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Inverse Scattering for the Nonlinear Schrödinger Equation and $L^p - L^{\acute{p}}$ Estimates [1]

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Abstract

In this paper we discuss the direct and the inverse scattering problems for the nonlinear Schrödinger equation on the line:

$$i\frac{\partial}{\partial t}u(t,x) = -\frac{d^2}{dx^2}u(t,x) + V_0(x)u(t,x) + \sum_{j=1}^{\infty} V_j(x)|u|^{2(j_0+j)}u(t,x).$$

The basis of our study is an $L^p - L^{\acute{p}}$ estimate for the linear Schrödinger equation with $V_j = 0, j = 1,2,\cdots$, that we proved recently. We prove, under appropriate conditions, that the small-amplitude limit of the scattering operator determines uniquely $V_j, j = 0,1,\cdots$. Our proof gives also a method for the reconstruction of the $V_j, j = 0,1,\cdots$.

1 Introduction

Let us consider the following nonlinear Schrödinger equation:

$$i\frac{\partial}{\partial t}u(t,x) = -\frac{d^2}{dx^2}u(t,x) + V_0(x)u(t,x) + F(x,u), u(0,x) = \phi(x), \tag{1.1}$$

where $t, x \in \mathbf{R}$, the potential, V_0, is a real-valued function and $F(x,u)$ is a complex-valued function.

Before we solve the inverse scattering problem we have, of course, to construct the scattering operator. Let us first first introduce some standard notations and definitions. We say that $F(x,u)$ is a C^k function of u in the real sense if for each $x \in \mathbf{R}$, $\Re F$ and $\Im F$ are C^k functions with respect to the real and imaginary parts of u. Below we assume that F is C^2 in the real sense and that $\left(\frac{\partial}{\partial x}F\right)(x,u)$ is C^1 in the real sense. If $F = F_1 + iF_2$ with F_1, F_2 real-valued, and $u = r + is, r, s \in \mathbf{R}$ we denote,

$$F^{(2)}(x,u) := \sum_{j=1}^{2}\left[\left|\frac{\partial^2}{\partial r^2}F_j(x,u)\right| + \left|\frac{\partial^2}{\partial r \partial s}F_j(x,u)\right| + \left|\frac{\partial^2}{\partial s^2}F_j(x,u)\right|\right], \tag{1.2}$$

$$\left(\frac{\partial}{\partial x}F\right)^{(1)}(x,u) := \sum_{j=1}^{2}\left[\left|\frac{\partial}{\partial r}\left(\frac{\partial}{\partial x}F_j\right)(x,u)\right| + \left|\frac{\partial}{\partial s}\left(\frac{\partial}{\partial x}F_j\right)(x,u)\right|\right]. \tag{1.3}$$

For any $\gamma \in \mathbf{R}$, L^1_γ denotes the Banach space of all complex-valued measurable functions, ϕ, defined on $\mathbf{R}$ and such that

$$\|\phi\|_{L^1_\gamma} := \int |\phi(x)|\,(1+|x|)^\gamma\,dx < \infty. \tag{1.4}$$

If $V_0 \in L^1_1$ the differential expression $\tau := -\frac{d^2}{dx^2} + V_0(x)$ is essentially self-adjoint on the domain

$$D(\tau) := \left\{\phi \in L^2_C : \phi \,\text{and}\, \frac{d}{dx}\phi \,\text{are absolutely continuous and}\, \tau\phi \in L^2\right\}, \tag{1.5}$$

where L^2_C denotes the set of all functions on L^2 that have compact support. We denote by H the unique self-adjoint realization of τ. It is known (see [5], [34]) that H has a finite number of negative eigenvalues, that it has no positive or zero eigenvalues, that it has no singular-continuous spectrum and that the absolutely-continuous spectrum is $[0,\infty)$. If moreover, $N(V_0) < \infty$ (see (1.11) below) the domain of H is the Sobolev space $W_{2,2}$[1]. By H_0 we denote the unique self-adjoint realization of $-\frac{d^2}{dx^2}$ with domain the $W_{2,2}$. The wave operators are given by:

$$W_\pm := s - \lim_{t\to\pm\infty} e^{itH}\, e^{-itH_0}. \tag{1.6}$$

In [21] it is proven that the limits in (1.6) exit in the strong topology in L^2 and that Range$W_\pm = \mathcal{H}_c$, the subspace of continuity of H. The scattering operator for the linear Schrödinger equation (equation (1.1) with $F=0$) is given by:

$$S_L := W_+^*\, W_-. \tag{1.7}$$

For any pair u, v of solutions to the stationary Schrödinger equation:

$$-\frac{d^2}{dx^2}u + V_0 u = k^2 u,\; k \in \mathrm{C}, \tag{1.8}$$

let $[u,v]$ denotes the Wronskian of u, v:

$$[u,v] := \left(\frac{d}{dx}u\right)v - u\frac{d}{dx}v. \tag{1.9}$$

Let $f_j(x,k), j = 1,2, \Im k \geq 0$, be the Jost solutions to (1.8) (see [6], [7], [5], [4] and [29]). A potential V_0 is said to be *generic* if $[f_1(x,0), f_2(x,0)] \neq 0$ and V_0 is said to be *exceptional* if $[f_1(x,0), f_2(x,0)] = 0$. If V_0 is *exceptional* there is a bounded solution to (1.8) with $k^2 = 0$ (a half-bound state or a zero-energy resonance). For these definitions and related issues see [15]. Note that the trivial potential, $V_0 = 0$, is *exceptional.* We denote: $V_0^{(l)} := \frac{d^l}{dx^l}V_0(x)$. Clearly, $V_0^{(0)} = V_0$. We define,

$$M := \left\{u \in C(\mathbf{R}, W_{1,p+1}) : \sup_{t\in\mathbf{R}}(1+|t|)^d \|u\|_{W_{1,p+1}} < \infty\right\}, \text{with norm :}$$

$$\|u\|_M := \sup_{t\in\mathbf{R}}(1+|t|)^d \|u\|_{W_{1,p+1}}, \tag{1.10}$$

where $p \geq 1$, and $d := \frac{1}{2}\frac{p-1}{p+1}$. For functions $u(t,x)$ defined in $\mathbf{R}^2$ we denote $u(t)$, for $u(t,\cdot)$. In the following theorem we construct the small-amplitude scattering operator.

THEOREM 1.1. *Suppose that $V_0 \in L^1_\gamma$, where in the* generic case $\gamma > 3/2$ *and in the* exceptional case $\gamma > 5/2$, *that H has no negative eigenvalues, and that*

$$N(V_0) := \sup_{x\in\mathbf{R}} \int_x^{x+1} |V_0(y)|^2 \, dy < \infty. \tag{1.11}$$

Furthermore, assume that F is C^2 in the real sense, that $F(x,0) = 0$, and that for each fixed $x \in \mathbf{R}$ all the first order derivatives, in the real sense, of F vanish at zero. Moreover, suppose that $\frac{\partial}{\partial x}F$ is $C^{(1)}$ in the real sense. We assume that the following estimates hold:

$$F^{(2)}(x,u) = O\left(|u|^{p-2}\right), \left(\frac{\partial}{\partial x}F\right)^{(1)}(x,u) = O\left(|u|^{p-1}\right), \; u \to 0, \; \textit{uniformly for } x \in \mathbf{R}, \tag{1.12}$$

for some $\rho < p < \infty$, and where ρ is the positive root of $\frac{1}{2}\frac{\rho-1}{\rho+1} = \frac{1}{\rho}$. Then, there is a $\delta > 0$ such that for all $\phi_- \in W_{2,2} \cap W_{1,1+\frac{1}{p}}$ with $\|\phi_-\|_{W_{2,2}} + \|\phi_-\|_{W_{1,1+\frac{1}{p}}} \leq \delta$ there is a unique solution, u, to (1.1) such that $u \in C(\mathbf{R}, W_{1,2}) \cap M$ and,

$$\lim_{t \to -\infty} \|u(t) - e^{-itH}\phi_-\|_{W_{1,2}} = 0. \tag{1.13}$$

Moreover, there is a unique $\phi_+ \in W_{1,2}$ such that

$$\lim_{t \to \infty} \|u(t) - e^{-itH}\phi_+\|_{W_{1,2}} = 0. \tag{1.14}$$

Furthermore, $e^{-itH}\phi_\pm \in M$ and

$$\left\|u - e^{-itH}\phi_\pm\right\|_M \leq C \left\|e^{-itH}\phi_\pm\right\|_M^p, \tag{1.15}$$

$$\|\phi_+ - \phi_-\|_{W_{1,2}} \leq C \left[\|\phi_-\|_{W_{2,2}} + \|\phi_-\|_{W_{1,1+\frac{1}{p}}}\right]^p. \tag{1.16}$$

The scattering operator, $S_V : \phi_- \hookrightarrow \phi_+$ is injective on $W_{1,1+\frac{1}{p}} \cap W_{2,2}$.

In Theorem 1.1 we do not need to restrict F in such a way that energy is conserved. Moreover, $\rho \approx 3.56$. There many results on scattering for the nonlinear Schrödinger equation in the case where $V_0 = 0$. See [24], [25], [26], [14], [12], [17], [3], [9], [2] and the references quoted there. In [11] the direct scattering for (1.1) with $F = F(u)$ was studied for $n \geq 3$. The corresponding inverse problem was considered in [28].

To reconstruct the potential, V_0, we introduce below the scattering operator that relates asymptotic states that are solutions to the linear Schrödinger equation with potential zero:

$$S := W_+^* \, S_V \, W_-. \tag{1.17}$$

In the following theorem we reconstruct S_L from S.

THEOREM 1.2. *Suppose that the assumptions of Theorem 1.1 are satisfied. Then for every* $\phi_- \in W_{2,2} \cap W_{1,1+\frac{1}{p}}$

$$\frac{d}{d\epsilon} S(\epsilon\phi)\Big|_{\epsilon=0} = S_L\phi, \tag{1.18}$$

where the derivative in the left-hand side of (1.18) exists in the strong convergence in $W_{1,2}$.

COROLLARY 1.3. *Under the conditions of Theorem 1.1 the scattering operator, S, determines uniquely the potential V_0.*

Proof: Theorem 1.2 implies that S_L is uniquely determined by S. From S_L we get the reflection coefficients for linear Schrödinger scattering on the line (see Section 9.7 of [16] and [29]). As H has no bound states we uniquely reconstruct V_0 from one of the reflection coefficients by using any method for inverse scattering on the line(see for example [6], [7], [5], [13], [4], [10]).

■

In the case where $F(x,u) = \sum_{j=1}^{\infty} V_j(x)|u|^{2(j_0+j)}u$ we can also reconstruct the $V_j, j = 1,2,\cdots$. Let us introduce some notation. For $\lambda > 0$ and $\acute{x} \in \mathbf{R}$ we denote by H_λ the following self-adjoint operator in L^2:

$$H_\lambda := H_0 + V_\lambda(x), \text{ where } V_\lambda(x) = \frac{1}{\lambda^2} V_0\left(\frac{x}{\lambda} + \acute{x}\right). \tag{1.19}$$

Since H has no eigenvalues, we have that H_λ has no eigenvalues, i.e., $H_\lambda > 0$.

THEOREM 1.4. *Suppose that the conditions of Theorem 1.1 are satisfied, and moreover, that $F(x,u) = \sum_{j=1}^{\infty} V_j(x)|u|^{2(j_0+j)}u$, where j_0 is an integer such that, $j_0 \geq (p-3)/2$, for $|u| \leq \eta$, for some $\eta > 0$, and where $V_j \in W_{1,\infty}$ with $\|V_j\|_{W_{1,\infty}} \leq C^j, j = 1,2,\cdots$, for* *some constant C. Then, for any $\phi \in W_{2,2} \cap W_{1,1+\frac{1}{p}}$ there is an $\epsilon_0 > 0$ such that for all $0 < \epsilon < \epsilon_0$:*

$$((S_V - I)(\epsilon\phi), \phi)_{L^2} = \sum_{j=1}^{\infty} \epsilon^{2(j_0+j)+1} \left[\int\int dt\, dx\, V_j(x) \left| e^{-itH}\phi \right|^{2(j_0+j+1)} + Q_j \right], \quad (1.20)$$

where $Q_1 = 0$ and $Q_j, j > 1$, depends only on ϕ and on V_k with $k < j$. Moreover, for any $\acute{x} \in \mathbf{R}$, and any $\lambda > 0$, we denote, $\phi_\lambda(x) := \phi(\lambda(x - \acute{x}))$. Then, if $\phi \neq 0$:

$$V_j(\acute{x}) = \frac{\lim_{\lambda\to\infty} \lambda^3 \int\int dt\, dx\, V_j(x) \left| e^{-itH}\phi_\lambda \right|^{2(j_0+j+1)}}{\int\int dt\, dx\, \left| e^{-itH_0}\phi \right|^{2(j_0+j+1)}}. \quad (1.21)$$

COROLLARY 1.5. *Under the conditions of Theorem 1.4 the scattering operator, S, determines uniquely the potentials $V_j, j = 0, 1, \cdots$.*

Proof: By Corollary 1.3, S determines uniquely V_0. Then the wave operators, $W_\pm$, are uniquely determined, and by (1.17), S determines uniquely S_V. Finally by (1.20) and (1.21) S_V determines uniquely $V_j, j = 1, 2, \cdots$.

The method to reconstruct the potentials $V_j, j = 0, 1, \cdots$, is as follows. First we obtain S_L from S using (1.18). By any standard method for inverse scatering for the linear Schrödinger equation on the line we reconstruct V_0. We then reconstruct S_V from S using (1.17). Finally (1.20) and (1.21) give us, recursively, $V_j, j = 1, 2, \cdots$.

Our formula (1.21) is an extension to our case of the reconstruction algorithm of [23]. In [23] Strauss proved that in the case $V_0 = 0$ and $F(x, u) = V(x)|u|^{p-1}u$, $x \in \mathbf{R}^n$, $p > 4$ if $n = 1, p > 3$ if $n = 2, p \geq 3$ if $n \geq 3$, and $V(x)$ a real-valued potential whose derivatives up to order l are bounded, with $l > 3n/4$, then, the scattering operator uniquely determines V.

For the proof of Theorems 1.1, 1.2, 1.4. Corollaries 1.3, 1.5 see [32]. The basic imput of the proofs is the following $L^p - L^{\acute{p}}$ estimate that we proved in [29].

THEOREM 1.6. *(The $L^p - L^{\acute{p}}$ estimate). Suppose that* $V \in L^1_\gamma$ *where in the* generic case $\gamma > 3/2$ *and in the* exceptional case $\gamma > 5/2$. *Then for* $1 \leq p \leq 2$ *and* $\frac{1}{p} + \frac{1}{\acute{p}} = 1$

$$\left\| e^{-itH} P_c \right\|_{\mathcal{B}\left(L^p, L^{\acute{p}}\right)} \leq \frac{C}{t^{\left(\frac{1}{p}-\frac{1}{2}\right)}},\ t > 0. \tag{1.22}$$

By P_c we denote the orthogonal projector onto the subspace of continuity of H. We also use in the proofs in [32] the following theorem on the continuiy of the wave operators on the Sobolev spaces $W_{k,p}$ that we proved in [30].

THEOREM 1.7. *(The $W_{k,p}$-continuity of the wave operators). Suppose that* $V \in L^1_\gamma$,*where in the* generic case $\gamma > 3/2$ *and in the* exceptional case $\gamma > 5/2$, *and that for some* $k = 1, 2, \cdots$, $V^{(l)} \in L^1$, *for* $l = 0, 1, 2, \cdots, k-1$. *Then* $W_\pm$ *and* $W_\pm^*$ *originally defined on* $W_{k,p} \cap L^2, 1 \leq p \leq \infty$, *have extensions to bounded operators on* $W_{k,p}$, $1 < p < \infty$. *Moreover, there are constants* $C_p, 1 < p < \infty$, *such that:*

$$\|W_\pm f\|_{k,p} \leq C_p \|f\|_{k,p};\ \left\|W_\pm^* f\right\|_{k,p} \leq C_p \|f\|_{k,p}, f \in W_{k,p} \cap L^2, 1 < p < \infty. \tag{1.23}$$

Furthermore, if V *is* exceptional *and* $a := \lim_{x \to -\infty} f_1(x, 0) = 1$, $W_\pm$ *and* $W_\pm^*$ *have extensions to bounded operators on* $W_{k,1}$ *and to bounded operators on* $W_{k,\infty}$, *and there are constants* C_1 *and* C_∞ *such that (1.23) holds for* $p = 1$ *and* $p = \infty$.

We also prove in [30] that in the general case the wave operators are bounded from $W_{k,1}$ into the weak $W_{k,1}$ space, and from $W_{k,\infty}$ into the space of functions of bounded mean oscillation, BMO, that have k derivatives in BMO.

A result in the continuity of the one-dimensional wave operators in $L^p, 1 < p < \infty$, was obtained by Galtbayar and Yajima in [8]. Galtbayar and Yajima proved their result under conditions on the potential that are more restrictive than ours. They require that $V^{(1)} \in L_2^1$ and that $V \in L_\gamma^1$, where in the *generic case* $\gamma = 3$ and in the *exceptional case* $\gamma = 4$.

For the extension of the results in this paper to the multidimensional case see [33].

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