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Semi-hyperbolic dynamics on $\overline{\mathbb{C}}$-bundles *

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Abstract

We consider dynamics on fiber bundles whose fibers are the Riemann spheres and the base spaces are compact metric spaces. We investigate entropy. We define the semi-hyperbolicity of dynamics on fiber bundles. We will show that if a dynamics on a fiber bundle is semi-hyperbolic, then we have that 2-dimensional Lebesgue measure of each fiberwise Julia set is equal to zero, that each fiberwise Julia set is uniformly perfect and that the dynamics has a kind of weak rigidity. Moreover if the fiberwise dynamics are polynomials, then the fiberwise basin of infinity is a c-John domain, where the constant c does not depend on any points in the base space.

1 Introduction

To investigate random 1-dimensional complex dynamics or fiber-preserving holomorphic dynamics on fiber bundles in several dimensions, we introduce the following dynamical systems on fiber bundles which preserve fibers. The notion of fibered rational maps were introduced by M.Jonsson in [J2]. The research on dynamics of semigroups generated by rational maps on the Riemann sphere ([HM1], [HM2], [HM3] [GR], [Bo], [St1], [St2], [S1], [S2], [S4], [S5]), the research of random iterations of rational functions([FS], [BBR]) and the research on polynomial skew products on $\mathbb{C}^2$ ([H1], [H2], [J1]) are

*1991 Mathematical Subject Classification: 30D05, 58F23 directly related to this subject. For the research of polynomial skew products whose base spaces are compact metric spaces, see [Se1] and [Se2]. For the research of ergodic theory of random diffeomorphisms, see [K].

In this paper applying some results of [S4] obtained by the author, we will get that semi-hyperbolicity along fibers of fibered rational maps implies that fiberwise Julia sets have 2-dimensional measure zero(Theorem 2.1), that the dynamics have a kind of weak rigidity (Theorem 2.2), that the fiberwise Julia sets are uniformly perfect such that the constants concerning the uniform perfectness do not depend on any points of base spaces (Theorem 2.4) and that if fiberwise maps are polynomials then fiberwise basins of infinity are c-John domains where c is a constant not depending on any points of base spaces(Theorem 2.6). This is a generalized result of a result in [CJY] to the version of skew products. For the research of semi-hyperbolicity of usual dynamics of rational functions, see [CJY] and [Ma].

To show those we need the potential theoritic stories(section 3), distortion lemmas for holomorphic proper maps and key results (continuity of fiberwise Julia sets with respect to the points in base spaces: this is very important property and not easy to show) from [S4].

In section 4 we also show some results on entropy of fibered rational maps, which are a kind of generalization of some results in [J2], without any conditions on (semi-)hyperbolicity.

Definition 1.1. ([J2]) A triplet (π, Y, X) is called a "$\overline{\mathbb{C}}$-bundle " if

1. Y and X are compact metric spaces,
2. $\pi : Y \to X$ is a continuous and surjective map,
3. There exists an open covering $\{U_i\}$ of X such that for each i there exists a homeomorphism $\Phi_i : U_i \times \overline{\mathbb{C}} \to \pi^{-1}(U_i)$ satisfying that $\Phi_i(\{x\} \times \overline{\mathbb{C}}) = \pi^{-1}(x)$ and $\Phi_j^{-1} \circ \Phi_i : (U_i \cap U_j) \times \overline{\mathbb{C}} \to (U_i \cap U_j) \times \overline{\mathbb{C}}$ is a Möbius map for each $x \in U_i \cap U_j$.

Remark 1. By the condition 3, each fiber $Y_x := \pi^{-1}(x)$ has a complex structure. We also have that given $x_0 \in X$ we may find a continuous family $i_x : \overline{\mathbb{C}} \to Y_x$ of homeomorphisms for x close to x_0. Such a family $\{i_x\}$ will be called a "lacal parameterization." Since X is compact, we may assume that there exists a compact subset M_0 of the set of Möbius transformations of $\overline{\mathbb{C}}$ such that $i_x \circ j_x^{-1} \in M_0$ for any two local parametrizatios $\{i_x\}$ and $\{j_x\}$. In this paper we always assume that.

Definition 1.2. ([J2]) We say that a $\overline{\mathbb{C}}$-bundle (π, Y, X) satisfies the "continuous forms condition" if for each $x \in X$ there exists a smooth $(1,1)$-form

$\omega_x > 0$ inducing the metric on Y_x and $x \mapsto \omega_x$ is continuous. That is, if $\{i_x\}$ is a local parametrization, then the pull back $i_x^* \omega_x$ is a positive smooth forms on $\overline{\mathbb{C}}$ depending continuously on x.

Definition 1.3. Let (π, Y, X) be a $\overline{\mathbb{C}}$-bundle. Let $f : Y \to Y$ and $g : X \to X$ be continuous maps. We say that f is a rational map fibered over g if

1. $\pi \circ f = g \circ \pi$
2. $f|_{Y_x} : Y_x \to Y_{g(x)}$ is a rational map for any $x \in X$. That is, $(i_{g_x})^{-1} \circ f \circ i_x$ is a rational map from $\overline{\mathbb{C}}$ to itself for any local parametrization i_x at $x \in X$ and $i_{g(x)}$ at $g(x)$.

Notation: If $f : Y \to Y$ is a rational map fibered over $g : X \to X$, then we put $f_x^n = f^n|_{Y_x}$ for any $x \in X$ and $n \in \mathbb{N}$. Furthermore we put $d_n(x) = \deg(f_x^n)$ and $d(x) = d_1(x)$ for any $x \in X$ and $n \in \mathbb{N}$.

Definition 1.4. Let (π, Y, X) be a $\overline{\mathbb{C}}$-bundle. Let $f : Y \to Y$ is a rational map fibered over $g : X \to X$. Then for any $x \in X$ we denote by F_x the set of points $y \in Y_x$ which has a neighborhood U in Y_x satisfying that $\{f_x^n\}_{n \in \mathbb{N}}$ is a normal family in U, that is, $y \in F_x$ if and only if the family $Q_x^n = i_{x_n}^{-1} \circ f_x^n \circ i_x$ of rational maps on $\overline{\mathbb{C}}$ (x_n denotes $g^n(x)$) is normal near $i_x^{-1}(y)$: note that by Remark 1, this does not depend on the choices local parametrizations at x and x_n. Still equivalently, F_x is the open subset of Y_x where the family $\{f_x^n\}$ of mappings from Y_x into Y is local equicontinuous. We put $J_x = Y_x \setminus F_x$. Furthermore, we put

$$\tilde{J}(f) = \overline{\bigcup_{x \in X} J_x}, \quad \tilde{F}(f) = Y \setminus \tilde{J}(f).$$

Remark 2. There exists a fibered rational map $f : Y \to Y$ satysfying that $\bigcup_{x \in X} J_x$ is NOT compact.

Example 1.5. 1. ([S4].) Let $h_1, \dots, h_m$ be non-constant rational maps. Let $\Sigma_m = \{1, \dots, m\}^{\mathbb{N}}$ be the space of one-sided infinite sequences of m symbols and $g : \Sigma_m \to \Sigma_m$ be the shift map: that is, g is defined by $g((w_1, w_2, \dots)) = (w_2, w_3, \dots)$. Let X be a compact subset of Σ_m such that $g(X) \subset X$. Let $Y = X \times \overline{\mathbb{C}}$ and $\pi : Y \to X$ be the natural projection. Then (π, Y, X) is a $\overline{\mathbb{C}}$-bundle with continuous forms condition. Let $f : Y \to Y$ be a map defined by: $f((w, y)) = (g(w),\ h_{w_1}(y))$. Then $f : Y \to Y$ is a rational map fibered over $g : X \to X$.

In the above if $X = \Sigma_m$ then we say that $f : Y \to Y$ is the skew product map assosiated with the generator system $\{h_1, \dots h_m\}$ of the rational semigroup $G = \langle h_1, \dots, h_m \rangle$, where we denote by $\langle h_1, \dots, h_m \rangle$ the semigroup generated by $\{h_1, \dots, h_m\}$ with the semigroup operation being composition of maps. We denote by $J(G)$ the Julia set of rational semigroup G: that is, the set of points $z \in \overline{\mathbb{C}}$ satisfying that z has no neighborhood where the family of maps G is normal. Then we have

$$\pi_{\overline{\mathbb{C}}}(\tilde{J}(f)) = J(G),$$

where $\pi_{\overline{\mathbb{C}}} : Y \to \overline{\mathbb{C}}$ is the projection. See [S4] for more details.

2. Let Y be a ruled surface over a Riemann surface X: that is, Y is a smooth projective variety of complex dimension 2 which is also a holomorphic $P^1(\mathbb{C})$-bundle over X. Every Y_x has a unique conformal structure and a positive form $\omega_x = \omega|_{Y_x}$, where ω is the Kähler form on Y. Let $\pi : Y \to X$ be the projection. Then (π, Y, X) is a $\overline{\mathbb{C}}$-bundle satisfying the continuous forms condition with $(\omega_x)_{x \in X}$.

   Dabija [D] showed that (almost) every holomorphic selfmap f of Y is a rational map fibered over a holomorphic map $g : X \to X$.

3. Let $p(x) \in \mathbb{C}[x]$ be a polynomial with degree at least two and $q(x, y) \in \mathbb{C}[x, y]$ a polynomial of the form: $q(x, y) = y^n + a_1(x)y^{n-1} + \cdots$. Let $f : \mathbb{C}^2 \to \mathbb{C}^2$ be a map defined by

$$f((x, y)) = (p(x), q(x, y)).$$

   This is called a polynomial skew product in $\mathbb{C}^2$. Such a kind of maps were investigated by S.-M.Heinemann in [H1] and [H2] and by M.Jonsson in [J1].

   Let X be a compact subset of $\overline{\mathbb{C}}$ such that $p(X) \subset X$. (e.g. the Julia set of p.) Let $(\pi, Y = X \times \overline{\mathbb{C}}, X)$ be a trivial $\overline{\mathbb{C}}$-bundle. Then the map $\tilde{f} : Y \to Y$ defined by $\tilde{f}((x, y)) = (p(x), q(x, y))$ is a rational map fibered over $p : X \to X$.

Notation :

- Let Z_1 and Z_2 be two topological spaces and $g : Z_1 \to Z_2$ be a map. For any subset A of Z_2, we denote by $c(g, A)$ the set of all connected components of $g^{-1}(A)$.

- for any $y \in \overline{\mathbb{C}}$ and $\delta > 0$, we put $B(y, \delta) = \{y' \in \overline{\mathbb{C}} \mid d(y, y') < \delta\}$, where d is the spherical metric. Similarly, for any $y \in \mathbb{C}$ and $\delta > 0$ we put $D(y, \delta) = \{y' \in \mathbb{C} \mid |y - y'| < \delta\}$.

Now we will define the semi-hyperbolicity of fibered rational maps.

Definition 1.6. (**semi-hyperbolicity**) Let (π, Y, X) be a $\overline{\mathbb{C}}$-bundle. Let $f : Y \to Y$ be a rational map fibered over $g : X \to X$. Let $N \in \mathbb{N}$. We say that a point $z \in Y$ belongs to $SH_N(f)$ if there exists a positive number δ, a neighborhood U of $\pi(z)$ and a local parametrization $\{i_x\}$ in U such that for any $x \in U$, any $n \in \mathbb{N}$, any $x_n \in g^{-1}(x)$ and any $V \in c(i_x(B(i_{\pi(z)}^{-1}(z), \delta)), f_x^n)$, we have

$$\deg(f_x^n : V \to i_x(B(i_{\pi(z)}^{-1}(z), \delta))) \leq N.$$

We set

$$UH(f) = Y \setminus \bigcup_{N \in \mathbb{N}} SH_N(f).$$

We say that f is semi-hyperbolic (along fibers) if for any point $z \in Y$ there exists a positive integer $N \in \mathbb{N}$ satisfying that $z \in SH_N(f)$.

Example 1.7.

1. Let $f : Y \to Y$ be a rational map fibered over $g : X \to X$. We set

$$P(f) = \overline{\bigcup_{n \in \mathbb{N}} \bigcup_{x \in X} f_x^n(\text{ critical points of } f_x)}.$$

   This is called the fiber post critical sets of fibered rational map f. If $f : Y \to Y$ is hyperbolic along fiberes: that is, $P(f) \subset F(f)$, then f is semi-hyperbolic along fiberes with the constant $N = 1$.

2. Let $\{h_1, \ldots, h_m\}$ be non-constant rational functions on $\overline{\mathbb{C}}$. Let $f : Y \to Y$ be the skew product map in Example 1.5.1. By easy arguments we can show that $f : Y \to Y$ is semi-hyperbolic along fiberes if and only if G is semi-hyperbolic: that is, for each $x \in J(G)$ there exists an open neighborhood U of x in $\overline{\mathbb{C}}$ and a number $\delta > 0$ such that for each $g \in G$ and $V \in c(B(x, \delta), g)$

$$\deg(g : V \to B(x, \delta) \leq N.$$

   In [S4], the following statement was shown:

   Assume that there exists an element of G with the degree at least two, that each element of Aut $\overline{\mathbb{C}} \cap G$(if this is not empty) is loxodromic and that $J(G) \neq \overline{\mathbb{C}}$. Then G is semi-hyperbolic if and only if all of the following conditions are satisfied.

   (a) for each $z \in J(G)$ there exists a neighborhood U of z in $\overline{\mathbb{C}}$ such that for any sequence $(g_n) \subset G$, any domain V in $\overline{\mathbb{C}}$ and any point $\zeta \in U$, we have that the sequence (g_n) does NOT converge to ζ locally uniformly on V

(b) for each $j = 1, \ldots, m$ each $c \in C(f_j) \cap J(G)$ satisfies

$$d(c,\ (G \cup \{id\})(f_j(c))) > 0$$

From this fact it was shown in [S4] that if we assume that there exists an element of G with the degree at least two, that each element of Aut $\overline{\mathbb{C}} \cap G$(if this is not empty) is loxodromic, that there is no super attracting fixed point of any element of G in $J(G)$ and $F(G) \neq \emptyset$, then G is semi-hyperbolic.

By this theorem we know that $G = \langle z^2 + 2,\ z^2 - 2 \rangle$ is semi-hyperbolic. This is NOT hyperbolic. See [S4].

We need some technical conditions.

Definition 1.8 (Condition(C1)). Let (π, Y, X) be a $\overline{\mathbb{C}}$-bundle. Let $f : Y \to Y$ be a rational fibered over $g : X \to X$. We say that f satisfies the condition (C1) if there exists a family $\{D_x\}_{x \in X}$ of topological discs with $D_x \subset Y_x$, $x \in X$ such that the following three conditions are satisfied:

1. $\overline{\bigcup_{n \geq 0} f_x^n(D_x)} \subset \tilde{F}(f)$ for each $x \in X$.

2. for any $x \in X$, we have that $\mathrm{diam}_Y(f_x^{(n)}(D_x)) \to 0$, as $n \to \infty$.

3. $\inf_{x \in X}$ diam $_Y(D_x) > 0$.

Definition 1.9 (Condition(C2)). Let (π, Y, X) be a $\overline{\mathbb{C}}$-bundle. Let $f : Y \to Y$ be a rational map fibered over $g : X \to X$. We say that f satisfies the condition (C2) if for each $x_0 \in X$ there exists an open neighborhood O of x_0 and a family $\{D_x\}_{x \in O}$ of topological discs with $D_x \subset Y_x, x \in O$ such that the following three conditions are satisfied:

1. $\overline{\bigcup_{n \geq 0} f_x^n(D_x)} \subset \tilde{F}(f)$ for each $x \in O$.

2. for any $x \in O$, we have that $\mathrm{diam}_Y(f_x^{(n)}(D_x)) \to 0$, as $n \to \infty$.

3. $x \mapsto D_x$ is continuous in O.

Example 1.10. 1. Let $\{h_1, \ldots h_m\}$ be non-constant rational functions on $\overline{\mathbb{C}}$ with $\deg(h_1) \geq 2$. Let $f : Y \to Y$ be the skew product map assosiated with the generator system $\{h_1, \ldots, h_m\}$ of rational semigroup $G = \langle h_1, \ldots, h_m \rangle$, which is described in Example 1.5.1. Suppose that f is semi-hyperbolic along fibers and that $\pi_{\overline{\mathbb{C}}}(\tilde{J}(f)) = J(G)$ is not equal to the Riemann sphere. Then we have that f satisfies the condition (C2). Actually, there exists an attracting periodic point a in $\overline{\mathbb{C}} \setminus J(G)$. Since

G is semi-hyperbolic, we have that setting $D_x = D(a, \epsilon)$ for each $x \in X$ where ϵ is a positive number, f satisfies the condition (C2) with the family of discs $(D_x)_{x \in X}$.

2. Let $(\pi,\ Y = X \times \overline{\mathbb{C}},\ X)$ be a trivial $\overline{\mathbb{C}}$-bundle. Let $f : Y \to Y$ be a rational map fibered over $g : X \to X$ satisfying that f_x is a polynomial mapping of degree at least two for each $x \in X$. Then setting $D_x = D$ where D is a small neighborhood of infinity for each $x \in X$, the rational map f satisfies the condition (C2) with the family of discs $(D_x)_{x \in X}$.

2 Results

In this section we intrduce some results which are deduced by semi-hyperbolicity.

Theorem 2.1. (**measure zero**) *Let (π, Y, X) be a $\overline{\mathbb{C}}$-bundle. Let $f : Y \to Y$ be a rational map fibered over $g : X \to X$. Suppose f is semi-hyperbolic along fibers and satisfies the condition (C2). Then for each $x \in X$, the 2-dimensional Lebesgue measure of J_x is equal to zero.*

Theorem 2.2. (**a rigidity**) *Let (π, Y, X) and $(\tilde{\pi}, \tilde{Y}, \tilde{X})$ be two $\overline{\mathbb{C}}$-bundles. Let $f : Y \to Y$ be a rational map fibered over $g : X \to X$ and $\tilde{f} : \tilde{Y} :\to \tilde{Y}$ a rational map fibered over $\tilde{g} : \tilde{X} \to \tilde{X}$. Let $u : Y \to \tilde{Y}$ be a homeomorphism which is a bundle conjugacy between f and $\tilde{f}$:i.e. u satisfies that $\tilde{\pi} \circ u = v \circ \pi$ for some homeomorphism $v : X \to X$ and $\tilde{f} \circ u = u \circ f$. Suppose that f is semi-hyperbolic along fiberes and satisfies the condition (C2). Suppose also that the restriction $u_x : Y_x \to \tilde{Y}_{v(x)}$ of u is holomorphic on F_x for all $x \in X$. Then we have that u_x is holomorphic on the whole Y_x for all $x \in X$.*

Definition 2.3. Let C be a positive number. Let K be a closed subset of $\overline{\mathbb{C}}$. We say that K is C-uniformly perfect if for any doubly connected domain A in $\overline{\mathbb{C}}$ satisfying that both two connected components of $\overline{\mathbb{C}} \setminus A$ have non-empty intersection with K, the modulus of A is less than C.

Theorem 2.4. (**uniform perfectness**) *Let $(\pi, Y.X)$ be a $\overline{\mathbb{C}}$-bundle with continuous forms condition. Let $f : Y \to Y$ be a rational map fibered over $g : X \to X$ with $d(x) \geq 2$ for any $x \in X$. Suppose that f is semi-hyperbolic along fiberes and satisfies the condition (C1). Then there exists a positive constant C such that J_x is C-uniformly perfect for any $x \in X$.*

Notation: Let $y \in \mathbb{C}$ and $b \in \overline{\mathbb{C}}$ be two distinct points. Let E be a curve in $\overline{\mathbb{C}}$ joining y to b satisfying that $E \setminus \{b\} \subset \mathbb{C}$. For any $c \geq 1$ we set

$$\text{car}\ (E, c, y, b) = \bigcup_{z \in E \setminus \{y, b\}} D(z,\ \frac{|y - z|}{c}).$$

This is called the c-carrot with core E and vertex y joining y to b.

Definition 2.5. Let V be a subdomain of $\overline{\mathbb{C}}$. Let $c \geq 1$ be a number. We say that V is a c-John domain if there exists a point $y_0 \in \overline{V}$ satisfying that for any $y \in V \setminus \{y_0\}$ there exists a curve E joining y_0 to y such that $E \setminus \{y_0\} \subset \mathbb{C}$ and

$$\text{car } (E, c, y, y_0) \subset V.$$

In the above the point y_0 is called the center of John domain V.

Remark 3. Johnness implies many good properties ([NV], [Jone]).For example, if V is a John domain, then the following facts hold.

- If $\infty \in \overline{V}$, then the center of V is ∞.
- Let $a \in \partial V \setminus \{\infty\}$ and $b \in V$. Then there exists a curve E joining a to b and a constant c such that car $(E, c, a, b) \subset V$. In particular, a is accessible from b.
- V is finitely connected at any point in ∂V: that is, if $y \in \partial V$, then there exists an arbitrary small open neighborhood U of y in $\overline{\mathbb{C}}$ such that $U \cap V$ has only finitely many connected components.
- If V is simply connected and $\partial V \subset \mathbb{C}$, then we have that ∂V is locally connected.
- If $\partial V \subset \mathbb{C}$ then ∂V is holomorphic removable: that is, if $\varphi : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a homeomorphim and is holomorphic on $\overline{\mathbb{C}} \setminus \partial V$, then φ is holomorphic on $\overline{\mathbb{C}}$. From this fact, we can deduce that the 2-dimensional Lebesgue measure of ∂V is equal to zero.

Theorem 2.6. **(Johnness**) *Let $(\pi, Y = X \times \overline{\mathbb{C}},\ X)$ be a trivial $\overline{\mathbb{C}}$-buundle. Let $f : Y \to Y$ be a rational map fibered over $g : X \to X$ satisfying that f_x is a polynomial with $d(x) \geq 2$ for any $x \in X$. Then there exists a positive constant c such that for any $x \in X$ the basin of infinity $A_x := \{y \in Y_x \mid f_x^n(y) \to \infty,\ n \to \infty\}$ in Y_x (here we identify f_x^n with a usual polynomial) satisfies that it is a c-John domain.*

Remark 4. In the Theorem 2.6 if X is a set consisting of one point, then f is semi-hyperbolic if and only if the basin of infinity is a John domain([CJY]).

3 Potential Theory and Measure Theory

We need some notations from [J2] and [S4], concerning potential theoritic aspects. Let (π, Y, X) be a $\overline{\mathbb{C}}$-bundle satisfying the continuous forms condition with a family $\{\omega_x\}_{x\in X}$ of positive $(1,1)$-forms. Let $f : Y \to Y$ be a rational map fibered over $g : X \to X$. Let $x \in X$ be a point. We set $x_n = g^n(x)$ for each $n \in \mathbb{N}$. The form ω_x on Y_x induces a measure which is also called ω_x on Y_x or even on Y. As measures on Y we have that $x \mapsto \omega_x$ is weakly continuous. For each continuous function φ on Y_x let $(f_x^n)_*\varphi$ be the continuous function on Y_{x_n} defined by $((f_x^n)_*\varphi)(z) = \sum_{f_x^n(w)=z} \varphi(w)$ for each $n \in \mathbb{N}$. We define pullbacks of measures by duality: $\langle (f_x^n)^*\nu,\ \varphi\rangle = \langle \nu,\ (f_x^n)_*\varphi\rangle$. Let $\mu_{x,n}$ be the probability measure on Y_x defined by $\mu_{x,n} = \frac{1}{d_n(x)}(f_x^n)^*\omega_{x_n}$.

We will lift $f_x : Y_x \to Y_{x_1}$ to self maps of $\overline{\mathbb{C}}$ and $\mathbb{C}^2_* := \mathbb{C}^2 \setminus \{0\}$. Let i_x and i_{x_1} be local parametrizations near x and x_1. Define $Q_x : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ to be a rational map and $R_x : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ to be a homogeneous polynomial map, both of degree $d(x)$, such that

$$\sup\{|R_x(z,w)| : |(z,w)| = 1\} = 1$$

and such that

$$f_x \circ i_x = i_{x_1} \circ Q_x,\ Q_x \circ \pi' = \pi' \circ R_x,$$

where we denote by π' the projection from $\mathbb{C}^2_*$ to $\overline{\mathbb{C}}$. Given the local parametrizations i_x and i_{x_1} these properties determine Q_x uniquely, and R_x uniquely up to multiplication by a complex number of unit modulus.

Now consider and orbit $(x_j)_{j\in\mathbb{N}}$ in X, select parametrizations at each point x_j and let R_{x_j} be the corresponding homogeneous selfmaps of $\mathbb{C}^2_*$. Let R_x^n be the composition $R_{x_n} \circ \cdots \circ R_x$. Then R_x^n is a homogeneous polynomial mapping of $\mathbb{C}^2_*$ of degree $d_n(x)$. Notice that R_x^n is determined, up to multiplication of by a complex number of unit modulus, by the local parametrizations at x and x_n.

Given a local parametrization $i_x : \overline{\mathbb{C}} \to Y_x$ there exists a smooth potential $G_{x,0}$ for ω_x in the sense that $\omega_x = dd^c(G_{x,0} \circ s \circ i_x^{-1})$, where s is any local section of π' and $d^c = \frac{i}{2\pi}(\overline{\partial} - \partial)$.

Define the plurisubharmonic function $G_{x,n}$ on $\mathbb{C}^2_*$ by

$$G_{x,n} = \frac{1}{d_n(x)} G_{x,0} \circ R_x^n.$$

If we change the local parametrizations at x_n and the potential $G_{x,0}$, then the modified plurisubharmonic function $\tilde{G}_{x,n}$ satisfies that there exists a constant

$C > 0$ such that

$$|G_{x,n}(z,w) - \tilde{G}_x^n(z,w)| \leq \frac{C}{d_n(x)}, \tag{1}$$

for all $x \in X$, (z,w) and $n \in \mathbb{N}$. By (1) and the arguments in [J2] and [S4], we get the following.

Proposition 3.1. *Let (π, Y, X) be a $\overline{\mathbb{C}}$-bundle satisfying the continuous forms condition with a family $\{\omega_x\}_{x \in X}$ of positive $(1,1)$-forms. Let $f : Y \to Y$ be a rational map fibered over $g : X \to X$. Assume that $d(x) \geq 2$ for each $x \in X$. Then we have the following.*

1. *$\mu_{x,n}$ converges to a probability measure μ_x on Y_x weakly as $n \to \infty$ for each $x \in X$.*

2. *$G_{x,n}$ converges to a continuous plurisubharmonic function G_x locally uniformly on $\mathbb{C}_*^2$ as $n \to \infty$ for each $x \in X$. This function does not depend on the choice of local parametrizations at $x_j, j \geq 1$ and potentials $G_{x,0}$.*

3. *$\mu_x = (i_x^{-1})_*(dd^c(G_x \circ s))$ where s is a local section of $\pi' : \mathbb{C}_*^2 \to \overline{\mathbb{C}}$. Further $G_x(z,w) \leq \log|(z,w)| + O(1)$ as $|(z,w)| \to \infty$ and $G_x(\lambda z, \lambda w) = G_x(z,w) + \log \lambda$ for each $\lambda \in \mathbb{C}$, for each $x \in X$.*

4. *$G_{x_1} \circ R_x = d(x) \cdot G_x$ for each $x \in X$.*

5. *if $x \to x'$ then $G_x \to G_{x'}$ uniformly on $\mathbb{C}_*^2$.*

6. *$(f_x)_*\mu_x = \mu_{x_1}$, $(f_x)^*\mu_{x_1} = d(x_1) \cdot \mu_x$ for each $x \in X$.*

7. *μ_x puts no mass on polar subsets of Y_x for each $x \in X$.*

8. *$x \mapsto \mu_x$ is continuous with respect to the weak topology of measures in Y.*

9. *$supp(\mu_x) = J_x$ for each $x \in X$.*

10. *J_x has no isolated points for each $x \in X$.*

11. *$x \mapsto J_x$ is lower semicontinuous with respect to the Hausdorff metric in the space of non-empty compact subsets of Y. That is, if $x, x^n \in X, x^n \to x$ as $n \to \infty$ and $y \in Y_x$, then there exists a sequence (y_n) of points in Y with $y_n \in Y_{x^n}$ for each $n \in \mathbb{N}$ such that $y_n \to y$ as $n \to \infty$.*

4 Entropy

Now we show some results on entropy of rational maps on $\overline{\mathbb{C}}$-bundles without any conditions on (semi-) hyperbolicity, using the arguments in [J2].

Notation: Let (Y, d) be a metric space. Let $f : Y \to Y$ be a continuous mapping. For any compact subset Z of Y we denote by $h(f, Z)$ the entropy of f on Z. We set $h(f) = h(f, Y)$. For any f-invariant probability measure ν on Y we denote by $h_\nu(f)$ the metric entropy of f with respect to ν. If $g : X \to X$ is a continuous mapping on a compact metric space X and $\pi : Y \to X$ is a continuous mapping such that $g \circ \pi = \pi \circ f$, then we denote by $h_\nu(f|g)$ the metric entropy of f relative to g with respect to ν. See [J2] for these notations and definitions.

Theorem 4.1. *Let (π, Y, X) be a $\overline{\mathbb{C}}$-bundle. Let $f : Y \to Y$ be a rational map fibered over $g : X \to X$. Then the following holds.*

1. $h(f, Y_x) \leq \limsup_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n-1} \log d(x_n)$ *for any* $x \in X$.

2. *If μ is an f-invariant probability measure on Y, then we have*
$$h_\mu(f|g) \leq \int_X \log d(x)\ d(\pi_*\mu)(x).$$

3. $h(f) \leq \sup\{h_{\pi_*\mu}(g) + \int_X \log d(x)\ d(\pi_*\mu)(x))\}$, *where the supremum is taken over all f-invariant probability measures μ on Y.*

Proof. The statement 1 is shown by the following lemma 4.2. The statement 2 and 3 follows from the statement 1, ergodic theorem, Abramov-Rohklin formula
$$h_\mu(f) = h_{\pi_*\mu}(g) + h_\mu(f|g)$$
and the variational principle: that is, if ν' is a g-invariant probability measure on X, then
$$\sup h_\nu(f|g) = \int_X h(f,\ Y_x)\ d\nu'(x),$$
where the supremum is taken over all f-invariant probability measure ν on Y such that $\pi_*\nu = \nu'$. $\square$

Lemma 4.2. *Let (π, Y, X) be a $\overline{\mathbb{C}}$-bundle. Let $f : Y \to Y$ be a rational map fibered over $g : X \to X$. Then for every $\delta > 0$ there exists a constant $C(\delta) > 0$ with the following property:for every $x \in X$ and every $n \in \mathbb{N}$ there exists an (n, δ)-spanning set in Y_x with at most $C(\delta)n^5 d_n(x)$ elements.*

Proof. This can be shown by the same arguments in the proof of Lemma 3.3 in [J2]. □

Theorem 4.3. *Let (π, Y, X) be a $\overline{\mathbb{C}}$-bundle satisfying the continuous forms condition with a family $(\omega_x)_{x\in X}$ of positive $(1,1)$-forms. Let $f : Y \to Y$ be a rational map fibered over $g : X \to X$. Assume that $d(x) \geq 2$ for any $x \in X$. Let μ' be a g-invariant Borel probability measure on X. Define the measure μ on Y by:*

$$\langle \mu, \varphi \rangle = \int_X \Big(\int_{Y_x} \varphi(y)\ d\mu_x(y)\Big) d\mu'(x)$$

for continuous funcitions φ on Y, where μ_x is the measure in Proposition 3.1. Then we have the following.

1. *μ is f-invariant.*
2. *if μ' is ergodic, then so is μ.*
3. *if μ' is (strongly)mixing, then so is μ.*
4. *$h_\mu(f|g) = \sup_\nu h_\nu(f|g) = \int_X \log d(x)\ d\mu'(x)$, where the supremum is taken over all f-invariant probability measures ν satisfying $\pi_*\nu = \mu'$.*

Proof. By the same arguments of the proof of Theorem 6.1 in [J2], Proposition 3.1 and Theorem 4.1. □

Remark 5. In some cases, the maximal entropy measure of f (or the measure μ with $\pi_*\mu = \mu'$ which gives us the equality in Therem 4.3.4) is unique. For example,

- the case that there exists a constant $d \geq 2$ satysfying that $d(x) = d$ for any $x \in X$. ([J2]).
- (π, Y, X) is the trivial bundle assosiated with a generator system of any (with a slight assumption) finitely generated rational semigroup. ([S5]) (In this case, each $d(x)$ may be different and might be equal to 1.)

It is a conjecture that for any fibered rational map f with $d(x) \geq 2$, $x \in X$, the maximal entropy measure of f (or the measure μ with $\pi_*\mu = \mu'$ which gives us the equality in Therem 4.3.4) is unique.

5 Tools and Proofs

To show theorems in the section2, we need the followings. For the research on semi-hyperbolicity of usual dynamics of rational functions, see [CJY] and [Ma].

Lemma 5.1 ([CJY]). (**distortion lemma for proper maps**) *For any positive integer N and real number r with $0 < r < 1$, there exists a constant $C = C(N, r)$ such that if $f : D(0,1) \to D(0,1)$ is a proper holomorphic map with* $\deg(f) = N$ *and $f(0) = 0$, then*

$$D(f(z_0), C) \subset f(D(z_0, r)) \subset D(f(z_0), r)$$

for any $z_0 \in D(0,1)$. Here we can take $C = C(N, r)$ independent of f.

The following is a generalized distortion lemma for proper maps.

Lemma 5.2 ([S4]). *Let V be a domain in $\overline{\mathbb{C}}$, K a continuum in $\overline{\mathbb{C}}$ with $diam_S K = a$. Assume $V \subset \overline{\mathbb{C}} \setminus K$. Let $f : V \to D(0,1)$ be a proper holomorphic map of degree N. Then there exists a constant $r(N, a)$ depending only on N and a such that for each r with $0 < r \leq r(N, a)$, there exists a constant $C = C(N, r)$ depending only on N and r satisfying that for each connected component U of $f^{-1}(D(0, r))$,*

$$diam_S\, U \leq C,$$

where we denote by $diam_S$ the spherical diameter. Also we have $C(N, r) \to 0$ as $r \to 0$.

The following theorem says that the backward dynamics of semi-hyperbolic dynamics on $\overline{\mathbb{C}}$-bundles are "contracting" in a sense. Moreover, we will show that the union of the fiberwise Julia sets is copact. This is very important and useful property. Note that there exists a rational map on a $\overline{\mathbb{C}}$-bundle which is not semi-hyperbolic satisfying that the union of the fiberwise Julia sets is not compact.

Theorem 5.3 ([S4]). (**Key theorem**) *Let (π, Y, X) be a $\overline{\mathbb{C}}$-bundle. Let $f : Y \to Y$ be a rational map fibered over $g : X \to X$. Assume f is semi-hyperbolic along fibers and satisfies the condition (C1). Then the following hold.*

1. *Let $z \in Y$ be any point with $z \in F_{\pi(z)}$. Then for any local parametrization (i_x) and any open connected neighborhood U of $i_{\pi(z)}^{-1}(z)$ in $\overline{\mathbb{C}}$, there exists no subsequence of $(i_{\pi f^n(z)}^{-1} \circ f_{\pi(z)}^n \circ i_{\pi(z)})_n$ converging to a non-constant map locally uniformly on U.*

2.
$$\tilde{J}(f) = \bigcup_{x \in X} J_x.$$

3. *Suppose the condition (C2) is satisfied. Then there exist positive constants* δ, L *and* $\lambda (0 < \lambda < 1)$ *such that for any* $n \in \mathbb{N}$,
$$\sup\{diam_{\,Y}U \mid U \in c(\tilde{B}(z,\delta),\ f_{x_n}^n),\ z \in \tilde{J}(f),\ x_n \in g^{-n}(\pi(z))\} \leq L\lambda^n,$$
where we denote by $\tilde{B}(z,\delta)$ *the ball in* $Y_{\pi(z)}$ *with the center* z *and the radius* δ *with respect to the metric in* $Y_{\pi(z)}$ *induced by the metric of* Y.

4. *Assume that* (π, Y, X) *satisfies the continuous forms condition and that* $d(x) \geq 2$ *for each* $x \in X$. *Then we have that* $x \mapsto J_x$ *is continuous with respect to the Hausdorff metric in the space of compact subsets of* Y.

5. *Assume that* (π, Y, X) *satisfies the continuous forms condition with a family* (ω_x) *of positive* $(1,1)$*-forms and that* $d(x) \geq 2$ *for each* $x \in X$. *Then for any compact subset* K *of* $\tilde{F}(f)$, *we have that* $\overline{\cup_{n\geq 0} f^n(K)} \subset \tilde{F}(f)$ *and there exist constants* $C > 0$ *and* $\tau < 1$ *such that for each* n, $\sup_{z \in K} \|(f^n)'(z)\| \leq C\tau^n$, *where we denote by* $\|(f^n)'(z)\|$ *the norm of the derivative measured from* $\omega_{\pi(z)}$ *to* $\omega_{g^n(\pi(z))}$. *In particular, the condition (C2) is satisfied.*

Proof. of Theorem 2.1. Suppose that there exists a point $x \in X$ such that J_x has positive measure. Then there exists a Lebesgue density point $y \in J_x$. Let $y_n = f_x^n(y)$ and $x_n = g^n(x)$ for any $n \in \mathbb{N}$. Let δ be a positive number which is sufficiently small. Let U_n be the element of $c(\tilde{B}(y_n,\delta),\ f_x^n)$ containing y, where we denote by $\tilde{B}(y_n,\delta)$ the ball in Y_{x_n} with respect to the metric induced by the metric of Y. By Lemma 5.1 and Lemma 5.2, we have that for any local parametrization i_x,
$$\lim \frac{m(i_x^{-1}(U_n \cap J_x))}{m(i_x^{-1}(U_n))} = 1,$$
where m denotes the spherical measure of $\overline{\mathbb{C}}$. This implies that
$$\lim \frac{m(i_{x_n}^{-1}(\tilde{B}(y_n,\delta) \cap F_{x_n}))}{m(i_{x_n}^{-1}(\tilde{B}(y_n,\delta)))} = 0, \tag{2}$$
where i_{x_n} denotes a local parametrization. There exists a subsequence (n_j) of (n), a point $y_\infty \in Y$ and a point $x_\infty \in X$ such that $y_{n_j} \to y_\infty$ and $x_{n_j} \to x_\infty$ as $j \to \infty$. By (2) and Theorem 5.3.2, we have that $\tilde{B}(y_\infty,\ \delta) \subset J_{x_\infty}$. On the

other hand, by the condition (C2) we have that for any $a \in X$ the Julia set J_a has no interior point. This is a contradiction.

□

Proof. of Theorem 2.2. By Lemma 5.1 and Lemma 5.2, we can show that there exists a constant $C > 0$ such that

$$\liminf_{r \to 0} \{ \frac{d(\tilde{u}_x(y),\ \tilde{u}_x(y'))}{d(\tilde{u}_x(y),\ \tilde{u}_x(y''))} \mid d(y,y') = d(y,y'') = r \} \leq C,$$

where $\tilde{u}_x = \mathrm{j}_{v(x)}^{-1} \circ u_x \circ i_x$ for some local parametrizations i_x and $j_{v(x)}$ and d denotes the spherical metric. By the theorem concerning the definition of qc maps (that we can replace "limsup" by " liminf" in the definition of qc map using the circular dilatation) in [HK], we can show that u_x is a quasiconformal mapping on Y_x. Since for any $x \in X$ the 2-dimensional Lebesgue measure of the fiberwise Julia set J_x of f is equal to zero, which is the consequence of Theorem 2.1, we have that u_x is holomorphic on Y_x. □

Proof. of Theorem 2.4. Suppose there exists a sequence (x_n) of points X and a sequence (B_n) of annulus with $B_n \subset Y_{x_n}$ such that B_n separates J_{x_n} and mod $(B_n) \to \infty$. We can assume diam $_Y(B_n) \to 0$. Let $y_n \in J_{x_n} \setminus B_n$ be a point for any $n \in \mathbb{N}$. By Lemma 5.1 and Lemma 5.2, for any $n \in \mathbb{N}$ there exists a positive integer m_n such that $f^{m_n}(B_n)$ contains an annulus $\tilde{B}_n$ satisfying that

- $f^{m_n}(y_n) \in J_{g^{m_n}x_n} \setminus \tilde{B}_n$,
- if we denote by $e_{n,1}$ the distance from $f^{m_n}(y_n)$ to the outer boundary of $\tilde{B}_n$ and we denote by $e_{n,2}$ the distance from $f^{m_n}(y_n)$ to the inner boundary of $\tilde{B}_n$ then $e_{n,1} \sim 1$ and
- $e_{n,2}/e_{n,1} \to 0$ as $n \to \infty$.

We can assume that $f^{m_n}(y_n)$ tends to some point $y \in \tilde{J}(f)$. By Theorem 5.3.2, we have $y \in J_{\pi(y)}$. Since $e_{n,1} \sim 1$ and $e_{n,2}/e_{n,1} \to \infty$, by Proposition 3.1.11 we conclude that $y \in J_{\pi(y)}$ is an isolated point of $J_{\pi(y)}$. But this contradicts to Proposition 3.1.10. □

Proof. of Theorem 2.6. We can show the statement in a similar way to that in [CJY] using Theorem 5.3.3, Theorem 2.4, Lemma 5.1 and Lemma 5.2.

The procedure is: first for any $x \in X$ we take Green's function H_x in A_x. Then by Propotion 3.1and the arguments in the previous paragraph of it, H_x can be extended to Y_x and $(x, y) \mapsto H_x(y)$ is continuous in Y.

Secondly we show that any Green's line in A_x lands to some point in J_x. Finally by distortion lemmas for proper maps we show that each point $y \in A_x$ can be joined with a c-carrot with core Green's line from ∞ to y. Since the constants in Theorem 5.3.3 and Theorem 2.4 are not depending on $x \in X$, we can choose c not depending on $x \in X$. $\square$

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