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Construction of the fundamental solution for a degenerate equation and a local version of Riemann-Roch theorem

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§1. Introduction. Let M be a compact Kaehler manifold whose complex dimension is n. The following Riemann-Roch theorem holds for M.

$$\sum_{q=0}^{n}(-1)^{q}\dim H_{q}(M)=\int_{M}(2\pi i)^{-n}[Td(T(M))]_{2n},$$

where

$$Td(T(M))=\det(\frac{\Omega}{e^{\Omega}-1})$$

with a curvature form Ω.

Analytical proofs of the above theorem are based on the following formula :

$$\sum_{q=0}^{n}(-1)^{q}\dim H_{q}(M)=\int_{M}\sum_{q=0}^{n}(-1)^{q}\operatorname{tr}e_{q}(t,x,x)dv, \tag{1.1}$$

where $e_{q}(t,x,y)$ denotes the kernel of the fundamental solution $E_{q}(t)$ for the heat equation for $\Delta_{q}=d"\vartheta"+\vartheta"d"$ acting on differental $(0,q)$-forms $A^{(0,q)}(M)$ $=\Gamma(\wedge^{q}T^{*(0,1)}(M))$. The author have shown the local version of Gauss-Bonnet-Chern theorem for a compact manifold with boundary, by constructing the fundamental solution according to the technique of symbolic calculus of pseudodifferential operators [10]. In this note, we show that the following formula, which is called a local version of Riemann-Roch theorem holds.

local version of Riemann-Roch Theorem

$$\sum_{q=0}^{n}(-1)^q \operatorname{tr}e_q(t,x,x)dv = (2\pi i)^{-n}[Td(T(M))]_{2n} + 0(t^{\frac{1}{2}}) \tag{1.2}$$

We shall give a rough sketch of a proof of the above formula, constructing the fundamental solution according to the method of symbolic caluculus for a degenerate parabolic operator as [8] instead of that of a parabolic operator. (See §4). Our point is that we can prove the above formula by only calculating the main term of the fundamental solution, introduing a new weight of symbols of pseudodifferential operators.

There are several papers for this problem. T.Kotake[7] proved this formula for manifolds of dimension 1. Then V.K.Patodi[12] has proved for Kaehler manifolds of any dimension. P.B.Gilkey[6] also has shown, using invariant theory. E.Getzler[5] treated this problem by different approach.

§2. The representation of Δ. Let M be a smooth Kaehler manifold with a hermitian metric g. Set $Z_1, Z_2, \cdots, Z_n$ be a local orthonormal frame of $T^{1,0}(M)$ in a local patch of chart U. And let $\omega^1, \omega^2, \cdots, \omega^n$ be its dual.

The differential d'' and its dual ϑ'' acting on $A^{0,q}(M)$ are given as follow, using the Levi-Civita connection ∇.

$$d'' = \sum_{j=1}^{n} e(\bar{\omega}^j)\nabla_{\bar{Z}_j}, \qquad \vartheta'' = -\sum_{j=1}^{n} \imath(\bar{Z}_j)\nabla_{Z_j}$$

where we use the following notations.

Notations.

$$Z_{\bar{j}} = \bar{Z}_j, \qquad \omega^{\bar{j}} = \bar{\omega}^j \qquad j = 1, \cdots, n,$$

$$e(\omega^\alpha)\omega = \omega^\alpha \wedge \omega, \ \imath(Z_\alpha)\omega(Y_1, \cdots, Y_{p-1}) = \omega(Z_\alpha, Y_1, \cdots, Y_{p-1}).$$

Let $c^\gamma_{\alpha,\beta}, \alpha, \beta, \gamma \in \{1, \cdots, n, \bar{1}, \cdots, \bar{n}\}$ be the following functions and let $R(Z_\alpha, \bar{Z}_\beta)$ be the curvature transformation.

$$\nabla_{Z_\alpha} Z_\beta = \sum_\gamma c^\gamma_{\alpha,\beta} Z_\gamma,$$

$$R(Z_\alpha, Z_\beta) = [\nabla_{Z_\alpha}, \nabla_{Z_\beta}] - \nabla_{[Z_\alpha, Z_\beta]},$$

$$R(Z_i, Z_{\bar{j}})Z_\beta = \sum_\gamma R^\gamma{}_{\beta i \bar{j}} Z_\gamma.$$

Remark 1. Owing to that $\{Z_j, \bar{Z}_j\}_{j=1,\cdots,n}$ is an orthnormal frame, we have $R^{\bar{i}}_{\bar{j}k\bar{\ell}} = R_{i\bar{j}k\bar{\ell}} = g(R(Z_k, \bar{Z}_\ell)\bar{Z}_j, Z_i) = R(Z_i, \bar{Z}_j, Z_k, \bar{Z}_\ell)$.

From the fact that our connection is the Riemannian connection and that M is a Kaehler manifold we have

Proposition 1. *The coefficients $c^\gamma_{\alpha,\beta}$ of connection form satisfy*

$$c^{\bar{i}}_{\alpha j} = c^{i}_{\alpha,\bar{j}} = 0, \qquad c^{i}_{\alpha,j} = -c^{\bar{j}}_{\alpha,\bar{i}},$$

$$\alpha \in \{1, \cdots, n, \bar{1}, \cdots, \bar{n}\}, i, j \in \{1, \cdots, n, \}$$

$$R(Z_i, Z_j) = 0, \quad R(\bar{Z}_i, \bar{Z}_j) = 0,$$

$$R^{\bar{k}}_{\ell i\bar{j}} = R^{k}_{\bar{\ell} i\bar{j}} = 0, \bar{R}^{k}_{\ell i\bar{j}} = R^{\bar{k}}_{\bar{\ell} \bar{i} j},$$

$$\bar{R}_{\ell \bar{k} i \bar{j}} = R_{\ell \bar{j} i \bar{k}},$$

$$[Z_\alpha, Z_\beta] = \sum_\gamma (c^\gamma_{\alpha,\beta} - c^\gamma_{\beta,\alpha})Z_\gamma.$$

We have the following representation for $\Delta = d''\vartheta'' + \vartheta'' d''$ which is known as Bocher-Kodaira formula.

Lemma 1.
On $A^{0,*}(M) = \sum_{q=0}^n A^{0,q}(M)$ we have

$$\Delta = -\frac{1}{2}\{\sum_{j=1}^n (\nabla_{Z_j}\nabla_{\bar{Z}_j} + \nabla_{\bar{Z}_j}\nabla_{Z_j}) - \sum_{i,j=1}^n (c^{\bar{j}}_{i,\bar{i}}\nabla_{\bar{Z}_j} + c^{j}_{\bar{i},i}\nabla_{Z_j}) - \sum_{j=1}^n R(Z_j, \bar{Z}_j)\}$$

We use the following notations in the rest of this paper.

$$e(\bar{\omega}^j) = a^*_j, \qquad \iota(\bar{Z}_m) = a_m,$$

$$a_I = a_{i_1}a_{i_2}\cdots a_{i_p}, \qquad a^*_I = a^*_{i_p}\cdots a^*_{i_1} \qquad \text{for } I = \{i_1 < i_2 < \cdots < i_p\},$$

$$\omega^{\bar{I}} = \bar{\omega}^{i_1} \wedge \bar{\omega}^{i_2} \wedge \cdots \wedge \bar{\omega}^{i_p} \qquad \text{for } I = \{i_1 < i_2 < \cdots < i_p\},$$

By the above notations we have

$$\nabla_{Z_\alpha}(\varphi_{\bar{J}}\omega^{\bar{J}}) = (Z_\alpha\varphi_{\bar{J}})\omega^{\bar{J}} - \sum_{k,\ell=1}^{n} \varphi_{\bar{J}} c^{\ell}_{\alpha,k} a^*_k a_\ell(\omega^{\bar{J}})$$

for $\varphi_{\bar{J}}\omega^{\bar{J}} \in A^{0,*}(M)$, $\omega^{\bar{J}} = \bar{\omega}^{i_1}\bar{\omega}^{i_2}\cdots\bar{\omega}^{i_p}$ $(J = (i_1, i_2, \cdots, i_p))$, using

$$\nabla_{Z_\alpha}(\bar{\omega}^{\ell}) = -\sum_{k=1}^{n} c^{\ell}_{\alpha,k}\bar{\omega}^{k}.$$

So we have the local representation of Δ.

$$\Delta = -\frac{1}{2}\sum_{j=1}^{n}\{(Z_j I - G_j)(\bar{Z}_j I - G_{\bar{j}}) + (\bar{Z}_j I - G_{\bar{j}})(Z_j I - G_j)\}$$

$$+\frac{1}{2}\sum_{i,j=1}^{n}\{c^{\bar{j}}_{i,\bar{i}}(\bar{Z}_j I - G_{\bar{j}}) + c^{j}_{\bar{i},i}(Z_j I - G_j)\} - \frac{1}{2}\sum_{j,k,\ell=1}^{n} R_{\ell\bar{k}j\bar{j}} a^*_k a_\ell$$

on $A^{0,*}(M)$. Here

$$G_\alpha = \sum_{\ell,m=1}^{n} c^{m}_{\alpha,\ell} a^*_\ell a_m$$

and I is an identity operator on $\wedge^*(T^*(M))$.

The following proposition is fundamental for a_i, a^*_j.

Proposition 2.

$$a_i a_j + a_j a_i = 0,$$

$$a^*_i a^*_j + a^*_j a^*_i = 0,$$

$$a_i a^*_j + a^*_j a_i = \delta_{ij},$$

$$[a^*_i a_j, a^*_k a_\ell] = \delta_{jk} a^*_i a_\ell - \delta_{i\ell} a^*_k a_j.$$

§3. Berezin-Patodi formula. Let V be a vector space of dimention n with inner product and let $\wedge^p(V)$ be its anti-symmetric p tensors. Set $\wedge^*(V) = \sum_{p=0}^{n} \wedge^p(V)$. Let $\{v_1, \cdots, v_n\}$ be an orthnormal basis for V. Set a^*_i be a linear transformation on $\wedge^*(V)$ defined by $a^*_i v = v_i \wedge v$ and set a_i be an adjoint operator of a^*_i on $\wedge^*(V)$. Then

$\{a_i^*, a_j\}$ satisfy Proposition 2. The following Theorem 1 and Corollary are shown in [3] under the above assumptions.

Theorem 1(Berezin-Patodi[3]). *For any linear operator A on $\wedge^*(V)$, we can write uniquely in the form $A = \sum_{I,J} \alpha_{I,J} a_I^* a_J$ and*

$$\sum_{p=0}^{n} \mathrm{tr}[(-1)^p A_p] = (-1)^n \alpha_{\{1,2,\cdots,n\}\{1,2,\cdots,n\}},$$

where $A_p = A|_{\wedge^p(V)}$.

Corollary . *(1)If multi index I and J satisfy $\sharp(I) < n$ or $\sharp(J) < n$, we have*

$$\sum_{p=0}^{n} \mathrm{tr}[(-1)^p a_I^* a_J] = 0.$$

(2)Let π and σ be elements of permutation of n. Then

$$\sum_{p=0}^{n} \mathrm{tr}[(-1)^p a_{\pi(1)}^* a_{\sigma(1)} a_{\pi(2)}^* a_{\sigma(2)} \cdots a_{\pi(n)}^* a_{\sigma(n)}] = (-1)^n sign(\pi) sign(\sigma).$$

§**4. Funfamental solution for a degenerate operator.** In this paper we use the pseudo-differential operators of Weyl symbols, that is, a symbol $p(x,\xi) \in S^m_{\rho,\delta}(\mathbf{R}^n)$ defines an operator as

$$Pu(x) = (2\pi)^{-n} \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} e^{i(x-y)\cdot\xi} p(\frac{x+y}{2}, \xi) u(y) dy d\xi, \qquad u \in \mathcal{S}(\mathbf{R}^n).$$

Definition 1. We denote the symbol multi-product $p_1(x,D)p_2(x,D) \cdots p_\ell(x,D)$ of pseudo-differential operators $p_j(x,D)$ with symbol $p_j(x,\xi)$ by $(p_1 \circ \cdots \circ p_\ell)(x,\xi)$. We use the notation $\sigma(P)$ to denote a symbol of a pseudo-differential operator P.

Theorem 2. *Let $p_j \in S^{m_j}_{\rho,\delta}(\mathbf{R}^n), (j = 1,2), \delta < 1, \rho \geq \delta$. Then for any $N \in \mathbf{N}$ we have the expansion*

$$p_1 \circ p_2 = \sum_{k=0}^{N-1} \frac{(2i)^{-k}}{k!} \sigma_k(p_1, p_2) + q_N, \tag{5.1}$$

where

$$\sigma_k(p,q) = \sum_{|\alpha|+|\beta|=k} (-1)^{|\beta|} \frac{k!}{\alpha!\beta!} p^{(\alpha)}_{(\beta)} q^{(\beta)}_{(\alpha)} \tag{5.2}$$

with $q_N \in S^{m_1+m_2-(\rho-\delta)N}_{\rho,\delta}(\mathbf{R}^n)$.

Definition 2. (1) ∇p means a vector

$$\binom{\nabla_x p}{\nabla_\xi p} = {}^t\left(\frac{\partial}{\partial x_1}p, \cdots, \frac{\partial}{\partial x_n}p, \frac{\partial}{\partial \xi_1}p, \cdots, \frac{\partial}{\partial x_n}p\right)$$

for a linear transformation $p(x,\xi)$ with parameter (x,ξ).
(2) J is a transformation on $\mathbf{C}^n \times \mathbf{C}^n$ defined by

$$J\binom{u}{v} = \binom{v}{-u}$$

for $u, v \in \mathbf{C}^n$. We also use the same nonation J in case $u = {}^t(u_1, \cdots, u_n), u_j$ is a linear tranformation on some vector space.
(3) H_p is the Hessian matrix of p.
(4) $< t, s >= \sum_{j=1}^k t_j s_j$ for a pair of vectors $t = {}^t(t_1, \cdots, t_k), s = {}^t(s_1, \cdots, s_k)$.

We consider the construction of the fundamental solution $U(t)$ for a degenerate parabolic system

$$\begin{cases} LU = (\frac{d}{dt} + P)U(t) = 0 & \text{in } (0, \mathrm{T}) \times \mathbf{R}^{\mathrm{m}}, \\ U(0) = I \quad \text{on } \mathbf{R}^{\mathrm{m}}, \end{cases}$$

for the Cauchy problem on $\mathbf{R}^m$ (See I-Iwasaki [8]). Here P is a differential operator of a symbol $p(x,\xi) = p_2(x,\xi) + p_1(x,\xi) + p_0(x,\xi)$, where $p_j(x,\xi)$ are homogeneous of order j with respect to ξ.

Condition (A).

$$p_2(x,\xi) = \sum_{j=1}^d b_j(x,\xi)c_j(x,\xi) \quad (c_j = \bar{b}_j), \tag{(A) - (1)}$$

where $b_j(\in S^1_{1,0})$ are scalar symbols.

$$p_1 + \mathrm{tr}_+(\frac{\mathcal{M}}{2}) \geq c|\xi| \tag{(A) - (2)}$$

for some positive constant c on the characteristic set $\Sigma = \{(x,\xi) \in \mathbf{R}^m \times \mathbf{R}^m;$ $b_j(x,\xi) = 0$ for any j $\}$, where $\mathrm{tr}_+ \mathcal{M}$ is the sum of all positive eigenvales of $\mathcal{M}$:

$$\mathcal{M} = i\Xi^* J \Xi.$$

Here $\Xi = (\nabla c_1, \cdots, \nabla c_d, \nabla b_1, \cdots, \nabla b_d)$.

Set $\mathbf{b} = {}^t(b_1, \cdots, b_d)$, $\mathbf{c} = {}^t(c_1, \cdots, c_d)$. Then we have

Theorem 3. *Let $p(x,\xi)$ satisfy Condition (A). Then the fundamental solution $U(t)$ is constructed as a pseudo-differential operator of a symbol $u(t)$ belonging to $S^0_{\frac{1}{2},\frac{1}{2}}$ with parameter t. Moreover $u(t)$ has the following expansion for any N:*

$$u(t) - \sum_{j=0}^{N-1} u_j(t) \text{ belongs to } S^{-\frac{N}{2}}_{\frac{1}{2},\frac{1}{2}},$$

$$u_0(t) = \exp\varphi, \quad u_j(t) = f_j(t) u_0(t) \in S^{-\frac{j}{2}}_{\frac{1}{2},\frac{1}{2}},$$

where

$$\varphi = -\frac{t}{2}(\begin{pmatrix} \mathbf{c} \\ \mathbf{b} \end{pmatrix}, F(\frac{\mathcal{M}t}{2}) \begin{pmatrix} \mathbf{b} \\ \mathbf{c} \end{pmatrix}) - \frac{1}{2}\mathrm{tr}[\log\{\cosh(\frac{\mathcal{M}t}{2})\}] - \mathrm{p}_1 t,$$

$$F(s) = s^{-1}\tanh s.$$

§5. Sketch of the proof. We shall apply the above discussion for our operator Δ. Fix a point $\hat{z} \in M$ and choose a local chart U of a neighborhood $\hat{z}$. Choose a local coordinate $z_1, z_2, \cdots, z_n$ of U such that

$$\hat{z} = 0, \qquad Z_j = \frac{\partial}{\partial z_j} + \sum_{k=1}^{n} \kappa_{jk}(z,\bar{z}) \frac{\partial}{\partial z_k}, \qquad \kappa_{jk}|_{z=0} = 0.$$

We use the following notation in the rest of paper.

$$\bar{z}_j = z_{\bar{j}}, \qquad \hat{G}_j = G_j|_{z=0}, \qquad \tilde{Z}_j = \sum_{k=1}^{n} \kappa_{jk}(z,\bar{z}) \frac{\partial}{\partial z_k}.$$

Then we have

$$\kappa_{\bar{j}\bar{k}} = \bar{\kappa}_{jk}, \qquad \kappa_{j\bar{k}} = \kappa_{\bar{j}k} = 0 \qquad (1 \leq j,k \leq n).$$

Proposition 3. *For the derivatives of $\kappa_{\alpha\beta}(z,\bar z)$ we have*

$$\{Z_\alpha(\kappa_{\beta\gamma}) - Z_\beta(\kappa_{\alpha\gamma})\}|_{z=0} = (c^\gamma_{\alpha,\beta} - c^\gamma_{\beta,\alpha})|_{z=0}, \qquad \alpha,\beta,\gamma \in \Gamma.$$

$$\sum_\gamma \{Z_\alpha(\kappa_{\beta\gamma}) - Z_\beta(\kappa_{\alpha\gamma})\}|_{z=0}\hat G_\gamma = (-\nabla_{[Z_\alpha,Z_\beta]} + [Z_\alpha, Z_\beta])|_{z=0}.$$

Set

$$W(z,\bar z) = \sum_{j=1}^n (z_j \hat G_j + \bar z_j \hat G_{\bar j}) = \sum_{\alpha\in\Lambda} z_\alpha \hat G_\alpha$$

$$\Lambda = \{1,\cdots,n,\bar 1,\cdots,\bar n\}.$$

Lemma 2. *We have*

$$(Z_\alpha - G_\alpha)e^W = e^W(Z_\alpha - F_\alpha), \qquad \alpha \in \Lambda$$

where

$$F_\alpha = G_\alpha - \hat G_\alpha - \sum_{\beta\in\Lambda} \kappa_{\alpha\beta}\hat G_\beta + \frac{1}{2}\sum_{\beta\in\Lambda}[\hat G_\alpha, \hat G_\beta] z_\beta + \tilde F_\alpha \tag{5.1}$$

with

$$\tilde F_\alpha = \sum_{\beta\in\Lambda} z_\beta [G_\alpha - \hat G_\alpha, \hat G_\beta] - \frac{1}{2}\sum_{\beta\in\Lambda}[\kappa_{\alpha_\beta}\hat G_\beta, W] \tag{5.2}$$

$$-I_3(1, C_2(Z_\alpha W : W) : W) + I_2(1, C_2(G_\alpha : W) : W),$$

where

$$I_j(t, B : A) = \int_0^t \frac{(t-s)^{j-1}}{(j-1)!} e^{-sA} B e^{sA} ds$$

and

$$C_2(B : A) = [[B, A], A].$$

Corollary . *For the same W we have*

$$\Delta e^W = e^W \tilde{\Delta},$$

where

(5.3)
$$\tilde{\Delta} = -\frac{1}{2}\sum_{j=1}^{n}\{(Z_jI - F_j)(\bar{Z}_jI - F_{\bar{j}}) + (\bar{Z}_jI - F_{\bar{j}})(Z_jI - F_j)\}$$
$$+\frac{1}{2}\sum_{i,j=1}^{n}\{c^{\bar{j}}_{\bar{i},i}(\bar{Z}_jI - F_{\bar{j}}) + c^{j}_{\bar{i},i}(Z_jI - F_j) - \frac{1}{2}R$$

with

(5.4)
$$R = e^{-W}(\sum_{j,k,\ell=1}^{n} R_{\ell\bar{k}j\bar{j}}a_k^* a_\ell)e^W.$$

So the symbol of $\tilde{\Delta}$ is given by

$$\sigma(\tilde{\Delta}) = -\frac{1}{2}\sum_{j=1}^{n}\{(p_jI - F_j)(q_jI - F_{\bar{j}}) + (q_jI - F_{\bar{j}})(p_jI - F_j)\} - \frac{1}{2}R,$$

where

$$r_1 = -\frac{1}{4}\sum_{k,j=1}^{n}\{Z_k(c^k_{\bar{j},j}) + \bar{Z}_k(c^{\bar{k}}_{j,\bar{j}})\}I + \frac{1}{2}\sum_{i,j=1}^{n}\{c^{\bar{j}}_{\bar{i},i}(q_jI - F_{\bar{j}}) + c^{j}_{\bar{i},i}(p_jI - F_j)\},$$

F_α and R are given by (5.1) and (5.4) respectively. Here

$$\sigma(Z_j) = p_jI \qquad \sigma(\bar{Z}_j) = q_jI.$$

By (5.1), (5.2) and Proposition 3 we have

Proposition 4. *For the symbols* $(p_kI - F_k), (q_kI - F_{\bar{k}})$ *,we have*

$$< J\nabla(p_kI - F_k), \nabla(p_jI - F_j) > |_{z=0} = i\sigma([Z_k, Z_j]),$$

$$< J\nabla(p_kI - F_k), \nabla(q_jI - F_{\bar{j}}) > |_{z=0} = i\{R(Z_k, \bar{Z}_j) + \sigma([Z_k, \bar{Z}_j])\},$$

$$< J\nabla(q_kI - F_{\bar{k}}), \nabla(q_jI - F_{\bar{j}}) > |_{z=0} = i\sigma([\bar{Z}_k, \bar{Z}_j]).$$

If we apply Theorem 3 to construction of the fundamental solution for $\frac{\partial}{\partial t} + \tilde{\Delta}$ considering $a_i a_j^*$ as $S^2_{1,0}$, we obtain, in this case, the fundamental solution with $\mathcal{M}$ whose main part is given by

$$\mathcal{M} = -\begin{pmatrix} \mathcal{M}_0 & 0 \\ 0 & \bar{\mathcal{M}}_0 \end{pmatrix},$$

where $(\mathcal{M}_0)_{ij} = R(Z_i, \bar{Z}_j)$. So the kernel $\tilde{u}_0(t,x,x)$ of pseudodifferential operator with symbol $u_0(t,x,\xi)$ is obtained as

$$\tilde{u}_0(t,x,x) = (2\pi t)^{-n} \det(\frac{t\mathcal{M}_0}{\exp(t\mathcal{M}_0)-1})$$

Then we have

$$\mathrm{str}\ \tilde{u}_0(t,x,x) = (2\pi t)^{-n} \mathrm{str}\ [\det(\frac{t\mathcal{M}_0}{\exp(t\mathcal{M}_0)-1})]$$

Applying Theorem 1 , we have

$$\mathrm{str}\ \tilde{u}_0(t,x,x)dv = (2\pi i)^{-n}[\mathrm{Td}(\mathrm{TM})]_{2n},$$

where

$$Td(T(M)) = \det(\frac{\Omega}{e^{\Omega}-1})$$

with curvature form Ω, that is,

$$\Omega = (\Omega^k_\ell), \quad \Omega^k_\ell = \sum_{i,j=1}^{n} R_{\bar{k},\ell,i,\bar{j}} \omega^i \wedge \bar{\omega}^j .$$

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