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Canonical representation and Markov property of Gaussian random fields

Si Si
Faculty of Information Science and Technology
Aichi Prefectural University,
Aichi-ken 480-1198, Japan

Abstract

We are interested in random fields $X(C)$ with parameter C, where C runs through the class

$$\mathbf{C} = \{C; C \in C^2, \text{diffeomorphic to } S^1\}.$$

Referring to the canonical representation theory of Gaussian processes, developed by T. Hida, we generalize the theory to the case of Gaussian random fields.

1 Introduction

We are interested in the representation of random field $X(C)$, where C runs through a class

$$\mathbf{C} = \{C; \text{diffeomorphic to } S^{d-1}, \text{convex}\}.$$

In particular, we consider a Gaussian random field $X(C); C \in \mathbf{C}$, with a representation in terms of R^d-parameter white noise.

First we breifly recall the canonical representation theory of Gaussian processes, developed by T. Hida (1960).

We give the definition of the canonical representation of Gaussian random field following the definition of canonical representation of Gaussian process $X(t)$, mentioned above. And the canonical criterion for Gaussian randon field is established in this note.

Canonical representation and non-canonical representation of Gaussian random fields are illusrated by examples.

2 Canonical representation of Gaussian processes

Let $X = \{X(t); t \in I\}$ be a Gaussian process. Denote by $\mathcal{B}_t(X)$ be the σ−field generated by the $X(s), s \leq t$.

Definition 2.1 *Let X be a Gaussian process. Assume that there exists a Gaussian process $B = \{B(t); t \in I\}$ with independent increments such that*

$$B(t) = (B_i(t), 1 \leq i \leq N; B_j^l(t), 1 \leq l \leq L_j, 1 \leq j \leq L_J) \qquad (2.1)$$

with $B_i(0) = 0, 1 \leq i \leq N$, $N \leq \infty$, $L_j \leq \infty, J \leq \infty$, satisfying the following conditions.

1. *Each $B_i(t)$ has independent increments and $E(|dB_i(t)|^2) = m_i(dt)$ defines a continuous measure. In addition, m_{i+1} is absolutely continuous with respective to $m_i : m_i(dt) \gg m_{i+1}(dt)$ for every i.*
2. *Each $B_j^l(t)$ is a process of the form*
$$B_j^l(t) = B_j^l, \quad t > t_j \text{ (or } t \geq t_j), \text{ and } = 0, \text{ otherwise}$$
where each B_j^l is subject to the standard Gausian distribution $N(0,1)$.
3. *The Gaussian processes B_i and the B_j^l are independent.*
4. *For every t, $X(t)$ has the same distribution as $\widetilde{X}(t)$ given by*
$$\widetilde{X}(t) = \sum_{i=1}^{N} \int_0^t F_i(t,u) dB_i(u) + \sum_{t_j \leq t} \sum_{l=1}^{L_j} b_j^l(t) B_j^l(t_j). \qquad (2.2)$$
where the kernel functions $F_i(t,u)$ satisfy the condition
$$\int F_i(t,u)^2 m_i(du) < \infty, \qquad i = 1, 2, \cdots N, \ \forall t,$$
and where the function $b_j^l(t)$ vanishes for $t_j < t$ and satisfies
$$\sum_{t_j \leq t} \sum_{l=1}^{L_j} b_j^l(t)^2 < \infty, \text{ for every } t.$$

Then $\{F_i(t,u), B_i(u); b_j^l(t), B_j^l(t)\}$ is called a representation of X.

Definition 2.2 *The representation* (2.2) *is called a generalized canonical representation, if*

$$E[X(t)|\mathcal{B}_s(X)] = \sum_{i=1}^{N} \int^s F_i(t,u) dB(u) + \sum_{t_j \leq s} \sum_{l} b_j^l(t) B_j^l(t_j). \qquad (2.3)$$

holds for all $s \leq t$.

To fix the idea, assume that $N = 1$ and that there is no discrete part of the spectrum. Thus the representation

$$X(t) = \int^{t} F(t,u)dB(u) \tag{2.4}$$

is called the canonical representation if

$$E[X(t)|\mathcal{B}_s(X)] = \int^{s} F(t,u)dB(u).$$

The kernel $F(t,u)$ is called a canonical kernel.

Definition 2.3 *The canonical representation* (2.4) *is called proper canonical representation if*

$$\mathcal{B}_t = \mathcal{B}_t(X), \ \textit{for every } t \in T, \tag{2.5}$$

where B_t *is the* σ*-field generated by* $\{B(S), s \leq t\}$.

There are many important cases which suggested to claim that $B(t)$ in the above expression is a standard Brownian motion, so that $dB(u)$ may be written as $\dot{B}(u)du$, where $\dot{B}(u)$ is a white noise.

3 Canonical representation of Gaussian random fields

Consider Gaussian random fields $\{X(C); C \in \mathbf{C}\}$ where

$$\mathbf{C} = \{C; C \in C^2, \text{diffeomorphic to } S^1, (C) \text{ is convex}\},$$
$$(C): \text{ being the domain enclosed by } C.$$

Assume that

1. $X(C) \neq 0$ for every C, and $E[X(C)] = 0$.
2. $E[X(C)^2] \neq 0$ for every C.

In particular, we consider the Gaussian random field $\{X(C); C \in \mathbf{C}\}$ with the assumption of causality. That means $X(C)$ can be expressed by a representation

$$X(C) = \int_{(C)} F(C,u)x(u)du, \tag{3.1}$$

in terms of R^2-parameter white noise $x(u)$ and $L^2(R^2)-$ kernel $F(C,u)$ for every C.

I. Uniqueness of canonical representation

Definition 3.1 Let $\mathrm{B}_{C'}(X)$ be the sigma field generated by $\{X(C), C < C'\}$. The representation (3.1) is called a canonical representation if

$$E[X(C)|\mathcal{B}_{C'}(X)] = \int_{(C')} F(C,u)x(u)du, \tag{3.2}$$

holds for any $C' < C$.

Theorem 3.1 *The canonical representation is unique if it exists.*

Proof. Take the variance of the conditional expectation, given in (3.2).

$$E\{E[X(C)|\mathbf{B}_{C'}(X)]^2\} = \int_{(C')} F(C,u)^2 du, \ C' < C. \tag{3.3}$$

We should note that the variance depends only on the probability distribution of $\{X(C)\}$ and is independent of the choice of representation.

If the representation is not unique, there are two canonical kernels F and F^* and then

$$\int_{(C')} F(C,u)^2 du = \int_{(C')} F^*(C,u)^2 du$$

holds for any $C' < C$. Hence we have

$$F(C,u) = \varepsilon(C,u)F^*(C,u); \quad |\varepsilon(C,u)| = 1, \tag{3.4}$$

where ε is a measurable function of u.
According to the two kernels F and F^*, the covariance of (3.2) is obtained as the covariance

$$E\left[E[X(C)|\mathcal{B}_{C''}(X)]E[X(C')|\mathcal{B}_{C''}(X)]\right] = \int_{C''} F(C,u)F(C',u)du.$$

On the other hand, we obtain

$$\begin{aligned} E\left[E[X(C)|\mathcal{B}_{C''}(X)]E[X(C')|\mathcal{B}_{C''}(X)]\right] &= \int_{C''} F^*(C,u)F^*(C',u)du \\ &= \int_{C''} F(C,u)F(C',u)\varepsilon(C,u)\varepsilon(C', \end{aligned}$$

by (3.5).

Similarly for any $C''' \in \mathbf{C}$; $C''' < C'$. we have

$$\int_{C'''} F(C,u)F(C',u)du = \int_{C'''} F(C,u)F(C',u)\varepsilon(C,u)\varepsilon(C',u)du.$$

Thus the equality

$$F(C,u)F(C',u) = F(C,u)F(C',u)\varepsilon(C,u)\varepsilon(C',u),$$

holds almost everywhere.

We can see that

$$\varepsilon(C,u)\varepsilon(C',u) = 1, \ \textit{on } C'.$$

Fix $C = C_0$, and determine $\varepsilon(C_0,u)(=\pm 1)$ as a function of u.

Thus

$$\varepsilon(C',u) = \frac{1}{\varepsilon(C_0,u)} = \varepsilon(C_0,u), \ \forall C'.$$

It means that $\varepsilon(C,u)$ is independent of C.

Thus it is proved that $F(C,u)$ is unique up to ± 1.

II. Kernel criterion for canonical representation

We now give the kernel criterion for canonical representation.

Assume that

1. *$X(C)$ has a causal representation*
2. *there is no open set G such that $\int_G F(C,u)\varphi(u)du = 0$ for any φ with $supp\{\varphi\} \subset G$.*

Theorem 3.2 *A random field $X(C)$,, satisfying the above assumption, has canonical representation if and only if $\forall C \subset C_1$; C_1 : fixed,*

$$\int_{(C)} F(C,u)\varphi(u)du = 0 \Rightarrow \varphi(u) = 0 \quad a.e.on(C_1).$$

Proof First we should note that $E[X(C)|\mathcal{B}_{C_0}]$ is the projection of $X(C)$ down to the closed linear space spanned by $\{X(C); C < C_0\}$, since we are concerned with Gaussian.

Let $M_{C_0}(X)$ and $M_{C_0}(x)$ denote the closed linear spaces spanned by $\{X(C); C < C_0\}$ and $\{x(u); u \in C_0\}$ respectively.

Claim that

$$M_{C_0}(X) \subset M_{C_0}(x)$$

since $X(C_0); C \leq C_0$ is a (linear) funcion of $x(u); u \in C_0$.

If $M_{C_0}(X) \neq M_{C_0}(x)$ then there exist $\varphi \neq 0$ such that $\int_{C_0} \varphi(u)x(u)du$ is orthogonal to $X(C); C < C_0$. It contradicts to the assumption. Thus the assertion is proved.

The followings are the examples for canonical representation.

Example 1. $X(C) = \int_{(C)} x(u)du, C \in \mathbf{C}$, given in (3.1) is a canonical representation.

Example 2. Consider a random field $X(C); C \in \mathbf{C}$, where $\mathbf{C}$ is a family of circles, with the representation

$$X(C) = X_0 \int_{(C_0)} e^{-k\rho(C,u)} \partial_u^* \nu(u)du,$$

where ρ denotes the distance, k is a constant, φ and ν are given continuous functions. We can prove that it is a canonical representation.

Indeed it is the solution of Langevin equation,

$$\delta X(C) = -X(C) \int_C k\delta n(s)ds + X_0 \int_C \nu(s)\partial_s^* \delta n(s)ds,$$

where $C \in \mathbf{C}_0$.

We give the example for non-canonical representation in the following..

Example 3. Let $\{C_R, R \in \mathbf{R}\}$ be a family of concentric circles with center at origin. Then

$$X(C) = \int_{C_R} (3R - 4|u|)x(u)du$$

is a canonical representation of $X(C)$, since there is a function $\varphi(u) = |u| \neq 0$ such that

$$\int_{C_R} (3R - 4|u|)\varphi(|u|)du = 0.$$

4 Multiple Markov Gaussian random fields

In this section we shall deal with the multiple Markov Gaussian random fields. Thus we recall the definition of Multiple Gaussian random field, given in [6],[7] and [8].

Definition *For any choice of C_i's such that $C_0 \leq C_1 < \cdots < C_N < C_{N+1}$, if*

1. $E[X(C_i)|\mathcal{B}_{C_0}(X)], i = 1, 2, \cdots, N$ *are linearly independent and*
2. $E[X(C_i)|\mathcal{B}_{C_0}(X)], i = 1, 2, \cdots, N+1$ *are linearly dependent*

then $X(C)$ is called N-ple Markov Gaussian random field.

Theorem 4.1 *If $X(C)$ is $N-$ple Markov and if it has a canonical representation, then it is of the form*

$$X(C) = \int_{(C)} \sum_{1}^{N} f_i(C) g_i(u) x(u) du, \tag{4.1}$$

where the kernel $\sum f_i(C) g_i(u)$ is a Goursat kernel and $\{f_i(C)\}, i = 1, \cdots, N$ satisfies

$$\det(f_i(C_j)) \neq 0, for\ any\ N \text{different } C_j \tag{4.2}$$

and $\{g_i(u)\}, i = 1, \cdots, N$ are linearly independent in L^2-space.

Proof. See [8].

Corollary. *If $N = 1$, then it is a (simple) Markov.*

Proof. It can be easily seen from the expression of canonical representation.

Remark. For a particular case of N-ple Markov Gaussian random field $X(C)$, where $C = C_r$ is a circle with radius r and center origin, the representation of $X(C_r)$ can be expressed in the form

$$X(C_r) = X(r) = \int_0^r \sum_{1}^{N} f_i(r) g_i(u) x(u) du, u \in R^2. \tag{4.3}$$

This representation can be expressed in terms of a one dimensional parameter white noise as a stochastic integral.

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