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On the qc rigidity of real polynomials

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The Fatou conjecture (or the HD conjecture) asserts that any rational function can be approximated by hyperbolic rational functions of the same degree and any polynomial can be approximated by hyperbolic polynomials of the same degree. The real Fatou conjecture asserts that a real polynomial can be approximated by hyperbolic real polynomials of the same degree.

A possible solution of these conjecture comes from solving the rigidity problem: any two combinatorial rational functions are quasiconformally conjugate (this statement is usually named the combinatorial rigidity conjecture); and a rational map other than a Lattès example, carries no invariant line field on the Julia set (this is named the quasiconformal rigidity conjecture, or the NILF conjecture following McMullen and Sullivan). In [18], McMullen and Sullivan reduced the Fatou conjecture to the qc rigidity problem. They stated the no invariant line field (NILF) conjecture and showed that the NILF conjecture implies the Fatou conjecture. We don't know whether the NILF conjecture implies the real Fatou conjecture. However, beside a solution to NILF conjecture, a solution to the combinatorial rigidity problem among real polynomials would imply the real Fatou conjecture.

These conjectures are far from being solved. All essential progress in this direction was only done for polynomials with only one critical point in its Julia set, to the author's knowledge. It is known that for a quadratic polynomial which is not infinitely renormalizable or which is real, there cannot be invariant line field supported on the Julia set due to Yoccoz and McMullen, see [5],[16]. It is also proved recently by Levin and van Strien that the no invariant line field conjecture holds for a real polynomial which has only a critical point, see [9], [10]. In [6] and [12], the real Fatou conjecture was solved in the quadratic case; more recently, Shishikura [23] has given a new proof of that theorem. Little is known about a polynomial which has more than a critical point. Branner and Hubbard has proved the rigidity conjecture for polynomials of degree 3 which has at most one non-escaping critical point and cannot be renormalized (infinitely many times) to a quadratic polynomial, see [2],[3]. In this result, although the polynomial is allowed to have several critical points, there is only one in the Julia set.

In [21], the NILF conjecture is proved for all real polynomials whose real critical points are all turning points and whose critical values are all on the real axis. In the space of real polynomials of a fixed degree with all critical points real, such polynomials form a dense set.

Main Theorem. *Let f be a real polynomial satisfying the following two conditions:*

(1) for any critical point c of f, $f(c) \in \boldsymbol{R}$; and

(2) any (real) persistently recurrent critical point c of f is a turning point (has even local degree).

Then f carries no invariant line field on the Julia set.

Towards the density of hyperbolicity, our result says the following:

Corollary. *Let f be a real polynomial of degree $d \geq 2$ whose all critical values are on the real axis. Suppose that f is structurally stable in the space of all (complex) polynomials of degree d. Then f is hyperbolic.*

Proof of Corollary: Since f is structurally stable, the Teichmuller space of f has dimension $d-1$. In particular, all the critical points are non-degenerate which implies that condition (2) in the main theorem holds. Thus by the main theorem, f carries no invariant line field on the Julia set. By [18], f has to be hyperbolic. Q.E.D.

An invariant line field can be seen as a measurable f-invariant Beltrami differential $\mu = \mu(z)d\bar{z}/dz$ on the Riemann sphere $\hat{\boldsymbol{C}}$ such that $\mu(z) = 0$ or $|\mu(z)| = 1$ for any z and the support $supp(\mu) = \{z : |\mu(z)| = 1\}$ has positive Lebesgue measure.

By the definition, if $J(f)$ has zero (two-dimensional Lebesgue) measure, then $J(f)$ carries no f-invariant line field. It is well-known that the Julia set of a hyperbolic rational map has measure zero. Moreover, due to Urbanski ([25]), if R is a rational function without non-periodic recurrent critical point, then $J(R)$ has zero measure. Another remarkable result on the measure of Julia sets is that for any quadratic polynomial f without indifferent periodic cycle, at most finitely renormalizable, $J(f)$ has measure zero, due to Lyubich ([11]) and Shishikura ([22]).

We shall first investigate when the Julia set has zero measure, and prove the following:

Theorem A. *Let f be a real polynomial such that all its critical values are on the real axis. Then for almost every point $z \in J(f)$, $\omega(z) = \omega(c)$ for some persistently recurrent critical point c.*

In particular, if f has no persistently recurrent critical point, then the Julia set has measure zero.

Here, we say a recurrent critical point c is *reluctantly recurrent* if $\omega(c)$ is not minimal (that is, there is an $x \in \omega(c)$ such that $\omega(x) \neq \omega(c)$) or there is a positive constant δ such that for any $n_0 \in \boldsymbol{N}$, there is a positive integer $n > n_0$ and an $x \in \omega(c)$, a neighborhood U of x such that $f^n : U \to B(f^n(x), \delta)$ is a diffeomorphism. A recurrent critical point c is *persistently recurrent* if it is not reluctantly recurrent. These concepts appear first on the work of Yoccoz on quadratic polynomials.

Since our object is a real polynomial, the non-recurrent critical points can be done in a very easy way. To deal with the reluctantly recurrent critical points, we will use puzzle partitions, constructed by Branner-Hubbard in the case that the Julia set is non-connected, and by Yoccoz in the case that the Julia set is connected.

The ω-limit set $\omega(c)$ of a persistently recurrent critical point c is a minimal set, so

$$E_c = \{z \in J(f) : \omega(z) = \omega(c)\} = \{z \in J(f) : \omega(z) \subset \omega(c)\}$$

is a measurable f-completely-invariant set for any persistently recurrent critical point c, so if $J(f)$ carries an invariant line field μ, then for some persistently recurrent critical point c, $\mu|E_c$ is also an invariant line field. This reduces the problem to the case that f has exactly one minimal set which contains critical points. Let $\mathcal{F}$ be the collection of all real polynomials satisfying the requirements in the main theorem.

Reduced Main Theorem. *Let g be a real symmetric generalized polynomial-like map induced by an $f \in \mathcal{F}$ such that g has exactly one minimal set which contains critical points. Suppose that g is non-renormalizable (if the domain of g is not connected) or infinitely renormalizable (if the domain of g is connected). Then $J(g)$ carries no g-invariant line field.*

Let $U_i(1 \le i \le m)$, V be topological disks. A map $g : \bigcup_{i=1}^m U_i \to V$ is called *generalized polynomial-like* if $g|U_i : U_i \to V$ is a branched covering for any $1 \le i \le m$ and for any critical point c of g and any positive integer k, $g^k(c) \in \overline{\bigcup_{i=1}^m U_i}$. g is called *real symmetric* if for any $z \in U_i$, we have $\bar{z} \in U_i$ and $g(\bar{z}) = \overline{g(z)}$. g is called *renormalizable* if there ia an interval I, a positive integer s such that the interiors of $I, g(I), \cdots, g^{s-1}(I)$ are pairwise disjoint, $g^s(I) \subset I$, $g^s(\partial I) \subset \partial I$.

Given a rational function f of degree $d \ge 2$ and μ an f-invariant line field. Let $\mathcal{H}(f)$ denote the full dynamics generated by f, that is the collection of holomorphic maps $h : U \to V$ with the following properties: U, V are open sets in $\hat{\boldsymbol{C}}$; and there exists $i, j \in \boldsymbol{N}$ such that $f^i \circ h = f^j$. Then for any element $h : U \to V$ in $\mathcal{H}(f)$, $h^*(\mu|V) = \mu|U$. Near an almost continuous point x of μ, such that $\mu(x) \ne 0$, the line field μ looks almost parallel. So if $\mathcal{H}(f)$ contains a sequence of functions $\{h_n : U_n \to V_n\}, n + 1, 2, \cdots$ with the following properties:

(1) U_n, V_n are topological disks, and

$$diam_s(U_n) \to 0, \; diam_s(V_n) \to 0$$

as $n \to \infty$;

(2) h_n is a proper map whose degree is ≥ 2 and $\le N$;

(3) For any $u \in U_n$ such that $h_n'(u) = 0$ we have

$$\max_{z \in \partial U_n} d_s(z, u) \le C d_s(u, \partial U_n)$$

and

$$\max_{z \in \partial V_n} d_s(z, u) \le C d_s(u, \partial V_n);$$

(4)

$$d_s(U_n, x) \le C diam_s(U_n), \; d_s(V_n, x) \le C diam_s(V_n),$$

where $diam_s, d_s$ denote the diameter, the distance in the spherical metric respectively, $N \in \boldsymbol{N}$ and $C > 0$ are constants independent of n. Then μ will have to fail to be almost continuous at x or $\mu(x)$ will have to be 0. This is an idea of McMullen ([16]) and will be the initial point for our proof of the nonexistence of invariant line field on the part E_c of the Julia set.

We will consider a particular subfamily of $\mathcal{H}(f)$, which consists of maps whose germs are holomorphic maps between the puzzle pieces in the non-infinitely renormalizable case or renormalizations in the renormalizable case. Functions in the subfamily are defined near a critical point, and can be pulled back to neighborhoods of a.e. $x \in J(f)$. To control the geometry of the domains, the images and the non-linearity, we need a "complex bound".

In our consideration, the "complex bound" will come from a "real bound":

Theorem B. *Let N be a compact interval or the unit circle and $f : N \to N$ be a C^3 map with non-flat critical points. Let c be a non-periodic recurrent critical point such that $\omega(c)$ is minimal and*

(1) there is a nice interval containing c; and

(2) the critical points in $\omega(c)$ are all turning points(local extrema).

Then there is a sequence of nice intervals I_n, $n = 1, 2, \cdots$, containing c such that

$$|I_n| \to 0 \text{ as } n \to \infty$$

and the δ-neighborhood of I_n is disjoint from $\omega(c) - I_n$, where $\delta > 0$ is positive constant depending only on f.

The real bound comes from the investigation of the real dynamics. The main analytic tool in this direction is cross-ratio estimate developed since 80's. To get the real bound, the non-infinitely renormalizable case and the infinitely renormalizable case have to be done in quite different ways.

If f has only one critical point, then theorem B was proved by Sullivan [24] (in the infinitely renormalizable case) and Martens [14](in other cases). When f has more than one critical points, the infinitely renormalizable case can be done in a similar way as Sullivan did. In the non-renormalizable case, Vargas ([26]) claimed that there is an arbitrarily small symmetric interval I containing c such that a definite neighborhood of I is disjoint from $\omega(c) - I$. (However, on pp175 of [26] Vargas said that "the intervals of the new covering chains cover Λ", which is confusing to me.) Our result says more, that is, we can take the small intervals to be nice, which is crucial when considering the first return maps. Our proof of theorem B in non-renormalizable case is based on Vargas's work and uses some ideas from renormalization theory.

As an immediate consequence, we know that under the assumption of theorem B, $\omega(c)$ has one-dimensional Lebesgue measure zero. The last statement is proved by Blokh and Lyubich in the infinitely renormalizable case and also claimed by Vargas ([26]) in the non-renormalizable case. We hope that theorem B may be useful in other places of one-dimensional dynamics.

To get a complex bound from a real bound was first done by Sullivan ([24]), see [9] also. Notice that however, our "complex bound" is in a weaker sense than usual. If an object can give an estimate of distortions and the shape of domains, images for a subfamily in $\mathcal{H}(f)$, we shall call it a "complex bound". In [9], [10], "complex bound" is used to control the geometry of a family of generalized polynomial-like elements in $\mathcal{H}(f)$.

The proof of the reduced main theorem is given in the same outline of McMullen [16]. We shall divide it into two cases: the case that an arbitrarily large real bound exists and the converse case. In the former case, we shall prove a "complex bound" in the usual sense. In the latter case, the postcritical set has essentially bounded geometry and hyperbolic geometry is widely used.

The need for a "complex bound" appears in many places of holomorphic dynamics, for example, the local connectivity of Julia set, combinatorial rigidity problem and renormalization theory, etc. However, how to get such a bound for a complex polynomial is still poorly understood. This is the essential part that our method requires the objects to be real.

The qc rigidity problem for real polynomials with inflection critical points remains open. Once we can prove that theorem B holds in that case too, the argument in this paper implies the non-existence of invariant line field as well. (In the case that there is only one inflection critical point, theorem B has been proved by Levin([8]).) The combinatorial rigidity problem is much more important and seems much more difficult. We should emphasize that the main theorem does not imply that the real Fatou conjecture. It is only a step towards the solution of the conjecture. Also, since we can only obtain a complex bound in a weak sense, we cannot conclude the local connectivity of Julia sets as in [9].

In the following, we shall restrict us to the non-renormalizable cubic case and expain the method of the proof of Theorem B and the main theorem in more details. The general case can be done using the same method, but with a more complicated argument. For the detailed proof of that part and the proof of theorem A, see [21]. Let $f(z) = \pm(z^3 - 3a^2 z) + b$, where $a, b \in \boldsymbol{R}$. f has exactly two critical points $a, -a$. We assume furthermore that $\omega(a) = \omega(-a) \ni a, -a$ and is minimal. Assume that f is not hyperbolic too. Then $a, -a$ are both recurrent critical points contained in the Julia set and the filled Julia set coincide with the Julia set.

1 Proof of Theorem B

A *chain* is a sequence of intervals $\{G_i\}_{i=0}^n$ such that $f(G_i) \subset G_{i+1}$ for any $0 \le i \le n-1$. The *order* is the number of G_i's containing a critical point. The *intersection multiplicity* is the maximal number of intervals G_i which have a common interior point. The chain is called *maximal* if G_i is a maximal interval such that $f(G_i) \subset G_{i+1}$ for any $0 \le i \le n-1$. If $\{G_i\}_{i=0}^n$ is a maximal chain we shall say that G_0 is a *pull-back* of G_n. If G_i does not contain a critical point for any $0 \le i \le n-1$, then the chain is called *monotone* and G_0 is called a *monotone pull back* of G_n. If G_i does not contains a critical point for any $0 < i < n$, but G_0 contains one, we shall say that the chain is *unimodal*, and G_0 is a *unimodal pull back* of G_n.

Recall that an interval T is called *nice* if $f^n(\partial T) \cap T^o = \emptyset$ for any $n \in \boldsymbol{N}$. In this manuscript, we require furthermore that ∂T is contained in the bachward orbit of a repelling periodic point of f. (Clearly, if G and G' are two pull-back of T then either they are disjoint or one is contained in the other.) There is an arbitarily short nice interval symmetric with respect to a (or $-a$).

A *homterval* is an interval I such that $f^n|I$ is monotone for any $n \in \boldsymbol{R}$. It is known that in our setting (f has no periodic attrctor), there is no homterval. Thus for any non-trivial interval I, $\inf_{n\geq 0}|f^n(I)| > 0$. Hence for each $\eta > 0$, there is a $\xi > 0$ such that if $|I| < \xi$, then for any $n \in \boldsymbol{N}$, any component of $f^{-n}(I)$ has length $< \eta$.

Schwarzian derivative and cross-ratio estimate: Note that

$$Sf = \frac{f'''}{f'} - \frac{3}{2}(\frac{f''}{f'})^2 < 0,$$

wherever $f' \neq 0$.

For ant two intervals $J \subset\subset I$, we define the cross-ratio

$$C(I,J) = \frac{|I||J|}{|L||R|},$$

where L, R are the components of $I - J$.

Accoding to [19], once $f|I$ is a monotone, $C(f(I), f(J)) \geq C(I,J)$. Thus the following holds:

Lemma 1.1 *Let $J \subset\subset I$ be two intervals such that $f^n|I$ is monotone and such that $f^n(I)$ conatins the δ-neighborhood of $f^n(J)$, then I contains the ϵ-neighborhood of J, where $\epsilon > 0$ is a constant depending only on $\delta > 0$.*

The following two propositions were proved by E.Vargas ([26]):

Proposition 1.1 *Suppose that I is a small symmetric nice interval containing $c = a$ or $-a$ and J is the component of the first return map to I which contains c. Then either I contains a ρ-neighborhood of J or if we denote T the ρ-neighborhood of I and k the minimal positive integer such that $f^k(c) \in I$, the chain $\{T_i\}_{i=0}^k$ with $T_k = T$, $T_i \supset g^i(J)$, $0 \leq i \leq k$ has intersection multiplicity bounded from above by 14 and order bounded from above by 4, where $\rho > 0$ is a constant depending only on f.*

Proposition 1.2 *There is an arbitrarily small symmetric nice interval I containing $c = a$ or $-a$, such that I contains the ρ_1-neighborhood of J, where J is the component containing c of the domain of the first return map to I and $\rho_1 > 0$ is a constant depending only on f.*

Next we will introduce a concept *box mappings*. These mapping were originally introduced by Swiatek in the study of the dynamics of real quadratic polynomials and then generalized by Swiatek and Vargas to study the dynamics of some real cubic polynomials.

Definition 1.1 *Let $I_1(I_2, resp.)$ be symmetric nice intervals around $a(-a, resp.)$ such that they are pairwise disjoint. Let J_i^j, $j = 0, 1, \cdots, r_i$ be pairwise disjoint intervals contained in I_i such that J_i^0 contains a critical point.*

A map $B : \bigcup_{i=1}^2 \bigcup_{i=0}^{r_i} J_i^j \to \bigcup_{i=1}^b I_i$ is called a (real) box mapping (induced by f) if for each $i \in \{1,2\}$ there is a positive integer s_i and symmetric nice intervals

$$I_i = K_i^0 \supset K_i^1 \supset \cdots \supset K_i^{s_i-1} \supset K_i^{s_i} = J_i^0$$

such that for any $i = 1, 2$ and each $j = 0, 1, \cdots, r_i$ there is a $k = k(i,j) \in \{1,2\}$ and $l = l(i,j) \in \{0, 1, \cdots, s_i\}$, $p = p(i,j) \in \boldsymbol{N}$ with the following properties:

(1) $B|J_i^j = f^p|J_i^j$;

(2) there is a maximal chain $\{G_k\}_{k=0}^p$ with $G_p = K_k^l$ and $G_0 = J_i^j$, moreover the chain is unimodal if $j = 0$ and monotone otherwise;

(3) $J_i^j \cap \omega(a) \neq \emptyset$;

(4) for any $x \in J_i^j$,

$$f(x), f^2(x), \cdots, f^{p-1}(x) \notin \bigcup_{i=1}^2 J_i^0;$$

(5) for any positive integer n,

$$B^n(\pm a) \in \bigcup_{i=1}^2 \bigcup_{j=0}^{r_i} J_i^j.$$

We shall call that $\max\{s_1, s_2\}$ *the order of the box mapping B.*

If for any $i \in \{1,2\}$ and each $j \in \{0, 1, \cdots, r_i\}$, $B(\partial J_i^j) \subset \bigcup_{i=1}^2 \partial I_i$ ($B(\partial J_i^j) \subset \bigcup_{i=1}^2 \partial I_i \cup \partial J_i^0$, resp.) we shall call that the box mapping B is of type I (type II, resp.).

Let $c \in \{a, -a\}$ be a critical point of f and c'be the other critical point of f. For any nice interval $I_1 \ni c$ sufficiently short, we can construct a box mapping naturally. Let I_2 be the component containing c' of the first return map to I of f. For any $x \in \omega(c) \cap I_1$, let $J(x)$ denote the component of the first return map to I_1. Let r_1 be the number of these intervals $J(x)$'s and let $J_1^0 \ni c, J_i^1, \cdots, J_1^{r_1}$ be these intervals $J(x)$'s. Let $r_2 = 0$ and $J_2^0 = I_2$. Finally define $B_I : \bigcup_{i=1}^2 \bigcup_{j=0}^{r_i} J_i^j \to \bigcup_{i=1}^2 I_i$ to be the first return map of f to $\bigcup_{i=1}^2 I_i$. It is not difficult to check that B_I is a box mapping of type I. We shall call B_I the box mapping *associated to I*. The map B_I will play a special role and is usually the initial point for our argument.

Give a box mapping, we can construct many other box mappings by taking appropriate restrictions of some iterates of the original box mapping. The procedure is called *"renormalizing"* and the new box mappings will be called *renormalizations* of the original one.

Let $B : \bigcup_{i=1}^2 \bigcup_{i=0}^{r_i} J_i^j \to \bigcup_{i=1}^2 I_i$ be a box mapping. Let Λ be a subset of $\{1,2\}$ and $\Lambda^c = \{1,2\} - \Lambda$. We can construct a new box mapping $B_1 = \mathcal{R}(B, \Lambda)$ as follows. Let $I_{i,1} = I_i$ for $i \in \Lambda_c$ and $I_{i,1} = J_i^0$ for $i \in \Lambda$. For any $x \in \omega(a) \cap (\bigcup_{i=1}^2 I_{i,1})$, let $k(x)$ be the minimal positive integer such that

$$B^{k(x)}(x) \in \bigcup_{i=1}^2 I_{i,1}$$

($k(x)$ exists since $\omega(a)$ is minimal and condition (5) in definition holds). Let $J(x)$ be the component of the domain of $B^{k(x)}$ containing x. Define

$$B_1 : \bigcup_{i=1}^{2} \bigcup_{x \in I_{i,1} \cap \omega(a)} J(x) \to \bigcup_{i=1}^{2} I_{i,1}$$

such that $B_1|J(x) = B^{k(x)}$. Roughly speaking, B_1 is the first return map of B to $\bigcup_{i=1}^{b} I_{i,1}$. It is easy to check that B_1 is a box mapping and if R is of type I, so is B_1.

It may happen that $\mathcal{R}(B, \Lambda) = B$, that is, we in fact have not constructed a new box mapping. This case happens if and only if for any $i \in \Lambda$, $r_i = 0$ and $J_i^0 = I_i$.

Let us introduce another renormalization operator for box mappings. Let $B : \bigcup_{i=1}^{2} \bigcup_{i=0}^{r_i} J_i^j \to \bigcup_{i=1}^{2} I_i$ be a box mapping of type II. Let $\Lambda = \{i : 1 \leq i \leq 2, J_i^0 = I_i\}$. Note that since we are assuming that f is non-renormalizable, Λ consists os at most one elemnt. Assuming that $\Lambda \neq \emptyset$, let us define a box mapping $\mathcal{L}(B, \Lambda)$ as follows. To fix the notations, let $\Lambda = \{2\}$. Since f is non-renormalizable, $B(I_2) \subset I_1$. For any $x \in \omega(a) \cap I_2$, if $B(x) \in J_1^j$, then define $J(x)$ to be the maximal interval such that $B(x) \subset J_1^j$. Define

$$B_1|J(x) = B|J(x) \text{ if } B(J(x)) \subset J_1^0; \text{ and}$$

$$B_1|J(x) = B^2|J(x) \text{ otherwise.}$$

Extend B_1 to be a box mapping from $\bigcup_{j=0}^{r_1} J_1^j) \cup (\bigcup_{x \in \omega(a) \cap I_2} J(x))$ to $\bigcup_{i=1}^{2} I_i$ be defining that

$$B_1| \bigcup_{j=0}^{r_1} J_1^j) = B|(\bigcup_{j=0}^{r_1} J_1^j).$$

Then B_1 is a box mapping of type II. We just define that $\mathcal{L}(B, \Lambda) = B_1$ in this case.

Proposition 1.3 *Suppose that I is a small symmetric nice interval containing c and J is the component of the domain of the first return map to I which contains c. Then there is a box mapping $B : (\bigcup_{j=0}^{r_1} J_1^j) \cup (I_2) \to \bigcup_{i=1}^{2} I_i$ such that $I_1 = I$ and and the following holds:*

(1) B is of type I;

(2) $B(I_2) \cap J \neq \emptyset$.

Proof. Let $B : (\bigcup_{j=0}^{r} J^j) \cup (I_2) \to I_1 \cup I_2$ be the box mapping associated to I. Then $J^0 = J$ by definition. If $B(I_2) \cap J \neq \emptyset$, then B satisfied the desired condition. Assume that $B(I_2) \cap J = \emptyset$. Let $R_1 = \mathcal{L}(B, \{2\})$.

R_1 is a box mapping of type I. Let $R_{n+1} = \mathcal{R}(R_n, \{2\})$ for any $n \in \boldsymbol{N}$. All these R_n's are box mappings of type I ($n \geq 2$). Since $\omega(a)$ is minimal, there is a positive integer n_1 such that $R_{n_1+1} = R_{n_1}$.

Let $\tilde{R}_1 = R_{n_1}$. Then $\tilde{R}_1$ has the form $(\bigcup_j J_{1,1}^j) \cup (I_{2,1})$ to $\bigcup_{i=1}^{2} I_{i,1}$ with $I_{1,1} = I_1 = I$. By the construction, $I_{2,1} \subset\subset I_2$.

If $\tilde{R}_1(I_{2,1}) \cap J_{1,1}^0 \neq \emptyset$, then $\tilde{R}_1$ satisfied the desired conditions. So we assume that we are not in this case. Then we can apply the argument for $\tilde{R}_1$ instead of

B. Continue the argument, we either obtain a box mapping as required, or obtain a sequence $\tilde{R}_n : (\bigcup_j J^j_{n,1}) \cup (I_{n,2}) \to I_{n,1} \cup I_{n,2}$ of box mappings of type I with the properties $I_{n,1} = I$ and $I_{n+1,2} \subset\subset I_{n,2}$. But since $\omega(a)$ is minmal, the latter case cannot happen. Q.E.D.

Proof of Theorem B:

Let I_1 be a small symmetric nice interval containing a such that $|I|/|J| > 1+\rho_1$, where J is the component of the first return map to I of f which contains a. Let $B : (\bigcup_{j=0}^r J_1^j) \cup I_2 \to I_1 \cup I_2$ be the box mapping as in the previous proposition. Then we have the following:

Proposition 1.4 *Let K be the component of the first return map to I_2 which contains $-a$, then $|I_2|/|K| > 1 + \rho_2$, where $\rho_2 > 0$ is a constant depending only on f.*

Proof. This is an observation of Vargas, as a corollary of Proposition 1.1. Q.E.D.

For $x \in \omega(a) \cap I_2$, let $k(x) \in \boldsymbol{N}$ ne the minimal positive integer such that $B^{k(x)}(x) \in J_1^0 \cup I_2$. Let $J(x)$ be the maximal interval containing x such that $B^{k(x)}|J(x)$ is well defined. Define $R|J(x) = B^{k(x)}$ and extend it to be a map from $(\bigcup_j J_1^j) \cup (\bigcup_{x\in\omega(a)\cap I_2} J(x)$ by defining $R|J_1^j = B|J_1^j$. Then R is a box mapping of type II.

We claim that $|I_2|/|J(-a)|$ is uniformly bounded from 1. In fact, if $R(J(-a)) \subset I_1$, then such a bound comes from the bound on $|I_1|/|J_1^0|$ and if $R(J(-a)) \subset I_2$, then $J(-a) \subset K$ and thus such a bound comes from the bound on $|I_2|/|K|$.

Let $\tilde{R} : \bigcup_{i=1}^2 \bigcup_{j=0}^{\tilde{r}_i} \tilde{J}_i^j \to \bigcup_{i=1}^2 \tilde{I}_i$ be the box mapping $\mathcal{R}(R, \{1,2\})$. Then $\tilde{R}$ is a box mapping of type I such that $|\tilde{I}_i|/|\tilde{J}_i^j|$ is bounded uniformly from below for any i, j such that $J_i^j \subset\subset I_i$.

Therefore thereom B holds because of the following proposition:

Proposition 1.5 *Let $B : \bigcup_{i=1}^2 \bigcup_{j=0}^{r_i} J_i^j \to \bigcup_{i=1}^2 I_i$ be a box mapping of type I such that for any i, j, $I_i = J_i^j$ or $|I_i|/|J_i^j| > 1 + \delta$. Assume that $|I_1|, |I_2|$ are sufficiently small. Then there is a pull back of I_1 or I_2 such that the ϵ-neighborhood of K is disjoint from $\omega(a) - K$, where $\epsilon > 0$ is a constant depending only f and $\delta > 0$.*

Proof. **Case 1.** We first assume that $J_i^0 \subset\subset I_i$ for $i = 1, 2$. For any $x \in \omega(a) \cap ((I_1 \cup I_2) - (J_1^0 \cup J_2^0))$, let $n(x)$ denote the minimal positive integer such that $B^{n(x)}(x) \in \bigcup_{i=1}^2 J_i^0$. Since $\omega(a)$ is minimal, $n(x)$ is uniformly bounded. Let $x_0 \in \omega(a) \cap (I_1 \cup I_2)$ be a point such that $n = n(x_0) = \max n(x)$. Suppose that $B^n(x) \in I_1$ without loss of generality. Consider the maximal chain $\{G_i\}_{i=0}^n$ of B such that $G_n = I_1$ and $G_0 \ni x$ (G_i is a maxiaml interval such that $B|G_i$ is well-defined and $B(G_i) \subset G_{i+1}$ for any $0 \le i \le n-1$). Let $G_0' \subset G_0$ be the maximal interval such that $B^n(G_0') \subset J_1^0$. One can easily check that the chain is a a montone chain and hence G_0 contains a definite neighborhood of G_0'. By the maximality of n, $\omega(a) \cap G_0 = \omega(a) \cap G_0'$. By pulling $G_0 \supset G_0'$ to the neighborhood of a critical point, we obtain a small symmetic nice interval K, such that a definite neighborhood of K is disjoint from $\omega(a) - K$.

Case 2. Now assume that $J_2^0 = I_2$, then we must have $J_1^0 \subset I_1$. Let c be the critical point in I_2 and let J be the component of the domain of B^2 conatining c.

If $B(c) \notin J_1^0$, let $\tilde{B}$ be the first return map (of f) to $J_1^0 \cup J$, and we return to case 1. Assume that $B(c) \in J_1^0$, then for any $x \in omega(a) \cap (I_1 - J_1^0)$ (note such an x exists), let $n(x)$ denote the minimal positive integer such that $B^{n(x)}(x) \in J_1^0 \cup J$. The argument in case 1 remains valid in this case. Q.E.D.

2 Proof of the Main Theorem

Case I: A large real bound exists.

Proposition 2.1 *Suppose that there exists a sequence of nice symmetric interval $I_n \ni a$ such that $|I_n| \to 0$ as $n \to \infty$ and $d(\omega(a) \cap I_n, omega(a) - I_n)/diam(\omega(a) \cap I_n) \to \infty$ as $n \to \infty$. Then for n sufficiently large, the box mapping associated to I_n has an "extension" to a holomorphic box mapping $F_n : (\bigcup_{j=0}^{r(n)} U^j(n)) \cup W(n) \to \overline{V(n)} \cup W(n)$ such that there is a topological disk $V'(n) \supset\supset V(n)$ with $mod(V'(n) - \overline{V(n)}) \to \infty$ as $n \to \infty$.*

A map $F : \bigcup_{j=0}^{r} U_j \cup W \to V \cup W$ is called a holomorphic box mapping if

(1) V, W are disjoint topological disks and U_j's are disjoint topological disks contained in V;

(2) $F|W : W \to V$ is a brached covering with a unique critical point c',;

(3) for each $0 \leq j \leq r$, $F(U_j) = V$ or W and $F|U_0$ is a banched covering with a unique critical point c, $F|U_j$ is conformal if $j \neq 0$; and

(4) for each $n \in \boldsymbol{N}$, $F^n(c), F^n(c') \in \bigcup_{j=0}^{r} U_j \cup W$;

(5) For any $x \in \omega(a) \cap (V \cup W)$, if $F(x) = f^s(x)$, then $f(x), f^2(x), \cdots, f^{s-1}(x) \notin V \cup W$.

Proof. Let $B : (\bigcup_{j=0}^{r} J_1^j) \cup I_2 \to I_1 \cup I_2$ be the box mapping associated to I_n. Using Yoccoz puzzle partition construction, B extends to a holomorphix box mapping $G_n : (\bigcup_{j=0}^{r} U_j') \cup W' \to V' \cup W'$. For such an extension, we cannot gurantee the existence of $V'(n)$. Let $B|I_2 = f^k$ and $B|J_1^j = f^{k_j}$.

Suppose $n >> 1$ and thus $d(\omega(a) \cap I_n)/diam(\omega(a) I_n) > M$, where $M > 1$ is a large number. For simplicity, let us assume $d(\omega(a) \cap I_n, \omega(a) - I_n)/|I_n| > M$. The other cases acn be done in the same way (with a little refinement, in that case, the holomorphic box mapping may fail to be an extension of the real box mapping in strict sense). Let D be a round disk centered at the critical point $c \in I_1$ with radius $\frac{M}{2}$. Let E be the component of $f^{-k}(D)$ containing I_2. Then E is "almost round" as seen from any point on I_2. For any $0 \leq j \leq r$, let D_j be the component of $f^{-k_j}(D)$ (or $f^{k_j}(E)$) containing J_1^j if $B(J_1^j) \subset I_1$ (or $B(J_1^j) \subset I_2$). Then D_j is "almost round" as seen from a point on J_1^j. Since the radius of D_j is at most $(1 + o(1))diam(I_1)$, $D_j \subset\subset D$.

Let U_j be the component of $D_j \cap U_j'$ containing J_1^j for $0 \leq j \leq r$, and V be the component of $D \cap V''$ conatining I_1, W be the component of $E \cap W'$ conatining I_2. Then the map $F : \bigcup_{j=0}^{r} U_j \cup W \to V \cup W$ defined by $F|W = f^k, F|U_j = f^{k_j}$ is the desired holomorphic box mapping. Q.E.D.

Corollary 2.2 *In this case, f carries no invariant line field on the Julia set.*

Proof. Let F_n be the holomorphic box mapping obtained in the previous proposition. To fix the notations, we assume that $a \in U^0(n)$ and $-a \in W(n)$. For any n, let s_n be the minimal positive integer such that $F_n(a) \in W(n)$, and $\tilde{W}(n)$ be the component of the domain of $F_n^{s_n}$ which contains a. Let $\tilde{V}(n) = W(n)$ and let $\tilde{F}_n$ be the first return map to $\tilde{V}(n) \cup \tilde{W}(n)$ of F_n, restristed to those components containing points in $\omega(a)$, then $\tilde{F}_n$ is also a holomorphic box mapping.

Take a point $x \in J(f)$ such that $\omega(x) = \omega(a)$ and such that x is not in the backward orbit of any critical point. Almost every point in the Julia set satisfy these conditions. We only need to show that for any f-invariant line field μ, $\mu(x) = 0$ or μ is not almost continuous at x.

Let $k_1^n = \{k \in \boldsymbol{N} \cup \{0\} : f^k(x) \in U^0(n)\}$ and $k_2^n = \{k \in \boldsymbol{N} \cup \{0\} : f^k(x) \in W(n)\}$.

Case 1. $k_1^n < k_2^n$ for infinitely many n.

For such n, there is a univalent branch g_n of $f^{-k_1^n}$ defined on $V(n)$ sending $f^{k_1^n(x)}$ to x. Let $h_n : g_n(U^0(n)) \to g_n(V(n)$ be the map $g_n \circ F_n \circ f^{k_1^n}$. Then $U_0(n), V(n) \ni x$. It is easy to see that $diam(V(n)) \to 0$ as $n \to \infty$. Therefore, after a little refinement, μ looks "almost parallel" on U_0^n and V_n, and $\{h_n\}$ forms a family of uniformly non-linerity (after approriate rescaling). These imply that $\mu(x) = 0$ or μ is not almost continous at x.

Case 2. $k_1^n > k_2^n$ for $n >> 1$.

In this case, for $n >> 1$, there is a univalent branch g_n of $f^{-k_2^n}$ defined on $\tilde{V}(n) = W(n)$, sending $f^{k_2^n}(x)$ to x.

Considering $\tilde{F}_n$ instead of F_n, we can define $\tilde{k}_i^n$, $i = 1, 2$ as before. If $\tilde{k}_1^n < \tilde{k}_2^n$ for infinitely many n, then we come back to Case 1. So assume that $\tilde{k}_2^n < \tilde{k}_1^n$ for $n >> 1$. We have then a univalent branch $\tilde{g}_n$ of $f^{-\tilde{k}_2^n}$ defined on $\tilde{W}(n)$, sending $f^{\tilde{k}_2^n}(x)$ to x. Let $h_n = g_n \circ \tilde{F}_n \circ f^{\tilde{k}_2^n} : \tilde{g}_n(\tilde{W}(n)) \to g_n(\tilde{V}(n))$. These maps provide a unformly non-linear family near x. The proof is completed. Q.E.D.

Case II: A large real bound does not exist.

From now on, we assume that there is an $M > 0$ such that for any symmetric nice interval I, $d(\omega(a) \cap I, \omega(a) - I)/diam(\omega(a) \cap I) \leq M$.

We shall call a symmetric nice interval I *delta-excellent* if the δ-neighborhood of I is disjoint from $\omega(a) - I$, and for any component J of the domain of the first return map to I (of f) intersecting $\omega(a)$, we have that the δ-neighborhood of J is contained in I. When we investigated the real dynamics, we in fact obtained the following:

Proposition 2.3 *For some $\delta > 0$, there exists an arbitarily small δ-excellent interval containing a (or $-a$).*

By defining an analogy of Yoccoz's τ-function, we obtained the following:

Proposition 2.4 *Let I be a sufficiently small symmetric δ-excellent interval. Then there is a constant $N > 0$ depending only on δ and M such that I has at most N unimodal pull back.*

For the proof, see[21]. In particular, if $\omega(a)$ is minimal but a is reluctantly recurrent, then $\omega(a)$ must has unbounded geometry.

Let $\mathcal{I}_\delta$ be the collection of δ-excellent intervals. Let $I \in \mathcal{I}_\delta$ with $|I| << 1$. Let $J_i, i = 1, \cdots, n$ be the components of the domain of the first return map to I intersecting $\omega(c) - I$ and let $J_0 = I$. The following is easy:

Lemma 2.1 *There is a constant $\delta_1 > 0$ such that for any $0 \leq i \leq n$,*

$$\frac{d_e(\omega(c) \cap J_i, \omega(c) - J_i)}{diam_e(\omega(c) \cap J_i)} \geq \delta.$$

Let us denote by γ_i the unique simple geodesic in $\boldsymbol{C} - \omega(c)$ separating $\omega(c) \cap J_i$ from $\omega(c) - J_i$. Denote by $l(\gamma_i)$ the length of γ_i in the hyperbolic metric of $\boldsymbol{C} - \omega(c)$. Let Ω_i denote the topological disk bounded by γ_i. Due to the previous lemma and the existence of the number M, there is two constants $0 < \delta_2 < \delta_3 < \infty$ such that for any i, $\delta_2 \leq l(\gamma_i) \leq \delta_3$.

From now on, we shall use d to denote the distance in the hyperbolic metric of $\boldsymbol{C} - \omega(a)$, and use d_e to denote the Euclean distance in $\boldsymbol{C}$.

Let $X = \boldsymbol{C} - \bigcup_i \Omega_i$. X is a finitely connected planar hyperbolic Riemann surface with one cusp, whose remaining ends are cut off by geodesic $\gamma_0, \gamma_1, \cdots, \gamma_n$, $n \geq 1$. Since the hyperbolic length of γ_i is uniformly bounded from zero, by a theorem of McMullen, there is a constant $\delta_4 > 0$ such that there are $i, j \in \{0, 1, \cdots, n\}$, $i \neq j$ with

$$d(\gamma_i, \gamma_j) \leq \delta_4.$$

For any $z \in \boldsymbol{C}$ such that $g^n(z) \notin P(g)$, let $\|(g^n)'(z)\|$ denote the norm of $(g^n)'$ measured in the hyperbolic metric of $\boldsymbol{C} - P(g)$. By considering pull back to the neighborhood of a critical point, we have the following:

Proposition 2.5 *There is positive constants $\delta_5, \delta_6, \delta_7, \delta_8$, a nonnegative integer p, two symmetric admissible intervals K, L which are pull backs of I with the following properties:*

(1)

$$\frac{d_e(\omega(c) \cap K, \omega(c) - K)}{diam_e(\omega(c) \cap K)} \geq \delta_5, \quad \frac{d_e(\omega(c) \cap L, \omega(c) - L)}{diam_e(\omega(c) \cap L)} \geq \delta_5;$$

(2) there exists a domain $\tilde{\Omega}_L$ such that $f^p : \tilde{\Omega}_L \to \Omega_L$ is a conformal map, and

$$d(\gamma_K, \partial\tilde{\Omega}_L) \leq \delta_6, \quad \delta_7 \leq l(\partial\tilde{\Omega}_L) \leq \delta_8,$$

where $\gamma_K(\gamma_L, resp.)$ is the simple geodesic separating $\omega(c) \cap K(\omega(c) \cap L, resp.)$ from $\omega(c) - K(\omega(c) - L, resp.)$, and $\Omega_K(\Omega_L)$ is the topological disk bounded by $\gamma_K(\gamma_L, resp.)$.

Let K and L be the symmetric nice intervals constructed in the last proposition. Let

$$B_K : (\bigcup_{j=0}^{r} K_1^j) \cup K_2 \to \bigcup_{i=1}^{2} K_i$$

denote the real box mapping associated to K. Since I is a δ-excellent interval, we know that both of K and L are δ'-excellent intervals for some $\delta' > 0$ depending only on δ. By proposition 2.4, either of K and L has at most $N = N(\delta)$ unimodal pull backs.

For the moment, let us fix a point $x \in J(f)$ such that $\omega(x) = \omega(a)$ and such that x is not in the backward orbit of any critical point of f.

Proposition 2.6 *There is a constant $C > 1$, a domain $\Omega \in \{\Omega_{K_1}, \Omega_{K_2}, \tilde{\Omega}_L\}$ and a nonnegative integer k such that there is a univalent branch h of g^{-k} defined on Ω and $d_e(x, h(\Omega)) \leq C\, diam_e(h(\Omega))$.*

Proof. The proof will be divided in two cases. Case 1 will be done similarly as the case that we have decay geometry (a large real bound), case 2 has to be done in a different way.

Let

$$R = R_K : (\bigcup_{j=0}^{r} K_1^j) \cup K_2) \to K_1 = K$$

denote the first return map to K. Write $R|K_1^j = f^{s_1^j}$, and denote by D_j the components of $f^{-s_1^j}(\Omega_K)$ containing $\omega(a) \cap K_1^j$. Write $R|K_2 = f^{s_2}|K_2$ and let $\tilde{\Omega}_{K_2}$ be the component of $f^{-s_i}(\Omega_K)$ containing $K_i \cap \omega(a)$. Let $s_i' \leq s_i$ be the positive integer such that $B_K|K_i = f^{s_i'}$, where B_K is the box mapping associated to K.

Lemma 2.2 *There is a constant $\delta_{13} > 0$ such that*

$$diam(\bigcup_{j=0}^{r} D_j - \Omega_K) \leq \delta_{13}.$$

proof. Any component E of $D_j - \Omega_K$ is a topological disk whose boundary is contained in $\partial D_j \cup \partial Omega_K$. Since $l(\partial D_j)$ is bounded from above uniformly, so is the diameter of E. Since $\partial E \cap \partial \Omega_K \neq \emptyset$, the proof is completed. qed of lemma

Continuation of proof of Proposition 2.6:
Case 1. For any nonnegative integer k, $d(f^k(x), \partial\Omega_K) > \delta_{13}$.
In particular this means that for any $k \geq 0$,

$$f^k(x) \notin (\bigcup_{j=0}^{r} D_j - \Omega_K).$$

This case can be done using a similar argument as in case I.

Case 2. There is a nonnegative integer k such that $d(\partial\Omega_K, f^k(x)) \leq \delta_{13}$.

In this case we shall show that there is a univalent branch h of f^{-k} defined on $\Omega = \Omega_K$ or $\tilde{\Omega}_L$ such that $d_e(a, h(\Omega)) \leq \delta_{14}\mathrm{diam}_e(h(\Omega))$.

Let us call a topological disk D *admissible* if $\partial D \cap omega(a) = \emptyset$ and there is an admissible interval I such that $D \cap \omega(a) = I \cap \omega(a)$. Let P_1, P_2 be two admissible topological disks and k be a nonnegative integer, we say that the triple (k, P_1, P_2) is *bounded by a constant* $C > 0$ if

$$d(f^k(x), \partial P_i) \leq C, \ \mathrm{l}(\partial P_i) \leq C$$

for $i = 1, 2$.

Convention: For a Jordan curve we always give it the anticlockwise orientation.

Lemma 2.3 *Let P be a topological disk such that $\partial P \cap \omega(a) = \emptyset$ and k be a non-negative integer. Let $\rho \subset \mathbf{C} - \omega(a)$ be a path with initial point $f^k(x)$ and endpoint $w \in \partial P$ such that if we denote ρ' the lift of ρ under f^k with initial point x and z the endpoint of ρ', then the lift of ∂P, considered as a loop based at w, with initial point z is a closed Jordan curve. Suppose*

$$l(\rho) \le C, \ C \ge l(\partial P) \ge \epsilon, \ l_e(\partial P)^2 \le C\, area(P)$$

and the injecticity radius of $f^k(x)$ in $\mathbf{C} - \omega(a)$ is $\ge \eta$, where $C > 0$, $\epsilon > 0$ and $\eta > 0$ are constants.

Then there is a constant C' depending only on C, ϵ and η and a univalent branch $h : P \to \mathbf{C}$ of f^{-k} such that

$$d_e(x, h(P)) \le C' diam_e(h(P)).$$

proof. Let γ denote the lift of ∂P with initial point z and let D denote the topological disk bounded by γ, then $f^k : D \to P$ is a conformal mapping. Let h denote the inverse of this conformal mapping. h has an analytic continuation along the path ρ^{-1}.

Since $\rho \cup \partial P$ has diameter bounded from above in the hyperbolic metric of $\mathbf{C} - \omega(a)$ and the point $f^k(x)$ has injectivity radius bounded from zero, there is a constant $\eta_1 > 0$ depending only on C and η such that each point in $\xi \cup \partial P$ has injectivity radius $\ge \eta_1$. Consequently, $\rho \cup \partial P$ can be covered by finitely many embedded disks in $\mathbf{C} - \omega(a)$ and the number of these disks is bounded from above. So it follows from Koebe's distortion theorem that $h|(\rho \cup \partial P)$ has bounded distortion, where we measure $|h'|$ in the Euclidean metric. Since h is conformal on P, it has bounded distortion on P. So $l_e(\partial h(P))^2/\text{area}_e(h(P))$ is bounded from above, and hence $l_e(\partial h(P))/\text{diam}_e(h(P))$ is bounded from above.

Since the injectivity radius is bounded from zero on $\rho \cup \partial P$ and $\text{diam}(\rho \cup \partial P)$ is bounded from above, by Koebe's distortion theorem the ratio of the Euclidean metric to the hyperbolic metric is comparable on the set. So $l_e(\rho) \le C_1 l_e(\partial P)$ and hence

$$d_e(x, h(P)) \le l_e(\rho') \le C_2 l_e(\partial h(P)) \le C' \text{diam}_e(h(P)).$$

qed of lemma 2.3.

Corollary 2.7 *Let T_i be a small symmetric nice interval and P_i be an admissible topological disk such that $P_i \cap \omega(a) = T_i \cap \omega(a)$, $i = 1, 2$. Assume that $T_1 \cap T_2 = \emptyset$. Let k be a positive integer such that the triple (k, P_1, P_2) is bounded by a constant $C > 0$. Then there is a constant $m(C)$ depending only on C and an $i \in \{1, 2\}$ such that one of the following holds:*

(1) A univalent branch h of f^{-k} can be defined on P_i such that $d_e(x, h(P_i)) \le C' diam_e(h(P_i))$; or

(2) There are two interval $T_{i,1}$, $T_{i,2}$, two topological disk $P_{i,1}$, $P_{i,2}$ a positive integer $k' < k$ such that

(2.i) $f(T_{i,1}) = f(T_{i,2})$, $f(P_{i,1}) = f(P_{i,2})$, $f^{k-k_1}(P_{i,1}) = P_i$;

(2.ii) $T_{i,j} \cap \omega(a) = P_{i,j} \cap \omega(a) \neq \emptyset$, $j = 1, 2$;

(2.iii) the triple $(k', P_{i,1}, P_{i,2})$ is bounded by $m(C)$;

(2.iv) for each $j = 1, 2$, there is a monotone maximal chain $\{G_m(j)\}_{m=0}^{k-k'}$ such that $G_m(j) = T_i$ and $G_0(j) = T_{i,j}$.

Proof. First let us show that the injectivity radius of $f^k(x)$ is bounded from zero. Since $l(\partial P_1) \leq C$, $d(\gamma_{T_1}, \partial P_1))$ is bounded from above, and therefore $d(\gamma_{T_1}, f^k(x))$ is bounded from above. Since $l_h(\gamma_{T_1})$ is not too small, a point on γ_{T_1} has injectivity radius bounded from zero, and hence so does $f^k(x)$.

Let ξ_i denote the shortest geodesic (in the hyperbolic surface $\boldsymbol{C} - \omega(a)$) from $f^k(x)$ to ∂P_i, $i = 1, 2$. Then

$$l(\xi_i) \leq C, \ i = 1, 2.$$

Let ξ_i' denote the lift of ξ_i with initial point x under f^k, $i = 1, 2$. Let z_i denote the endpoint of ξ_i'. Let ζ_i denote the lift of ∂P_i under f^k with initial point z_i.

If either of ζ_1 and ζ_2 is a closed Jordan curve then we are in Case (1) by lemma 2.3 since $l(\partial P_i) \geq l(\gamma_{T_i})$ is bounded from zero.

Assume we are not in this case. Let k_i be the maximal integer such that $f^{k_i}(\zeta_i)$ is not a closed Jordan curve. Without loss of generality, assume that $k_1 \geq k_2$. Let Δ_i denote the domain bounded by $f^{k_i+1}(\xi_i')$, then both Δ_1 and Δ_2 contains a critical value of f. It is not difficult to see that $f^{k_1+1}(\zeta_1 \cup \zeta_2 \cup \xi_1' \cup \xi_2')$ has small diameter in the Euclidean metric provided that both $|T_1|$ and $|T_2|$ are small. So we may assume that $k_1 > k_2$. By the same reason, the set $f^{k_1}(\zeta_1 \cup \zeta_2 \cup \xi_1' \cup \xi_2')$ is close to a critical point c which is contained in the set $\omega_f(x)$. Let U be a small neighborhood of c such that $f|U : U \to f(U)$ is a brached covering with a unique critical point c and $\phi : U \to U$ is the prime transformation of the branched covering such that the lift of $\partial f(U)$, considered as a loop based at $f(z)(z \in \partial U)$, with initial point z under f is ended by $\phi(z)$.

Let

$$\rho = \xi_2 * \xi_1^{-1} * \partial P_1 * \xi_1.$$

ρ is a piece smooth Jordan curve from $f^k(x)$ to ∂P_2 whose hyperbolic length is bounded from above. Let ρ' be the lift of ρ with initial point x under f^k and z' the endpoint of ρ'. The endpoint $f^{k_1}(z')$ of $g^{k_1}(\rho')$ is $\phi(f^{k_1}(z_2))$. The lift of ∂P_2 with initial point $\phi(f^{k_1}(z_2))$ is the Jordan curve $\phi(f^{k_1}(\zeta_2))$, which bounds a topological disk $\phi(f^{k_1-k_2-1}(\Delta_2))$.

Let $P_{2,1} = f^{k_1-k_2-1}(\Delta_2)$ and $P_{2,2} = \phi(P_{2,1})$. Then $f(P_{2,1}) = f(P_{2,2})$. If $P_{2,2} \cap \omega(a) = \emptyset$, then the lift of ∂P_2 under g^k with initial point z' is obviously a closed Jordan curve and hence by lemma 2.3, we are in Case (1). So let us assume $P_{2,2} \cap \omega(a) \neq \emptyset$. Obviously both $P_{2,1}$ and $P_{2,2}$ are admissible topological disks, and the only non-trivial part to be verified is (2iii). It follows from the Schwarz lemma. qed of Corollary 2.7

We can complete the proof of Proposition 2.6 now.

Continuation of Proposition 2.6:

If $\tilde{\Omega}_L \cap \omega(a) = \emptyset$, then it follows from lemma 2.3 that we can take $\Omega = \tilde{\Omega}_L$ to conclude the proof. So assume that $\tilde{\Omega}_L \cap \omega(a) \neq \emptyset$. Then $\tilde{\Omega}_L$ is an admissible topological disk. Let $P_1 = \Omega_K$ and $P_2 = \tilde{\Omega}_L$. Let T_i be the admissible interval such that $T_i \cap \omega(a) = P_i \cap \omega(a)$, $i = 1, 2$. The triple (k, P_1, P_2) is bounded by $C_0 = \delta_2 + \delta_6 + \delta_8 + \delta_{13}$. Notice that $T_1 = K$ and T_2 is a monotone pull back of L.

Apply corollary 2.7 to the triple (k, P_1, P_2), we have two possiblities. If we are in Case (1) in that corollary, then the proof is completed. Assume that we are in Case (2). Then we have $i_0 \in \{1,2\}$ and another triple $(k_1, P_{i_01}, P_{i_02})$ which is bounded by some constant $C_1 = m(C_0)$. Let T_{i_0j} be the admissible interval such that $T_{i_0j} \cap \omega(a) = P_{i_0j} \cap \omega(a) \neq \emptyset$, $j = 1, 2$. Both of T_{i_01} and T_{i_02} are monotone pull back of T_{i_0}.

Apply corollary 2.7 to the triple $(k_1, P_{i_01}, P_{i_02})$ and so on. Either we complete the proof within $N+1$ steps, or we will have $i_0, i_1, \cdots, i_N \in \{1,2\}$ and admissible intervals $T_i, T_{i_0j}, T_{i_0i_1j}, \cdots, T_{i_0i_1\cdots i_Nj}$, $j = 1, 2$, intersecting $\omega(a)$ such that for any $0 \leq s \leq N$, $T_{i_0i_1\cdots i_s1} \neq T_{i_0i_1\cdots i_s2}$ are both monotone pull back of $T_{i_0i_1\cdots i_s}$. For any $i \in \{1,2\}$, let i' denote the element of $\{1,2\} - \{i\}$. Let

$$\mathbf{S} = \{T_{i_0i_1'}, \cdots, T_{i_0i_1\cdots i_N'}, T_{i_0i_1\cdots i_N1}, T_{i_0i_1\cdots i_N2}\}.$$

Then $\mathbf{S}$ has $N+1$ elements, which are all monotone pull back of T_{i_0}. For each $S \in \mathbf{S}$, let $k(S)$ be the minimal positive integer such that $f^{k(S)}(c(S)) \in S$ for some $c(S) \in \{a, -a\}$ and let $T(S)$ be the pull back of S along the orbit $\{f^j(c(S))\}_{j=0}^{k(S)}$. Then $T(S)$ is a unimodal pull back of $T_1 = K$ (if $i_0 = 1$) or L (if $i_0 = 2$). It is easy to see that for $S, S' \in \mathbf{S}$ with $S \neq S'$, $T(S)$ and $T(S')$ are different. So we know that either K or L has at least $N+1$ children, which is a contradiction. Q.E.D.

Corollary 2.8 *For any f-invariant line field μ, $\mu(x) = 0$ or μ is not almost continuous at x.*

Proof. For any $I \in \mathcal{I}_\delta$ sufficiently short, we have proved in proposition 2.6 that there exists a pull back $T \in \mathcal{I}_{\delta'}$ of I and an admissible topological disk Ω such that
(1) $\Omega \cap \omega(a) = T \cap \omega(a)$; and
(2) there exists a univalent branch h of some f^{-k} defined on Ω such that

$$d_e(x, h(\Omega)) \leq C diam_e(h(\Omega)).$$

Let T' be the maximal open interval containing T such that $T' \cap \omega(a) = T \cap \omega(a)$ and let $D = \boldsymbol{C} - (\boldsymbol{R} - T')$. Then h extends to a univalent function on D. Also by the possibilities of Ω we have that mod$(D - \bar{\Omega})$ is bounded from zero uniformly.

Let T_j, $j = 0, 1, \cdots$ be the components intersecting $T \cap \omega(a)$ of the domain of the first return map of f to T. By the existence of the bound M, it is not difficult to show that there exists j_0 such that $R|T_j$ is not monotone and $|T_{j_0}|$ is comparable to $|T|$. Let $s \in \boldsymbol{N}$ be such that $R|T_{j_0} = f^s|T_{j_0}$ and let Ω_0 be the component of $f^{-s}(\Omega)$ containing $T_{j_0} \cap P(g)$. Then $\Omega_0 \subset D$ and $f^s : \Omega_0 \to \Omega$ is a proper map whose degree is bounded from above uniformly and bounded from below by 2.

We can then use pull back to define a sequence $\{h_n : U_n \to V_n\}$ in $\mathcal{H}(f)$ such that U_n, V_n has uniformly "good shape", $diam_e(U_n), diam_e(V_n) \to 0$ as $n \to \infty$, $d_e(x, U_n)/diam_e(U_n)$, $diam_e(x, V_n)/diam_e(V_n)$ are uniformly bounded and such that after approriate rescaling, h_n's are uniformly linear. Thus the corollary holds. Q.E.D. Since for a.e. $x \in J(f)$, $\omega(x) = \omega(a)$ and $f^k(x) \notin \{a, -a\}$ for any $k \in \boldsymbol{N} \cup \{0\}$, we have

Corollary 2.9 *If there is a positive constant M such that for any nice interval I,*

$$\frac{d_e(\omega(a) \cap I, \omega(a) - I)}{diam_e(\omega(a) \cap I)} \leq M,$$

then $J(f)$ carries no invariant line field.

Combined with Corollary 2.2, we have proved the main theorem.

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References

[1] L. Ahlfors. Lectures on quasi-conformal maps. Van Nostrand Co, 1966

[2] B. Branner and J.H. Hubbard. The iteration of cubic polynomials I. Acta Math. 160,(1988), 143-206.

[3] B. Branner and J.H. Hubbard. The iteration of cubic polynomials II. Acta Math. 169, (1992), 229-325.

[4] A. Douady and J.H. Hubbard. On the dynamics of polynomial-like mappings. Ann. Scient., Ec. Norm. Sup. 4^e series, 18 (1985) 287-343.

[5] J.H. Hubbard. Local connectivity of Julia sets and bifurcation loci: three theorems of J.-C. Yoccoz. In: "Topological Methods in Modern Mathematics, A Symposium in Honor of John Milnor's 60th Birthday". Publish or Perish, 1993.

[6] J. Graczyk and G. Swiatek. Generic hyperbolicity in the logistic family. Annals of Math. 146(1997), 1-52.

[7] O. S. Kozlovski. Structurally stability in one-dimensional dynamics. Thesis, Amersterdam University (1998).

[8] G. Levin. Bounds for maps of an interval with one reflecting critical point. I. Fundamenta Mathematicae. 157 (1998)

[9] G. Levin and S. van Strien. Local connectivity of Julia sets of real polynomials. Annals of Math, 1998.

[10] G. Levin and S. van Strien. Total disconnectedness of the Julia set of real polynomials. Preprint IHES, 1996/68(1996). To appear in Asterique.

[11] M. Lyubich. On the Lebesgue measure of the Julia set of a quadratic polynomial. Stony Brook IMS Preprint 1991/10.

[12] M. Lyubich. Dynamics of quadratic polynomials, I and II. Acta Math. 178(1997)185-297.

[13] M. Lyubich and J. Milnor. The unimodal Fibonacci maps. Journal of the A.M.S. 6, 425-457,(1993).

[14] M. Martens. Interval dynamics. Thesis, Delft (1990).

[15] M. Martens, W. de Melo and S. van Strien. The Julia-Fatou-Sullivan theory in one-dimensional dynamics. Acta Math., 1992

[16] C. McMullen. Complex dynamics and renormalization. Annals of Math. Studies, v. 135, Princeton University Press, 1996.

[17] C. McMullen. Renormalization and 3-manifolds which fiber over the circle. Annals of Math. Studies, v. 142, Princeton University Press, 1994.

[18] C. McMullen and D. Sullivan. Quasiconformal homeomorphisms and dynamics III: The Teichmuller space of a holomorphic dynamical system. Adv. Math.

[19] W. de Melo and S. van Strien. One-dimensional dynamics. Springer, 1993.

[20] J. Milnor. Dynamics in One Complex Variable: Introductory Lectures. Stony Brook IMS Preprint, 1990/5.

[21] W. Shen. On the qc rigidity of real polynomials. In preparation.

[22] M. Shishikura. Unpublished

[23] M. Shishikura. Manuscript, 1998.

[24] D. Sullivan. Bounds, quadratic differentials and renormalization conjectures. In "A.M.S. Centennial Publications, vol. 2, Mathematics into The Twenty-first Century (1988)".

[25] Urbanski. Rational functions with no recurrent critical points. Erg. Th. and Dynam. Sys. 14 (1994), 391-414.

[26] E. Vargas. Measure of minimal sets of polymodal maps. Ergod. Th. and Dynam. Sys. 16(1996), 159-178.