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Dynamics of Polynomial Automorphisms of $\mathbb{C}^2$: Stable and unstable manifolds.

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Abstract

We study the structure of stable and unstable manifolds. Let a be a saddle point and let $W^u(a)$ be its unstable manifold. There exists a biholomorphic mapping $H : \mathbb{C} \to W^u(a)$. Then each of $H = (h_1, h_2)$ becomes a transcendental entire function. Such a function has many interesting properties, so we are able to show the followings in this paper. An arbitrary 1-dimensional algebraic variety intersects with $W^u(a)$ infinitely countable times. Yoccoz inequality for $H^{-1}(K)$ is satisfied under a weaker condition than connectivity. K^+ and $W^u(a)$ has an interesting relation. Moreover since the method to prove Yoccoz inequality is new to polynomial dynamics of one variable, we can obtain a new proof of an improved Yoccoz inequality which holds with less assumptions.

1 Introduction

In this paper we use a notation $z = (x, y) \in \mathbb{C}^2$ and define $\pi_1(z) = x$, $\pi_2(z) = y$. Let $p_j(y)$ be monic polynomials of $\deg d_j > 1$ for $j = 1, \ldots, m$. We call $g_j(x, y) = (y, p_j(y) - \delta_j x)$ generalized Hénon mappings, where $\delta_j \neq 0$. Moreover we define

$$F = g_m \circ \cdots \circ g_1, \quad \delta = \delta_1 \cdots \delta_m, \quad d = d_1 \cdots d_m.$$

For convenience, we define $F_j = g_j \circ \cdots \circ g_1$.

In [FM] Friedland and Milnor classified polynomial automorphisms of $\mathbb{C}^2$ into three types: affine mapping, elementary mapping, composite of generalized Hénon mappings. They investigated the former two mappings completely. So we study the last one, i.e. F which we have defined.

Easily we obtain $g_j^{-1}(x, y) = \left(\frac{1}{\delta} p_j(x) - \frac{1}{\delta} y, x\right)$. It is similar to $g_j(x, y)$ if x and y are exchanged. Therefore once we obtain a property about F, immediately we can apply it to the case of F^{-1} with a little modification. So we will frequently omit the proofs.

1.1 Definitions and basic properties

We define $K^\pm = \{z \in \mathbb{C}^2 \mid \{F^{\pm n}(z) \mid n \in \mathbb{N}\} \text{ is bounded}\}$, $J^\pm = \partial K^\pm$, $K = K^+ \cap K^-$, $J = J^+ \cap J^-$. According to [BS1], they are closed invariant sets.

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Let a be a k-periodic point and let eigenvalues of $DF^k(a)$ be λ, λ' ($|\lambda| \geq |\lambda'|$). We call a

- a source if $|\lambda|, |\lambda'| > 1$,
- a sink if $0 < |\lambda|, |\lambda'| < 1$,
- a saddle point if $0 < |\lambda'| < 1 < |\lambda|$.

Katok showed in [Ka] that there exist saddle points.

We can assume a is a fixed point, since we can replace F^k by F.

Let $d(\ , \ )$ be an appropriate distance in $\mathbb{C}^2$. For $X \subset \mathbb{C}^2$, define the stable set $W^s(X)$ and the unstable set $W^u(X)$ as follows:

$$W^s(X) = \{z \in \mathbb{C}^2 \mid d(F^n(z), F^n(X)) \to 0 \quad (n \to \infty)\},$$
$$W^u(X) = \{z \in \mathbb{C}^2 \mid d(F^n(z), F^n(X)) \to 0 \quad (n \to -\infty)\}.$$

The next theorem is well-known. See [MNTU, Theorem 6.4.3] for example. The following equations act a main role in applying Nevanlinna theory to dynamical systems.

Theorem 1.1. *Assume a is a fixed point of saddle type and λ, λ' ($|\lambda'| < 1 < |\lambda|$) are eigenvalues of $DF(a)$. Then there exists a biholomorphic mapping $H : \mathbb{C} \to W^u(a)$ such that*

$$F \circ H(t) = H(\lambda t) \quad (t \in \mathbb{C}).$$

Similarly there is a biholomorphic mapping $H' : \mathbb{C} \to W^s(a)$ such that

$$F \circ H'(t) = H'(\lambda' t) \quad (t \in \mathbb{C}).$$

By the theorem we can call stable/unstable set stable/unstable manifold when a is a saddle point.

Remark 1.2. *Even if a is not saddle type, sometimes similar theorems hold. For example see [MNTU, Theorem 6.4.3] and [MNTU, Theorem 6.3.1]. Therefore many assertions in this paper are true for such fixed points.*

Let us recall the notion of *access*. In general, let a be a fixed point and Λ be a component of the compliment of (filled) Julia set (e.g. $K^{+/-}$). Suppose $a \in \partial\Lambda$. Then we say that a is *accessible* from Λ if and only if there exists a curve $\gamma : [0,1] \to \overline{\Lambda}$ which suffices:

$$\gamma(0) = a \quad \text{and} \quad \gamma((0,1]) \subset \Lambda.$$

We call such γ an *access*. Moreover we call γ a *periodic access*, if it satisfies $F^q(\gamma) \subset \gamma$ or $F^q(\gamma) \supset \gamma$, where $q \in \mathbb{N}$ is the period of Λ.

1.2 The main theorems

Of course we compute the order of H at first.

Theorem 2.1. *Each of $H = (h_1, h_2)$ is a transcendental entire function and they are of mean type of order $\rho = \log d / \log |\lambda|$.*

Using the transcendence, we obtain the following.

Theorem 2.11. *Let $P(x,y)$ be a non-constant polynomial of two variables. Then $P \circ H$ has no Picard's exceptional values, i.e. an arbitrary 1-dimensional algebraic variety intersects with $W^{u/s}(a)$ infinitely countable times.*

Suppose $\widetilde{K} = H^{-1}(K^+)$. Then we have the following famous inequality under a weaker assumption.

Theorem 3.9. (Yoccoz inequality). *Assume $\widetilde{K}$ is bridged i.e. the component of $\widetilde{K}$ containing 0 is not a point. Then the following holds.*

$$\frac{\operatorname{Re}\log\lambda}{|\log\lambda - 2\pi i p/q|^2} \geq \frac{Nq}{2\log d},$$

where we choose an appropriate branch of $\log\lambda$.

The above Yoccoz inequality does not need connectivity. Instead, we introduce the notion of *bridge* whose property will be described in Proposition 3.8. It can be conclude that any components of $\widetilde{K}$ are compact unless Yoccoz inequality holds.

In the sequel, we will investigate a relation between K^+ and $\widetilde{K}^+$.

Proposition 4.1. *If a point $z_0 \in W^s(a)$ is accessible from* $\operatorname{int} K^+$ *then $\widetilde{K}^+ = H^{-1}(K^+)$ is bridged. Therefore Yoccoz inequality holds there.*

By the argument, we will show in Example 4.2 that there exists $W^s(a)$ such that any points on it are not accessible from $\operatorname{int} K^+$ though $W^s(a)$ is a dense subset of $\partial \operatorname{int} K^+$.

At last we will prove an improved Yoccoz inequality in dynamics of one variable. Let $P(x)$ be a monic polynomial of one variable and let a be a repelling fixed point whose multiplier is λ. Then there exists an entire function ϕ such that

$$P \circ \phi(t) = \phi(\lambda t) \quad \text{and} \quad \phi(0) = a.$$

Let K be the filled Julia set and define $\widetilde{K} = \phi^{-1}(K)$. Then the following holds.

Theorem 5.5. (Yoccoz inequality). *Assume that $\widetilde{K}$ is bridged, i.e. the component of $\widetilde{K}$ containing 0 is not a point. Then*

$$\frac{\operatorname{Re}\log\lambda}{|\log\lambda - 2\pi i p/q|^2} \geq \frac{Nq}{2\log d}$$

holds, where we choose an appropriate branch of $\log\lambda$.

2 Transcendental entire function

We denote $H = (h_1, h_2)$ and $H' = (h'_1, h'_2)$.

2.1 Transcendence

We recall that the *order* ρ of $f \in \mathcal{O}(\mathbb{C})$ is:

$$\rho = \operatorname{ord} f = \limsup_{r\to\infty} \frac{\log\log \sup_{|x|=r} |f(x)|}{\log r}.$$

Moreover if ρ is finite, the *type* τ is:

$$\tau = \limsup_{r\to\infty} \frac{\log \sup_{|x|=r} |f(x)|}{r^\rho}.$$

We say f is of minimum type, mean type, maximum type when $\tau = 0$, $0 < \tau < \infty$, $\tau = \infty$, respectively.

Theorem 2.1. *h_1, h_2, h_1', h_2' are transcendental entire functions. They are of mean type of orders:*

$$\rho = \operatorname{ord} h_1 = \operatorname{ord} h_2 = \frac{\log d}{\log|\lambda|}, \quad \rho' = \operatorname{ord} h_1' = \operatorname{ord} h_2' = \frac{\log d}{-\log|\lambda'|}.$$

To prove the theorem, we quote the following.

Lemma 2.2. [BS1]. *For $R > 0$, define $V^+ = \{(x,y) \in \mathbb{C}^2 \mid |x| > R, |x| > |y|\}$, $V^- = \{(x,y) \in \mathbb{C}^2 \mid |y| > R, |y| > |x|\}$, $V = \{(x,y) \in \mathbb{C}^2 \mid |x| \le R, |y| \le R\}$. Then for sufficiently large $R > 0$,*

$$\begin{aligned} &K^+ \subset V \cup V^+, \quad F_j(K^+) \subset V \cup V^+ \quad (j = 1, \dots, m-1), \\ &K^- \subset V \cup V^-, \quad F_j(K^-) \subset V \cup V^- \quad (j = 1, \dots, m-1). \end{aligned}$$

Corollary 2.3. *For arbitrary $\varepsilon > 0$, there exists $M > 0$ such that*

$$\begin{aligned} (1-\varepsilon)|x|^{d_m} - M \le |y| \le (1+\varepsilon)|x|^{d_m} + M \qquad & (x,y) \in K^-, \\ (1-\varepsilon)|x|^{d_j} - M \le |y| \le (1+\varepsilon)|x|^{d_j} + M \qquad & (x,y) \in F_j(K^-), \end{aligned}$$

for $j = 1, \dots, m-1$.

Proof. For $(x_1, y_1) \in F_j(K^-)$, there is $(x_0, y_0) \in F_{j-1}(K^-)$ such that $(x_1, y_1) = g_j(x_0, y_0)$. By Lemma 2.2, for arbitrary $\varepsilon > 0$ there is $M > 0$ such that

$$\begin{aligned} |y_1| &= |p_j(y_0) - \delta_j x_0| \le |p_j(y_0)| + |\delta_j| \max\{|y_0|, R\} \\ &\le (1+\varepsilon)|y_0|^{d_j} + M = (1+\varepsilon)|x_1|^{d_j} + M, \\ |y_1| &= |p_j(y_0) - \delta_j x_0| \ge |p_j(y_0)| - |\delta_j| \max\{|y_0|, R\} \\ &\ge (1-\varepsilon)|y_0|^{d_j} - M = (1-\varepsilon)|x_1|^{d_j} - M. \end{aligned}$$

□

To compute the order, we prepare the following. Take $(x_0, y_0) \in K^-$ and define

$$y_n = \pi_2 \circ F^n(x_0, y_0), \quad n = 1, 2, \dots.$$

For simplicity, we set $C_\varepsilon = (1+\varepsilon)^{d_2 \cdots d_m + d_3 \cdots d_m + \cdots + 1}$.

Lemma 2.4. *For arbitrary $\varepsilon > 0$, there exists $M > 0$ such that*

$$|y_n| \le C_\varepsilon^{d^n} \max\{|y_0|^{d^n}, M^{d^n}\} \quad \textit{for any } (x_0, y_0) \in K^-.$$

Proof. We define a more detailed sequence as follows.

$$(x_0, y_0) \xmapsto{g_1} (\widetilde{y}_0, \widetilde{y}_1) \xmapsto{g_2} \cdots \xmapsto{g_m} (\widetilde{y}_{m-1}, \widetilde{y}_m).$$

Note that $(\widetilde{y}_{j-1}, \widetilde{y}_j) \in F_j(K^-)$ for $j = 1, \ldots, m$. According to Corollary 2.3 there exists $M > 1$ such that

$$|\widetilde{y}_j| \le (1+\varepsilon) \max\{|\widetilde{y}_{j-1}|^{d_j}, M^{d_j}\} \quad j = 1, \ldots, m.$$

Then

$$\begin{aligned}
|\widetilde{y}_2| &\le (1+\varepsilon) \max\{|\widetilde{y}_1|^{d_2}, M^{d_2}\} \\
&\le (1+\varepsilon) \max\{((1+\varepsilon)|\widetilde{y}_0|^{d_1})^{d_2}, ((1+\varepsilon)M^{d_1})^{d_2}, M^{d_2}\} \\
&= (1+\varepsilon)^{d_2+1} \max\{|\widetilde{y}_0|^{d_1 d_2}, M^{d_1 d_2}\}.
\end{aligned}$$

By repeating the procedure, we obtain

$$|\widetilde{y}_m| \le C_\varepsilon \max\{|\widetilde{y}_0|^d, M^d\}, \quad \text{i.e.} \quad |y_1| \le C_\varepsilon \max\{|y_0|^d, M^d\}.$$

We have by repeating similarly,

$$|y_n| \le C_\varepsilon^{d^{n-1}+d^{n-1}+\cdots+1} \max\{|y_0|^{d^n}, M^{d^n}\} \le C_\varepsilon^{d^n} \max\{|y_0|^{d^n}, M^{d^n}\}.$$

□

Lemma 2.5. *For arbitrary $0 < \varepsilon < 1$, there exists $M > 0$ such that*

$$\max\{|y_n|, C_{-\varepsilon}^{d^n} M^{d^n}\} \ge C_{-\varepsilon}^{d^n} |y_0|^{d^n} \quad \textit{for any } (x_0, y_0) \in K^-.$$

Proof. We use the same notation $\{\widetilde{y}_0, \ldots, \widetilde{y}_m\}$ as above. According to Corollary 2.3, there exists $M > 1/(1-\varepsilon)^d$ such that

$$(1-\varepsilon)|\widetilde{y}_{j-1}|^{d_j} \le \max\{|\widetilde{y}_j|, (1-\varepsilon)M^{d_j}\} \quad j = 1, \ldots, m.$$

Then

$$\begin{aligned}
(1-\varepsilon)((1-\varepsilon)|\widetilde{y}_0|^{d_1})^{d_2} &\le (1-\varepsilon)(\max\{|\widetilde{y}_1|, (1-\varepsilon)M^{d_1}\})^{d_2} \\
&= \max\{(1-\varepsilon)|\widetilde{y}_1|^{d_2}, (1-\varepsilon)^{d_2+1} M^{d_1 d_2}\} \\
&\le \max\{|\widetilde{y}_2|, (1-\varepsilon)M^{d_2}, (1-\varepsilon)^{d_2+1} M^{d_1 d_2}\} \\
&= \max\{|\widetilde{y}_2|, (1-\varepsilon)^{d_2+1} M^{d_1 d_2}\}.
\end{aligned}$$

By repeating the procedure, we obtain

$$C_{-\varepsilon}|\widetilde{y}_0|^d \le \max\{|\widetilde{y}_m|, C_{-\varepsilon} M^d\}, \quad \text{i.e.} \quad C_{-\varepsilon}|y_0|^d \le \max\{|y_1|, C_{-\varepsilon} M^d\}.$$

Then we have by repeating similarly,

$$\begin{aligned}
C_{-\varepsilon}^{d^{n-1}+d^{n-2}+\cdots+1} |y_0|^{d^n} &\le \max\{|y_n|, C_{-\varepsilon}^{d^{n-1}+d^{n-2}+\cdots+1} M^{d^n}\}. \\
\therefore \quad C_{-\varepsilon}^{d^n} |y_0|^{d^n} &\le \max\{|y_n|, C_{-\varepsilon}^{d^n} M^{d^n}\}.
\end{aligned}$$

□

Proof of Theorem 2.1. We assume the order of h_2 is $\rho = \log d / \log|\lambda|$ and compute the type. If it is of mean type, we see the tentative order is true and h_2 is transcendental.

At first, we show that the type is bounded. By the maximum principle and Theorem 1.1 and Lemma 2.4 we can compute as follows because $H(t) \in K^-$.

$$
\begin{aligned}
\rho &= \limsup_{r\to\infty} \frac{\log \max_{|t|=r} |h_2(t)|}{r^\rho} \\
&\leq \limsup_{n\to\infty} \frac{\log \max_{|t|=|\lambda^{n+1}t_0|} |h_2(t)|}{|\lambda^n t_0|^\rho} \\
&= \limsup_{n\to\infty} \frac{\log \max_{|t|=|\lambda t_0|} |\pi_2 \circ F^n \circ H(t)|}{|\lambda^n t_0|^\rho} \\
&\leq \limsup_{n\to\infty} \frac{\log \max_{|t|=|\lambda t_0|} C_\varepsilon^{d^n} \max\{|h_2(t)|^{d^n}, M^{d^n}\}}{d^n |t_0|^\rho} \\
&= \frac{\log C_\varepsilon + \log \max_{|t|=|\lambda t_0|} \max\{|h_2(t)|, M\}}{|t_0|^\rho} < \infty.
\end{aligned}
$$

In the calculation it is employed that $|\lambda|^\rho = d$.

Secondly, we show that the type is positive. h_2 is not constant because of Corollary 2.3, so we can take the modulus of $y_0 = h_2(t_0)$ as large as we like. Therefore it can be assumed that

$$
|y_n| \geq C_{-\varepsilon}^{d^n} |y_0|^{d^n} \quad n = 1, 2, \ldots
$$

by Lemma 2.5. Then we can compute similarly to the above.

$$
\begin{aligned}
\rho &\geq \limsup_{n\to\infty} \frac{\log |h_2(\lambda^n t_0)|}{|\lambda^n t_0|^\rho} = \limsup_{n\to\infty} \frac{\log |\pi_2 \circ F^n \circ H(t_0)|}{|\lambda^n t_0|^\rho} \\
&\geq \limsup_{n\to\infty} \frac{\log C_{-\varepsilon}^{d^n} |y_0|^{d^n}}{d^n |t_0|^\rho} = \frac{\log C_{-\varepsilon} + \log |y_0|}{|t_0|^\rho} > 0.
\end{aligned}
$$

The rest of the proof is completed by the following lemma. □

Lemma 2.6. *Let τ, τ' be the types of h_1, h_1' respectively. Then $0 < \tau, \tau' < \infty$ and for $j = 0, 1, \ldots, m$*

$$
\begin{aligned}
\limsup_{r\to\infty} \frac{\log \max_{|t|=r} |\pi_1 \circ F_j \circ H(t)|}{r^\rho} &= d_{j-1} \cdots d_0 \tau, \\
\limsup_{r\to\infty} \frac{\log \max_{|t|=r} |\pi_2 \circ F_j \circ H(t)|}{r^\rho} &= d_j \cdots d_0 \tau, \\
\limsup_{r\to\infty} \frac{\log \max_{|t|=r} |\pi_1 \circ F_j \circ H'(t)|}{r^\rho} &= \frac{\tau'}{d_j \cdots d_1}, \\
\limsup_{r\to\infty} \frac{\log \max_{|t|=r} |\pi_2 \circ F_j \circ H'(t)|}{r^\rho} &= \frac{\tau'}{d_{j+1} \cdots d_1},
\end{aligned}
$$

where $d_0 = d_m, d_{m+1} = d_1$. Especially, all are of mean type.

Proof. For $j = 0, \dots, m$, put

$$\alpha_j = \limsup_{r\to\infty} \frac{\log \max_{|t|=r} |\pi_1 \circ F_j \circ H(t)|}{r^\rho},$$
$$\beta_j = \limsup_{r\to\infty} \frac{\log \max_{|t|=r} |\pi_2 \circ F_j \circ H(t)|}{r^\rho}.$$

By Corollary 2.3,

$$\begin{aligned}\beta_j &\le \limsup_{r\to\infty} \frac{\log \max_{|t|=r}\{(1+\varepsilon)|\pi_1 \circ F_j \circ H(t)|^{d_j} + M\}}{r^\rho} \\ &= d_j \limsup_{r\to\infty} \frac{\log \max_{|t|=r} |\pi_1 \circ F_j \circ H(t)|}{r^\rho} = d_j\alpha_j.\end{aligned}$$

Moreover by definition of g_j, we obtain $\beta_j = \alpha_{j+1}$. Therefore we have

$$d_0\alpha_0 \ge \beta_0 = \alpha_1,\ d_1\alpha_1 \ge \beta_1 = \alpha_2, \dots, d_{m-1}\alpha_{m-1} \ge \beta_{m-1} = \alpha_m.$$

On the other hand,

$$\begin{aligned}\alpha_m &= \limsup_{r\to\infty} \frac{\log \max_{|t|=r} |\pi_1 \circ F \circ H(t)|}{r^\rho} = \limsup_{r\to\infty} \frac{\log \max_{|t|=r} |h_1(\lambda t)|}{r^\rho} \\ &= d \limsup_{r\to\infty} \frac{\log \max_{|t|=r} |h_1(t)|}{r^\rho} = d\alpha_0.\end{aligned}$$

By putting the above inequalities and equations together, we have

$$\alpha_0 \ge \frac{1}{d_0}\alpha_1 \ge \cdots \ge \frac{1}{d_{m-1}\cdots d_0}\alpha_m = \frac{d}{d_{m-1}\cdots d_0}\alpha_0 = \alpha_0.$$

It has already known that $0 < \beta_0 < \infty$. It concludes the assertion. □

Remark 2.7. *Similarly, we can compute the lower order of h_2, and it is the same as the order. i.e.*

$$\liminf_{r\to\infty} \frac{\log\log \max_{|t|=r} |h_2|}{\log r} = \frac{\log d}{\log|\lambda|}.$$

2.2 Compositions with functions on $\mathbb{C}^2$

We shall investigate compositions of some kinds of functions and H.

Proposition 2.8. *Let $f \in \mathcal{O}(\mathbb{C}^2)$ be non-constant. Then $f \circ H$ is a transcendental entire function.*

Proof. First we show that $f \circ H(t)$ does not have $t = \infty$ as a pole. By Picard's theorem, for some $y_0 \in \mathbb{C}$ there exist infinitely many $t \in \mathbb{C}$ satisfying $h_2(t) = y_0$. By Corollary 2.3, we see that $\{h_1(t) \mid h_2(t) = y_0\}$ is bounded. Then there is a sequence $\{t_j\}$ such that $t_j \to \infty$ and a limit of $H(t_j)$ exists. Therefore the limit of $f \circ H(t_j)$ also exists. This implies $t = \infty$ is not a pole.

Secondly we prove that $f \circ H$ is not constant. Assume $f \circ H$ is constant. By Picard's theorem, for any $y \in \mathbb{C}$ except for at most one point $y_0 \in \mathbb{C}$ there exist infinitely many t satisfying $h_2(t) = y$. By Corollary 2.3, $\{h_1(t) \mid h_2(t) = y\}$ is bounded. Therefore the set has at least one limit point. On the other hand, $f(h_1(t), y)$ is constant where $h_2(t) = y$. By uniqueness theorem we obtain that $f(\cdot, y)$ is constant for each fixed y. Then $f(\cdot, h_2(t))$ becomes constant, so it is concluded that f is a constant. □

Proposition 2.9. *Let f be a non-constant rational function of two variables, i.e. there are relatively prime polynomials $P(x,y), Q(x,y)$ ($Q \neq 0$) which satisfy $f(x,y) = P(x,y)/Q(x,y)$. Then $f \circ H$ is a transcendental meromorphic function.*

To prove the proposition we prepare a lemma.

Lemma 2.10. *Let $P(x,y)$ be a non-constant polynomial. Then $K^- \cap P^{-1}(0)$ is compact unless empty.*

Proof. Bedford and Smillie showed in [BS1, Proposition 4.2] that for sufficiently large any $n \in \mathbb{N}$, the terms of highest total degree of $P \circ F^n(x,y)$ consist of only power of y and some non-zero coefficient. Then we can have $\{(x,y) \mid P \circ F^n(x,y) = 0\} \subset V \cup V^+$. Hence $K^- \cap \{(x,y) \mid P \circ F^n(x,y) = 0\}$ is compact. Therefore

$$\begin{aligned} K^- \cap \{(x,y) \mid P(x,y) = 0\} &= K^- \cap F^n\left(\{(x,y) \mid P \circ F^n(x,y) = 0\}\right) \\ &= F^n\left(K^- \cap \{(x,y) \mid P \circ F^n(x,y) = 0\}\right) \end{aligned}$$

is compact, too. □

Proof of Proposition 2.9. When Q is constant it reduces to Proposition 2.8. So we assume Q is non-constant. We will show that $t = \infty$ is neither a pole nor a regular point.

At first we prove $f \circ H(t)$ doesn't have $t = \infty$ as a pole. Since $Q \circ H$ is transcendental by Proposition 2.8, there exists $q_0 \neq 0$ such that infinitely many $t \in \mathbb{C}$ satisfies $Q \circ H(t) = q_0$ by Picard's theorem. Because the image of H is included in K^-, it can be seen that $\{H(t) \mid Q \circ H(t) = q_0\}$ is bounded according to the previous lemma. $P \circ H$ is bounded on the set though t can tend to ∞. Therefore $t = \infty$ isn't a pole.

Similarly it can be shown that $t = \infty$ is not zero point of $f \circ H(t)$.

Otherwise assume that $\lim_{t \to \infty} f \circ H(t) = c$, ($c \neq 0, \infty$). Then if we define $\widetilde{f}(x,y) = f(x,y) - c$, we see that $\widetilde{f} \circ H(t)$ has $t = \infty$ as a zero point. It contradicts with the previous statement. □

Theorem 2.11. *Let $P(x,y)$ be a non-constant polynomial of two variables. Then $P \circ H$ has no Picard's exceptional values, i.e. an arbitrary 1-dimensional algebraic variety intersects with $W^{u/s}(a)$ infinitely countable times. Further the intersection is bounded.*

The theorem insists that $\mathbb{C}^2$ is filled with $W^{u/s}(a)$ in the sense of algebraic variety.

To prove the theorem, we use the following lemma obtained easily by combining [Nr, 21.] and [Nr, 59.] (or [O, Theorem 3.3] and [O, Theorem 9.2]).

Lemma 2.12. *Let $f_1, \ldots, f_n$ be meromorphic functions on $\mathbb{C}$. Assume all of them have finite zero points and finite poles. If $c_1, \ldots, c_n \in \mathbb{C} \setminus \{0\}$ satisfies*

$$c_1 f_1 + \cdots + c_n f_n = 0,$$

then for some $h \neq k$, f_h/f_k becomes a rational function.

Proof of Theorem 2.11. We imitate a technique used in [Nt, Chapter 5]. It sufficient to show that a composition of each non-constant irreducible polynomial and H has infinite zero points.

First we show that at most two non-constant irreducible and relatively prime polynomials can have 0 as Picard's exceptional value when they are composed with H.

Let P_1, P_2, P_3 be non-constant, irreducible and relatively prime polynomials of two variables. Assume that $P_1 \circ H, P_2 \circ H, P_3 \circ H$ have finite zero points. Put

$$w_1 = P_1(h_1, h_2), \quad w_2 = P_2(h_1, h_2), \quad w_3 = P_3(h_1, h_2).$$

Then w_1, w_2, w_3 are entire functions which have finite zero points. On the other hand, we can utilize the polynomial ring's theory to eliminate h_1, h_2 in the above equations. In fact, by the system of resultants there exists a non-constant polynomial Q which satisfies

$$Q(w_1, w_2, w_3) = 0.$$

Then we have by expanding Q,

$$Q(w_1, w_2, w_3) = \sum_{i,j,k} q_{ijk} w_1^i w_2^j w_3^k = 0.$$

Since each term has finite zero points and no poles, we can use the previous lemma and obtain that there exist $(i_1, j_1, k_1) \neq (i_2, j_2, k_2)$ such that

$$\frac{w_1^{i_1} w_2^{j_1} w_3^{k_1}}{w_1^{i_2} w_2^{j_2} w_3^{k_2}} = P_1(H)^{i_1 - i_2} P_2(H)^{j_1 - j_2} P_3(H)^{k_1 - k_2}$$

is rational. But it contradicts with Proposition 2.9.

Secondly we prove that the composition of each non-constant irreducible polynomial and H can not have 0 as Picard's exceptional value.

Assume P is a non-constant irreducible polynomial and $P \circ H$ has finite zero points. Then

$$P \circ F^n \circ H(t) = P \circ H(\lambda^n t) \quad (n \in \mathbb{Z})$$

also have 0 as Picard's exceptional value. Moreover $P \circ F^n$ are irreducible because F is bijective. Bedford and Smillie showed in [BS1, Proposition 4.2] that there is $n \in \mathbb{N}$ such that the terms of highest total degree of $P \circ F^n(x, y)$ consist of only power of y and some non-zero coefficient. Therefore $P \circ F^n$, $P \circ F^{n+1}$, $P \circ F^{n+2}$ are different and irreducible and relatively prime. It contradicts with the first assertion.

The last statement is clear because of Lemma 2.10. $\square$

3 Unstable slice

We denote $\widetilde{K} = H^{-1}(K^+) = H^{-1}(K)$ and call it unstable slice. $\widetilde{K}$ is invariant under $t \mapsto \lambda t$ by Theorem 1.1.

3.1 Preliminaries

In [BS1] Bedford and Smillie showed that $G^+(z) = \lim_{n\to\infty} \frac{1}{d^n} \log^+ ||F^n(z)||$ is plurisubharmonic and continuous. It is positive and pluriharmonic in $\mathbb{C}^2 \setminus K^+$, and vanishes on K^+. Therefore $\widetilde{K} = \{t \in \mathbb{C} \mid G^+ \circ H(t) = 0\}$. For simplicity we denote $u^+(t) = G^+ \circ H(t)$. $\widetilde{K}$ has positive capacity in any neighborhood of arbitrary point in $\widetilde{K}$ because u^+ is a non-negative subharmonic function. Since $G^+ \circ F = d \cdot G^+$, we have

$$u^+(\lambda t) = d \cdot u^+(t) \quad (t \in \mathbb{C}).$$

We can see that $\max_{|t|=r} u^+(t) > 0$ for any $r > 0$.

Proposition 3.1. *For a subharmonic function u on $\mathbb{C}$, we define its order as* $\operatorname{ord} u = \limsup_{r\to\infty} \log \max_{|t|=r} u(t)/\log r$. *Then*

$$\rho = \operatorname{ord} u^+ = \frac{\log d}{\log |\lambda|}.$$

Proof. Using the maximum principle, we can compute simply as follows.

$$\rho \leq \limsup_{n\to\infty} \frac{\log \max_{|t|=|\lambda|^{n+1}} u^+(t)}{\log |\lambda|^n} = \limsup_{n\to\infty} \frac{\log \max_{|t|=1} d^{n+1} u^+(t)}{n \log |\lambda|} = \frac{\log d}{\log |\lambda|},$$

$$\rho \geq \limsup_{n\to\infty} \frac{\log \max_{|t|=|\lambda|^n} u^+(t)}{\log |\lambda|^n} = \limsup_{n\to\infty} \frac{\log \max_{|t|=1} d^n u^+(t)}{n \log |\lambda|} = \frac{\log d}{\log |\lambda|}.$$

□

Throughout this section, we use the following quoted from [H].

Suppose $u(re^{i\theta})$ be an upper semi-continuous function of real θ. Let D be the set of all points $re^{i\theta_0}$ such that $u(re^{i\theta})$ has a positive lower bound in a real neighborhood of θ_0. Then we define $\theta(r)$ as follows. If $u(t) > 0$ for some $t \in \{|t| = r\} \setminus \overline{D \cap \{|t| = r\}}$, then $\theta(r) = 0$. If $\{|t| = r\} \subset D$ or $u|_{\{|t|=r\}} \equiv 0$, then $\theta(r) = +\infty$. Otherwise, we define

$$\theta(r) = \frac{1}{r}(\text{the maximun length of the components of } D \cap \{|t| = r\}).$$

Then the following holds. We call it *Tsuji inequality* according to Hayman.

Theorem 3.2. [H, Theorem 8.3.] **(Tsuji inequality).** *Let u be a non-negative and non-constant subharmonic function on $\mathbb{C}$. Define $\theta(r)$ as above.*

Then for $1/e \leq \kappa < 1$ and $r_0 \leq \kappa^2 r$, we have

$$\log \max_{|t|=r} u(t) \geq \pi \int_{r_0/\kappa}^{\kappa r} \frac{dr}{r\theta(r)} + \log \max_{|t|=r_0} u(t) + \log \frac{(1-\kappa)^{3/2}}{6}.$$

In the previous versions of this paper, [T, Theorem III. 68.] and the raw h_2 were used instead of the above inequality and u^+. The author replaced them since Bedford advised to use $G^+ \circ H$, and Maegawa told that K may be unbounded in $\mathbb{C}^3$ case.

3.2 The case of broken $\widetilde{K}$

Let us investigate the simplest case. For $A \subset \mathbb{C}$ and $r > 0$, define $1_A(r)$ as follows:

$$1_A(r) = \begin{cases} 1 & \text{if } A \cap \{|t| = r\} \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 3.3. *Assume $\rho < 1/2$. Then for any $r_0 > 0$,*

$$\frac{1}{\log|\lambda|} \int_{r_0}^{|\lambda| r_0} \frac{1_{\widetilde{K}}(r)}{r} dr \leq 2\rho.$$

Especially any components of $\widetilde{K}$ are compact and $\mathbb{C} \setminus \widetilde{K}$ is connected.

Proof. The essence of the proof is Wiman's theorem. We use Tsuji inequality which implies it. In fact, a precise form of Wiman's theorem is proved in [T, Theorem III. 72.].

If we apply Tsuji inequality to u^+, we obtain

$$\log \max_{|t|=r} u^+(t) \geq \pi \int_1^{\kappa r} \frac{dr}{r\theta(r)} - \text{const.} \geq \int_1^{\kappa r} \frac{1_{\widetilde{K}(r)}}{2r} dr - \text{const.}$$

Take $l \in \mathbb{N}$ such that $|\lambda|^l r_0 \geq 1$. Then

$$\begin{aligned} \rho &= \limsup_{r\to\infty} \frac{\log \max_{|t|=r} u^+(t)}{\log r} \geq \limsup_{r\to\infty} \frac{1}{\log r} \int_1^{\kappa r} \frac{1_{\widetilde{K}(r)}}{2r} dr \\ &\geq \limsup_{n\to\infty} \frac{1}{\log |\lambda|^n r_0} \int_{|\lambda|^l r_0}^{\kappa |\lambda|^n r_0} \frac{1_{\widetilde{K}(r)}}{2r} dr \\ &= \limsup_{n\to\infty} \frac{1}{n \log|\lambda| + \log r_0} \sum_{j=l}^{n-1} \int_{|\lambda|^j r_0}^{|\lambda|^{j+1} r_0} \frac{1_{\widetilde{K}(r)}}{2r} dr \\ &= \limsup_{n\to\infty} \frac{n-l}{n \log|\lambda| + \log r_0} \int_{r_0}^{|\lambda| r_0} \frac{1_{\widetilde{K}(r)}}{2r} dr = \frac{1}{\log|\lambda|} \int_{r_0}^{|\lambda| r_0} \frac{1_{\widetilde{K}(r)}}{2r} dr. \end{aligned}$$

□

3.3 Yoccoz inequality

We examine the structure of the components of $\mathbb{C} \setminus \widetilde{K}$.

If the number of components of $\mathbb{C} \setminus \widetilde{K}$ is infinite, the structure is too complex. But fortunately we can obtain the following theorem.

Theorem 3.4. *The number of components of $\mathbb{C} \setminus \widetilde{K}$ never exceeds* $\max\{2\rho, 1\}$. *Therefore every component of $\mathbb{C} \setminus \widetilde{K}$ is periodic.*

The theorem is clear because of a variation of Denjoy-Carleman-Ahlfors Theorem, i.e. for an arbitrary non-negative subharmonic function u on $\mathbb{C}$ the number of the components of $\{t \in \mathbb{C} \mid u(t) > 0\}$ does not exceed $\max\{2\rho, 1\}$. But we prepare the following as a guide to prove Yoccoz inequality.

Proof. Assume the number of the components is greater than 1. We take n components $U_1, \ldots, U_n$, where $1 < n < \infty$. We want to show $n \leq 2\rho$.

Note that $\mathbb{C} \setminus \widetilde{K} = \{u^+(t) > 0\}$. We put $u_j(t) = u^+(t)$ for $t \in U_j$, and $u_j(t) = 0$ otherwise. Then $\{u_j\}_{j=1}^n$ are subharmonic. Define $\theta_j(r)$ for each u_j. We apply Tsuji inequality to u_j, we have

$$\log \max_{|t|=r} u^+(t) \geq \frac{1}{n} \sum_{j=1}^{n} \log \max_{|t|=r} u_j(t) \geq \frac{1}{n} \sum_{j=1}^{n} \pi \int_1^{\kappa r} \frac{dr}{r\theta_j(r)} - \text{const.}$$

Since $n \geq 2$, $\sum \theta_j \leq 2\pi$. By Schwarz inequality, we have

$$n^2 = \left(\sum \frac{\sqrt{\theta_j}}{\sqrt{\theta_j}} \right)^2 \leq \left(\sum \theta_j \right) \left(\sum \frac{1}{\theta_j} \right) \leq 2\pi \sum \frac{1}{\theta_j}.$$

Therefore we obtain

$$\log \max_{|t|=r} u^+(t) \geq \frac{n}{2} \log \kappa r - \text{const.}$$

By the definition of ρ, we can conclude $\rho \geq n/2$ easily. □

Corollary 3.5. 0 *is accessible from arbitrary component of* $\mathbb{C} \setminus \widetilde{K}$. *Moreover the access* γ *can be periodic, i.e. if* q *is the period of the component,* γ *satisfies* $\gamma([0,1]) \subset \lambda^q \cdot \gamma([0,1])$.

Especially, every saddle point is accessible from $\mathbb{C}^2 \setminus K^{\pm}$.

Proof. Take arbitrary $t_0 \in \mathbb{C} \setminus \widetilde{K}$ and fix it. We will show there exists the above γ such that $\gamma(1) = t_0$.

Theorem 3.4 implies that all components of $\mathbb{C} \setminus \widetilde{K}$ are periodic of the same period. Let q be the period. Then t_0/λ^q is also contained in the same component. Therefore there exists a curve $\gamma : [\frac{1}{2}, 1] \to \mathbb{C} \setminus \widetilde{K}$ such that $\gamma(1) = t_0$ and $\gamma(\frac{1}{2}) = t_0/\lambda^q$.

Let us extend γ as follows. For $0 \leq \xi \leq 1$, we define

$$\gamma(\xi) = \begin{cases} 0 & \text{if } \xi = 0, \\ \frac{1}{\lambda^{qn}} \gamma(2^n \xi) & \text{for } n \in \mathbb{N} \cup \{0\} \text{ such that } \frac{1}{2} \leq 2^n \xi \leq 1. \end{cases}$$

Then γ is well-defined and continuous, and it satisfies $\gamma([0,1]) \subset \lambda^q \gamma([0,1])$. □

Let us prepare for *Yoccoz inequality.* Theorem 3.4 tells that all components of $\mathbb{C} \setminus \widetilde{K}$ are periodic and have the same period under $t \mapsto \lambda t$.

Definition 3.6. [BxH]. *Assume* $\mathbb{C} \setminus \widetilde{K}$ *has* q' *components. Because all components have the same period, suppose a component move to* p'*-th component under* $t \mapsto \lambda t$, *counting counterclockwise, where* $0 \leq p' < q'$. *Then let* $p'/q' = p/q$ *by reduction and let* N *be the greatest common divider of* p' *and* q'.

For example, see the figure 1. The bold curves show a model of $\widetilde{K}$. $\mathbb{C} \setminus \widetilde{K}$ has $q' = 6$ components. The three arrows represent $t \mapsto \lambda t$. Each component moves to $p' = 2$-nd component counting counterclockwise. Then the period of each component is $q = 3$ and there are $N = 2$ cycles. Each component moves $p = 1$-st component in the cycle.

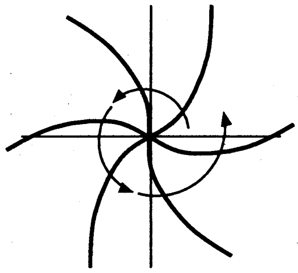

Figure 1: A model of an unstable slice.

Definition 3.7. *Let A be a subset of $\mathbb{C}$. When the component of A containing 0 is unbounded, we call the component a* bridge. *If A has a bridge, we say that A is* bridged.

Proposition 3.8. *The following three conditions are equivalent.*

(1) *$\widetilde{K}$ is bridged.*

(2) *The component of $\widetilde{K}$ containing 0 is not a point.*

(3) *Some component of $\widetilde{K}$ is unbounded.*

Proof. In (2), let A be the component containing 0. Since A is invariant under $t \mapsto \lambda t$, the left side of

$$\bigcup_{j=0}^{\infty} \lambda^j A \subset \widetilde{K}$$

is connected and unbounded. It implies (1).

In (3), let A be an unbounded component. Define

$$B = \overline{\bigcup_{j=1}^{\infty} \frac{1}{\lambda^j} A} \subset \widetilde{K}.$$

B contains 0. Let $B(0)$ be its component containing 0. Let us show $B(0)$ is unbounded. Assume $B(0)$ is compact, then there is a closed curve Γ which surrounds $B(0)$ and never intersects B. But $\frac{1}{\lambda^j} A \cap \Gamma \neq \emptyset$ for sufficiently large $j \in \mathbb{N}$. It contradicts. $\square$

In [BxH] Buff and Hubbard have proved Yoccoz inequality on $W^u(a)$ when $\widetilde{K}$ is connected. The following theorem is slightly improved because it does not need the connectivity. Instead, we need the notion of *bridge*. The figure 2 is a case in which an unstable slice is not connected but bridged. The right vertex is the origin. Ushiki gave the Hénon map in a videocassette. But its bridgedness and non-connectedness have not been proved mathematically yet.

Theorem 3.9. (Yoccoz inequality). *Assume that $\widetilde{K}$ is bridged, i.e. the component of $\widetilde{K}$ containing 0 is not a point. Then*

$$\frac{\operatorname{Re}\log\lambda}{|\log\lambda - 2\pi i p/q|^2} \geq \frac{Nq}{2\log d}$$

holds, where we choose an appropriate branch of $\log\lambda$.

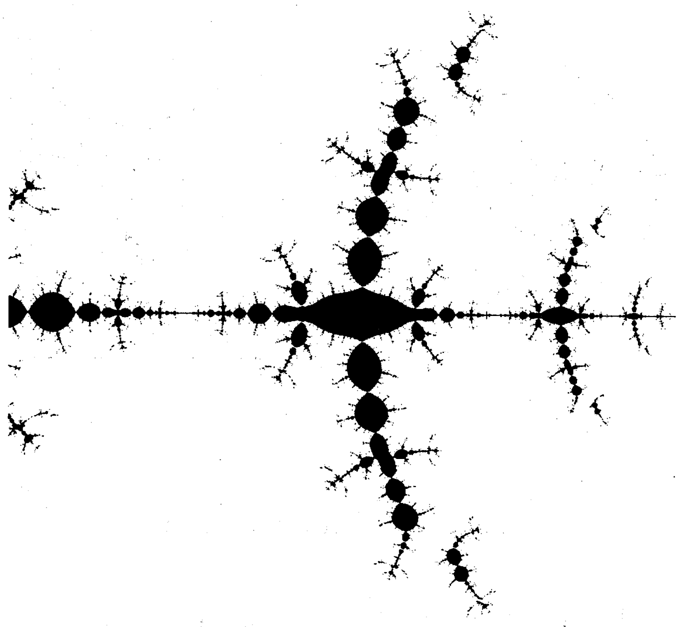

Figure 2: An unstable slice for $F(x,y) = (y, y^2 - 0.9 + 0.4x)$.

Remark 3.10. *In Theorem 3.3, we have shown a sufficiency criterion that all components of $\widetilde{K}$ become compact. The above theorem improves the criterion slightly. In fact, given d, λ. If any p, q, N cannot satisfy Yoccoz inequality, there exist no bridges, i.e. any components of $\widetilde{K}$ are compact.*

After the author had proved Theorem 3.4, Shishikura advised to generalize the method to prove Yoccoz inequality. Therefore the following proof is similar to the proof of Theorem 3.4 and is independent of proofs by torus.

Proof. The way to prove is to transform t-plane into s-plane by logarithm and apply Tsuji inequality on s-plane.

$\mathbb{C} \setminus \widetilde{K}$ can be classified into cycles. Then we take one component from each cycle and name them $U_1, \ldots, U_N$.

Define $v(t) = \max\{u^+(t) - 1, 0\}$ and $D = \{t \in \mathbb{C} \mid v(t) > 0\}$. Let $D_1, \ldots, D_N$ be components of D such that D_j is a subset of U_j. Define $t = e^s$, i.e. $s = \log t$. Let D'_j be a connected image of each D_j. Since $\widetilde{K}$ is bridged, the transformation is well-defined.

Then

$$D'_j \ni s \mapsto s + q \log \lambda - 2\pi i p \in D'_j \tag{3.1}$$

is well-defined for an appropriate branch of $\log \lambda$. In fact, see Figure 3. In the right, the orbit of D'_j and their other branches are illustrated. Suppose A moves to B in the same component on t-plane by multiplying by λ^q. To move A to B on s-plane, we should add $q \log \lambda$ to s. Moreover, if we subtract $2\pi i p$, B moves to B' and returns to the same component involving A.

Therefore, each D'_j is a domain distributing along a line whose direction is $\log \lambda - 2\pi i p/q$, i.e.

$$\left| \operatorname{Re} s - \frac{\operatorname{Re} \log \lambda}{|\log \lambda - 2\pi i p/q|} |s| \right| \text{ is bounded for } s \in \bigcup D'_j. \tag{3.2}$$

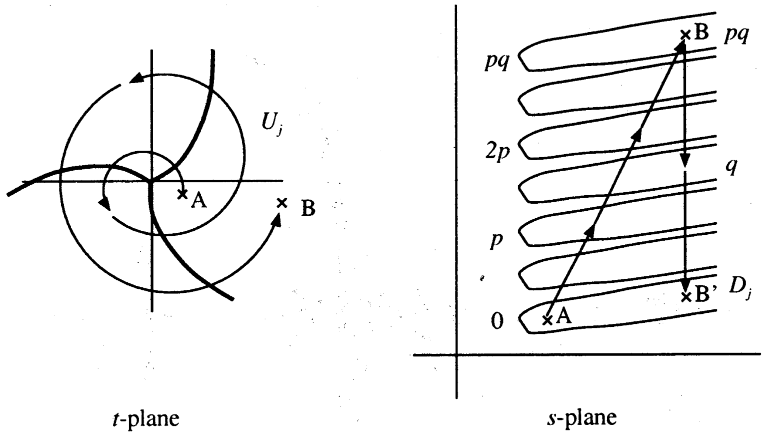

Figure 3: t-plane and s-plane.

On the other hand, since a circle in t-plane centered at 0 is mapped to a 2π-length segment parallel to the imaginary axis in s-plane, the ordinary line measure of $\{\mathrm{Re}\, s = \text{const.}\} \cap \bigcup_j D'_j$ is at most $2\pi/q$ in average. Precisely speaking, if we let len() be the line measure, for any $\xi \in \mathbb{R}$

$$\mathrm{len}\left(\{\mathrm{Re}\, s = \xi\} \cap \bigcup_{j=1}^{N} \bigcup_{n=0}^{q-1} \log(\lambda^n U_j)\right) \leq 2\pi,$$

because $\{\lambda^n U_j\}$ are mutually disjoint for $n = 0, \ldots, q-1, j = 1, \ldots, N$. Further we employ (3.1), we can see that U_j is invariant under $s \mapsto s + q \log \lambda - 2\pi i p$, so we obtain by integration

$$\begin{aligned}
2\pi \cdot q \,\mathrm{Re} \log \lambda &\geq \int_{\xi}^{\xi + q\, \mathrm{Re}\log\lambda} \mathrm{len}\left(\{\mathrm{Re}\, s = \xi\} \cap \bigcup_{j=1}^{N} \bigcup_{n=0}^{q-1} \log(\lambda^n U_j)\right) d\xi \\
&= q \int_{\xi}^{\xi + q\, \mathrm{Re}\log\lambda} \mathrm{len}\left(\{\mathrm{Re}\, s = \xi\} \cap \bigcup_{j=1}^{N} \log(U_j)\right) d\xi \\
&\geq q \int_{\xi}^{\xi + q\, \mathrm{Re}\log\lambda} \mathrm{len}\left(\{\mathrm{Re}\, s = \xi\} \cap \bigcup_{j=1}^{N} D'_j\right) d\xi.
\end{aligned}$$

Therefore

$$\frac{1}{q \,\mathrm{Re}\log\lambda} \int_{\xi}^{\xi + q\, \mathrm{Re}\log\lambda} \mathrm{len}\left(\{\mathrm{Re}\, s = \xi\} \cap \bigcup_{j=1}^{N} D'_j\right) d\xi \leq \frac{2\pi}{q}.$$

We define subharmonic functions $\{w_j\}_{j=1}^{N}$ on s-plane. $w_j(s) = v(e^s)$ for $t \in D'_j$, and $w_j(s) = 0$ otherwise. Now, we apply Tsuji inequality to $w_j(s)$.

$$\sum_{j=1}^{N} \log \max_{|s|=r} w_j(s) \geq \sum_{j=1}^{N} \pi \int_{1}^{\kappa r} \frac{dr}{r\theta_j(r)} - \text{const.}, \tag{3.3}$$

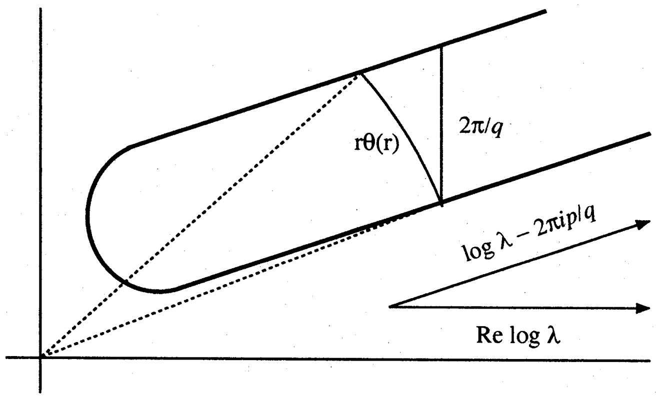

Figure 4: The area of $\bigcup D'_j$

where we define $\theta_j(r)$ for each D'_j. Note that $r = |s|$. We can have $\theta_j(r) < \infty$ for sufficiently large r.

We compute the right side at first. By Schwarz inequality, we obtain

$$N^2 = \left(\sum_{j=1}^{N} \frac{\sqrt{r\theta_j}}{\sqrt{r\theta_j}}\right)^2 \le \left(\sum r\theta_j\right)\left(\sum \frac{1}{r\theta_j}\right),$$

$$(\kappa r - 1)^2 = \left(\int_1^{\kappa r} \frac{\sqrt{\sum r\theta_j}}{\sqrt{\sum r\theta_j}} dr\right)^2 \le \left(\int_1^{\kappa r} \sum r\theta_j \, dr\right)\left(\int_1^{\kappa r} \frac{dr}{\sum r\theta_j}\right).$$

Therefore

$$\sum \pi \int_1^{\kappa r} \frac{dr}{r\theta_j(r)} \ge \pi N^2 \int_1^{\kappa r} \frac{dr}{\sum r\theta_j(r)} \ge \frac{\pi N^2 (\kappa r - 1)^2}{\int_1^{\kappa r} \sum r\theta_j(r) dr}.$$

Recall (3.2) and $\mathrm{len}\left(\{\mathrm{Re}\, s = \mathrm{const.}\} \cap \bigcup D'_j\right) \le 2\pi/q$ in average. See the figure 4. Then we find that the area of $\bigcup D_j$ up to the radius r is approximately bounded by the area of $\bigcup D_j$ up to the real coordinate $\frac{\mathrm{Re}\log\lambda}{|\log\lambda - 2\pi i p/q|} r$. Therefore we obtain

$$\begin{aligned} \int_1^{\kappa r} \sum r\theta_j(r) dr &\le \int_0^{\frac{\mathrm{Re}\log\lambda}{|\log\lambda - 2\pi i p/q|}\kappa r} \mathrm{len}\left(\{\mathrm{Re}\, s = \xi\} \cap \bigcup D'_j\right) d\xi + \mathrm{const.} \\ &\le \frac{2\pi}{q} \frac{\mathrm{Re}\log\lambda}{|\log\lambda - 2\pi i p/q|} \kappa (r + \mathrm{const.}). \end{aligned}$$

Hence, the right side of (3.3) can be estimated as:

$$\sum_{j=1}^{N} \pi \int_1^{\kappa r} \frac{dr}{r\theta_j(r)} \ge \frac{N^2 q |\log\lambda - 2\pi i p/q|}{2\,\mathrm{Re}\log\lambda} \frac{(\kappa r - 1)^2}{\kappa (r + \mathrm{const.})} - \mathrm{const.} \tag{3.4}$$

Secondly, let us estimate the left side of (3.3). Note that $|e^s| = e^{\mathrm{Re}\, s}$.

$$\begin{aligned} \sum_{j=1}^{N} \log \max_{|s|=r} w_j(s) &\le N \log \max_{s \in \bigcup D'_j, |s|=r} v(e^s) \\ &\le \max_{s \in \bigcup D'_j, |s|=r} N\, \mathrm{Re}\, s \cdot \frac{\log|u^+(e^s)|}{\log|e^s|}. \end{aligned} \tag{3.5}$$

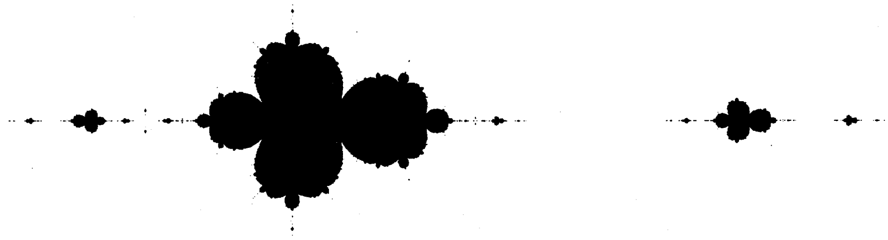

Figure 5: An unstable slice for $F(x,y) = (y, y^2 - 1.37 - 0.36x)$.

We put the above inequalities (3.3), (3.4), (3.5) together and obtain

$$\max_{s \in \bigcup D'_j, |s|=r} N \operatorname{Re} s \cdot \frac{\log u^+(e^s)}{\log |e^s|} \geq \frac{N^2 q |\log \lambda - 2\pi i p/q|}{2 \operatorname{Re} \log \lambda} \frac{(\kappa r - 1)^2}{\kappa (r + \text{const.})} - \text{const.}$$

Divide the both sides by r and let $r \to \infty$, we have

$$N \frac{\operatorname{Re} \log \lambda}{|\log \lambda - 2\pi i p/q|} \operatorname{ord} u^+ \geq \frac{N^2 q |\log \lambda - 2\pi i p/q|}{2 \operatorname{Re} \log \lambda} \kappa,$$

because $\operatorname{Re} s/|s|$ tends to $\operatorname{Re} \log \lambda / |\log \lambda - 2\pi i p/q|$. Then we employ that $\operatorname{ord} u^+ = \log d / \operatorname{Re} \log \lambda$ and that κ is arbitrary $(1/e \leq \kappa < 1)$, we obtain

$$\frac{N \log d}{|\log \lambda - 2\pi i p/q|} \geq \frac{N^2 q |\log \lambda - 2\pi i p/q|}{2 \operatorname{Re} \log \lambda}.$$

It reduces to Yoccoz inequality. □

It should be mentioned that Yoccoz inequality has a deep relation with Ahlfors' Spiral Theorem. See [H, Theorem 8.21.].

We have shown that if $\widetilde{K}$ is bridged then Yoccoz inequality holds. But the converse is not true. In fact, Buff and Hubbard gave an example in [BxH] in which Yoccoz inequality holds and any components of $\widetilde{K}$ are compact. The Hénon map is

$$F(x,y) = (y, y^2 - 1.37 - 0.36x).$$

It has two fixed points, $x = y \doteqdot -0.674$ and $x = y \doteqdot 2.034$. When $x = y \doteqdot 2.034$, the eigenvalues of DF are $\lambda \doteqdot 3.977$ and $\lambda' \doteqdot 0.091$. Therefore the point is of saddle type. On the other hand, the *order* is:

$$\rho = \frac{\log d}{\log |\lambda|} \doteqdot 0.502 > \frac{1}{2}.$$

We can see that Yoccoz inequality holds in this case. But according to the figure 5, any component of $\widetilde{K}$ are compact. The right vertex is the origin.

4 Collision

Let a be a fixed point of saddle type. Suppose a connected closed subset of K^+ meets $W^s(a)$. Then by iteration the set runs to a along $W^s(a)$ and collides with $W^u(a)$ as a limit. Marks of the set will be left on the unstable manifold. The marks are subsets of K^+. In this section we investigate how the set collides.

4.1 Explanation

Let us describe precisely. Assume $z_0 \in W^s(a)$ is accessible from $\operatorname{int} K^+$, i.e. there exists a curve $\gamma : [0,1] \to K^+$ such that

$$\gamma(0) = z_0 \quad \text{and} \quad \gamma((0,1]) \subset \operatorname{int} K^+.$$

The z_0 runs to a along $W^s(a)$ by iteration.

On the other hand, F can be regularized around a as follows. Refer to [MNTU, Theorem 6.4.1]. There exists a local biholomorphic mapping Φ such that $\Phi(0) = a$ and it satisfies

$$\widetilde{F}(x,y) = \Phi^{-1} \circ F \circ \Phi(x,y) = (\lambda' x + xy\alpha(x,y), \lambda y + xy\beta(x,y)) \tag{4.1}$$

in a neighborhood of 0, where λ, λ' $(0 < |\lambda'| < 1 < |\lambda|)$ are eigenvalues of $DF(a)$ and α, β are holomorphic functions around 0. We may assume $\widetilde{F}$ is holomorphic in a neighborhood of $\overline{\mathbb{D}}_{r_0}^2$ for some $r_0 > 0$, where $\mathbb{D}_{r_0} = \{x \in \mathbb{C} \mid |x| < r_0\}$. Clearly we have $\Phi(\mathbb{D}_{r_0} \times \{0\}) \subset W^s(a)$ and $\Phi(\{0\} \times \mathbb{D}_{r_0}) \subset W^u(a)$.

Let us study the behavior of $F^n(\gamma)$. For some n_0, $F^{n_0}(z_0) \in \Phi(\mathbb{D}_{r_0} \times \{0\})$. We denote $(x_0, 0) = \Phi^{-1} \circ F^{n_0}(z_0)$. Define $L_j \subset \overline{\mathbb{D}}_{r_0}^2$ $(j = 0, 1, \ldots)$ as follows.

$$\begin{aligned} L_0 &= \text{the component of } \Phi^{-1}(F^{n_0}(\gamma) \cap \Phi(\overline{\mathbb{D}}_{r_0}^2)) \text{ containing } (x_0, 0), \\ L_{j+1} &= \text{the component of } \widetilde{F}(L_j) \cap \overline{\mathbb{D}}_{r_0}^2 \text{ containing } \widetilde{F}^{j+1}(x_0, 0). \end{aligned}$$

Suppose $r_0 > 0$ is sufficiently small. By the regular form (4.1), it can be seen that L_j approaches y-axis uniformly when j tends to ∞. In fact, choose small $\varepsilon > 0$ and $r_0 > 0$ so that $|\lambda'| + r_0|\alpha(x,y)| < 1 - \varepsilon$ holds on $\overline{\mathbb{D}}_{r_0}^2$. Then

$$|\lambda' x + xy\alpha(x,y)| \le (|\lambda'| + r_0|\alpha(x,y)|)|x| < (1 - \varepsilon)|x|.$$

It reduces to the assertion. Furthermore L_j stretches, i.e. there is j_0 for any $j \ge j_0$, $\max |\pi_2(L_j)| = r_0$. It can be shown similarly.

Define $L \subset \{0\} \times \overline{\mathbb{D}}_{r_0}$ as follows.

$$z \in L \Longleftrightarrow \liminf_{j \to \infty} d(z, L_j) = 0 \Longleftrightarrow z \in \bigcap_{n=0}^{\infty} \overline{\bigcup_{j=n}^{\infty} L_j}.$$

It is clear that $\Phi(L) \subset K^+$. The following holds.

Proposition 4.1. *Assume a point $z_0 \in W^s(a)$ is accessible from* $\operatorname{int} K^+$. *Then L is a connected subset of* $\{0\} \times \overline{\mathbb{D}}_{r_0}$. *Hence $\widetilde{K}^+ = H^{-1}(K^+)$ is bridged. Therefore Yoccoz inequality holds there.*

Proof. Assume L is disconnected. Because L is a compact set contained in y-axis, the components can be separated by a closed curve Γ contained in y-axis. Take $z_1 \in L$ such that z_1 and 0 are in opposite sides of Γ each other. By definition we can choose a subsequence $\{L_{j_k}\}$ so that

$$d(z_1, L_{j_k}) \le \frac{1}{k} d(z_1, \Gamma) \quad (k \in \mathbb{N}).$$

By the way, because $\{L_{j_k}\}$ is connected, we have

$$(\overline{\mathbb{D}}_{r_0} \times \pi_2(\Gamma)) \cap L_{j_k} \neq \emptyset$$

for any $k \in \mathbb{N}$. Therefore it can be concluded that $\Gamma \cap L \neq \emptyset$ because L_j approaches y-axis uniformly. It contradicts.

Because $\Phi(L) \subset K^+$ and $\Phi(L) \subset W^u(a)$ we have

$$H^{-1} \circ \Phi(L) \subset H^{-1}(K^+) = \widetilde{K}^+.$$

$H^{-1} \circ \Phi(L)$ is a connected set which contains 0 and is not a point. Therefore by Proposition 3.8, $\widetilde{K}^+$ is bridged. $\square$

4.2 Example

Example 4.2. Let us study the Hénon map:

$$F(x, y) = (y, y^2 - 2 - 0.7x).$$

We show that it has a stable manifold $W^s(a)$ of a fixed point a of saddle type, which satisfies that:

(1) No points on $W^s(a)$ are accessible from $\operatorname{int} K^+$,

(2) $W^s(a)$ is dense in $\partial \operatorname{int} K^+$.

Especially a is not accessible from $\operatorname{int} K^+$. It contrasts sharply with Corollary 3.5.

In fact, the Hénon map has two fixed points, $x = y = -0.8$ and $x = y = 2.5$.

When $x = y = -0.8$, the eigenvalues of DF are $\lambda \doteqdot -0.8 + 0.245i$ and $\lambda' \doteqdot -0.8 - 0.245i$. The point is a sink. Let U be the basin of the sink. In [BS2] Bedford and Smillie showed that $J^+ = \partial U$ and $J^+ = \overline{W^s(a)}$. Therefore $W^s(a)$ is dense in $\partial \operatorname{int} K^+$.

When $x = y = 2.5$, $\lambda \doteqdot 4.856$, $\lambda' \doteqdot 0.144$. Therefore the point is of saddle type. On the other hand, the *order* is:

$$\rho = \frac{\log d}{\log |\lambda|} \doteqdot 0.439 < \frac{1}{2}.$$

Theorem 3.3 tells that the unstable slice is not bridged. By Proposition 4.1, every point on $W^s(a)$ is not accessible from $\operatorname{int} K^+$.

The figure 6 is the slice of the K^+ by $x = -0.8$.

5 Yoccoz inequality in dynamics of one variable

In the third section, we have shown Yoccoz inequality on $W^u(a)$ without the hypothesis of connectivity. In this section we apply the method to prove an improved Yoccoz inequality of one variable.

Figure 6: The slice of K^+ of $F(x,y) = (y, y^2 - 2 - 0.7x)$ by $x = -0.8$.

5.1 Statements

Suppose $P(x)$ is a monic polynomial of one variable of degree $d \geq 2$. Let us study the dynamics $P : \mathbb{C} \to \mathbb{C}$. Let a be a repelling fixed point whose multiplier $DP(a) = \lambda \in \mathbb{C} \setminus \overline{\mathbb{D}}$, where $\mathbb{D} = \{x \in \mathbb{C} \mid |x| < 1\}$. In the sequel, most proofs are omitted.

It is well-known that there exists $\phi \in \mathcal{O}(\mathbb{C})$ such that

$$P \circ \phi(t) = \phi(\lambda t) \quad \text{and} \quad \phi(0) = a.$$

Theorem 5.1. *ϕ is a transcendental entire function and*

$$\rho = \operatorname{ord} \phi = \frac{\log d}{\log |\lambda|}.$$

Moreover ϕ is of mean type.

Let K be the filled Julia set of P and define $\widetilde{K} = \phi^{-1}(K)$. $\widetilde{K}$ is invariant under $t \mapsto \lambda t$.

Theorem 5.2. *Assume $\rho < 1/2$. Then for any $r_0 > 0$*

$$\frac{1}{\log |\lambda|} \int_{r_0}^{|\lambda| r_0} \frac{1_{\widetilde{K}}(r)}{r} dr \leq 2\rho,$$

where

$$1_{\widetilde{K}}(r) = \begin{cases} 1 & \text{if } \widetilde{K} \cap \{|t| = r\} \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Especially any components of $\widetilde{K}$ are compact and $\mathbb{C} \setminus \widetilde{K}$ is connected.

Theorem 5.3. *The number of components of $\mathbb{C} \setminus \widetilde{K}$ never exceeds* $\max\{2\rho, 1\}$. *Therefore every component of $\mathbb{C} \setminus \widetilde{K}$ is periodic.*

Corollary 5.4. 0 *is accessible from arbitrary component of $\mathbb{C} \setminus \widetilde{K}$. Therefore a is accessible from $\mathbb{C} \setminus K$.*

The above assertions have been already shown by Erëmenko and Levin in [EL]. They also use the equation $P \circ \phi(t) = \phi(\lambda t)$.

The previous theorem implies that all components of $\mathbb{C} \setminus \widetilde{K}$ have the same period q. Choose q components which move each other under $t \mapsto \lambda t$. Let a component move to p-th in q components, counting counterclockwise. Let N be the number of cycles.

Theorem 5.5. (Yoccoz inequality). *Assume that $\widetilde{K}$ is bridged, i.e. the component of $\widetilde{K}$ containing* 0 *is not a point. Then*

$$\frac{\operatorname{Re}\log\lambda}{|\log\lambda - 2\pi ip/q|^2} \geq \frac{Nq}{2\log d}$$

holds, where we choose an appropriate branch of $\log\lambda$.

The proof is the same as Theorem 3.9. Refer to Proposition 3.8, too.

5.2 Applications

We have stripped the connectivity criterion from Yoccoz inequality. Therefore we can obtain several assertions.

Theorem 5.6. *Let P be a polynomial and let a be a repelling fixed point. If P does not satisfy Yoccoz inequality at a then the component of the filled Julia set containing a consists of a point.*

Proof. It is clear because of the fact that ϕ is locally conformal at 0 and Theorem 5.5. □

Proposition 5.7. *Let $P(x)$ be a monic cubic polynomial. Then all non-trivial components of filled Julia set are preperiodic. If any fixed points do not satisfy Yoccoz inequality, then no non-trivial components of filled Julia set are invariant, i.e. period* ≥ 2.

Proof. In [BbH], Branner and Hubbard have shown the followings. When P is cubic, all non-trivial components of filled Julia set are preperiodic . Let X be a non-trivial periodic component of filled Julia set of P and let k be its period. Then there exists a polynomial-like mapping (V, U, P^k) such that the filled Julia set of (V, U, P^k) is X and (V, U, P^k) is hybrid equivalent to $z^2 + c$ where c is in Mandelbrot set.

Therefore if X is invariant, X has a repelling or parabolic fixed point. The hypothesis is that any fixed points don't suffice Yoccoz inequality, so X can contain neither repelling nor parabolic fixed points by the previous theorem. Hence no non-trivial components of filled Julia set are invariant. □

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