α. Let $\varepsilon > 0$ be given. Consider

$$\begin{pmatrix} a_0 + a_1 u \\ b_0 + b_1 u \end{pmatrix} = \begin{pmatrix} a_0 & a_1 \\ b_0 & b_1 \end{pmatrix} \begin{pmatrix} 1 \\ u \end{pmatrix}.$$

Since $sr(M_2(A)) = 1$ by Lemma 2, there exists an invertible element $\begin{pmatrix} c_0 & c_1 \\ d_0 & d_1 \end{pmatrix} \in M_n(A)$ such that

$$|| \begin{pmatrix} a_0 & a_1 \\ b_0 & b_1 \end{pmatrix} - \begin{pmatrix} c_0 & c_1 \\ d_0 & d_1 \end{pmatrix} || < \frac{\varepsilon}{2}.$$

Consider

$$\begin{pmatrix} c_0 & c_1 \\ d_0 & d_1 \end{pmatrix} \begin{pmatrix} 1 \\ u \end{pmatrix} = \begin{pmatrix} c_0 + c_1 u \\ d_0 + d_1 u \end{pmatrix}.$$

Then, $(c_0 + c_1 u, d_0 + d_1 u) \in Lg_2(A \times_\alpha \mathbf{Z_2})$, and $||a_0 + a_1 u - (c_0 + c_1 u)|| < \varepsilon$, $||b_0 + b_1 u - (d_0 + d_1 u)|| < \varepsilon$. Hence, $sr(A \times_\alpha \mathbf{Z_2}) \leq 2$. □

Corollary 4 *Let $1 \in A \subset B$ be a pair of unital C*-algebras, and $E : B \to A$ be a faithful conditional expectation of index-finite type. That is, there exists a quasi-basis $\{v_i^*, v_i\}_{i=1}^n$ such that $x = \sum_{i=1}^{n} E(xv_i^*)v_i$, $\forall x \in B$. Then, $sr(B) \leq sr(A) \times n$.*

Corollary 5 *Let $1 \in A$ be a unital C*-algebra and G be a finite group. Then,*

$$sr(A \times_\alpha G) \leq sr(A) \times |G|.$$

3 Application

Using Corollary 5 we can present an affirmative data to a question of Blackdar[2]:

Question 6 *Let A be a AF C^*-algebra and G be a finite group. Then*

$$sr(A \times_\alpha G) \leq 1.$$

Theorem 7 (Jeong-Osaka[11]) *Let A be a simple unital C^*-algebra with $sr(A) = 1$ and SP-property. If G is a finite group and α is an action of G on A then the crossed product $A \times_\alpha G$ has cancellation.*

Here, a C^*-algebra has SP-*property* if each of its non-zero hereditary C^*-subalgebras contains a non-zero projection.

In particular,

Corollary 8 *Under the assumptions of the above theorem, if $A \times_\alpha G$ has real rank zero, that is, any self-adjoint element can be approximated by a self-adjint element with finite spectra, then $sr(A \times_\alpha G) = 1$.*

Remark 9 *Generally, we can not hope that a given simple crossed product algebra $A \times_\alpha G$ has real rank zero, even if A is UHF , and $G = \mathbf{Z_2}$ [7].*

If one consider a crossed product by the integer group $\mathbf{Z}$ then there is no conditional expectation of index-finite type from the crossed product $A \times_\alpha \mathbf{Z}$ onto A, but we have the following cancellation theorem:

Theorem 10 (Jeong-Osaka[11]) *Let A be a simple unital C^*-algebra with $sr(A) = 1$ and SP-property. If α is an outer action of the integer group $\mathbf{Z}$ on A such that $\alpha_* = id$ on $K_0(A)$ then the crossed product $A \times_\alpha \mathbf{Z}$ has cancellation.*

Example 11 *Simple AF C^*-algebras and non-commutative tori A_θ are examples for C^*-algebras in Theorems 7 and 10.*

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