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Diffusive Property of Historical Catalytic Occupation Density Measures†

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Abstract. We consider the historical version of catalytic processes. We define the corresponding historical catalytic occupation time measure (HCOTM) and give its historical characterization. By using HCOTM we can define the historical occupation density measure (HODM) and show that HODM may diffuse almost surely.

1. Historical Catalytic Process (HCP)

First of all we will begin with introducing catalytic processes $X^{L(\rho)}$ with branching rate functional $L(\rho) = L_{[W,\rho]}$, where $\rho \equiv \rho^{\gamma} = X^K = X^{\gamma dr}$ is a Dawson-Watanabe superprocess (or super-Brownian motion) with constant branching rate $\gamma > 0$. The study of catalytic processes has been initiated by Dawson-Fleischmann (1997) [3]. Roughly speaking, $X^{L(\rho)}$ is the special continuous super-Brownian motion which is constructed under the branching mechanism that the branching occurs only at the place where exists the catalyst whose time evolution is given as a continuous super-Brownian motion (SBM) X^K with $K = \gamma dr$, $\gamma > 0$. In other words, it is nothing but a continuous measure-valued branching Markov process under the framework that the collision local time $L_{[W,\rho]}$ in the sense of Barlow-Evans-Perkins [15] of a Brownian path W and catalytic mass process $\rho \equiv \rho^{\gamma}$ governs the branching mechanism. As a matter of fact, the setup and formulation of this catalytic process is understood rigorously in terms of the general theory of additive functionals due to E. B. Dynkin (e.g. [11],[12]). In this section we consider the HP representation (= expression as historical process) which just corresponds to the catalytic process $X^{L(\rho)}$. In what follows, for simplicity, we put $I \equiv I_T = [0, T]$, $0 < T < \infty$, and denote by $\mathbf{C} \overset{\text{or}}{=} C(I, \mathbf{R})$ the Banach space of real valued continuous paths on I. Let $\tilde{w} := \{w_{s\wedge t}; s \in I\}$ denote the stopped path at time $t \in I$ for a path $w \in \mathbf{C}$, and we write the totality of such those stopped paths $\{\tilde{w}\}$ by $\mathbf{C}^t$. Then notice that the set

$$\hat{C}(I, \mathbf{C}) := \{w \in C(I, \mathbf{C}); w_t \in \mathbf{C}^t, \forall t \in I\}$$

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becomes a closed subspace of $C(I, \mathbf{C})$. Set

$$I \hat{\times} \mathbf{C}^{\cdot} := \{[t, w]; t \in I, w \in \mathbf{C}^t\}.$$

For this space $I \hat{\times} \mathbf{C}^{\cdot}$, there exists a continuous mapping : $\mathbf{C} \ni w \mapsto \tilde{w} \in \hat{C}(I, \mathbf{C})$, and this mapping allows the graph of $w \in \hat{C}(I, \mathbf{C})$ to be a subset of $I \hat{\times} \mathbf{C}^{\cdot}$. When a path $w \in \mathbf{C}$ is given, then we can regard $\tilde{w} := \{\tilde{w}_t; t \in I\}$ as its path trajectory. Under the above setup we can discuss the HP representation of the catalytic process. For $z \in \mathbf{R}$ let Π_z be the law on $\mathbf{C}$ of the Brownian path W with diffusion coefficient κ starting at z at time $t = 0$. Then we denote by $\tilde{W} = [\tilde{W}, \tilde{\Pi}_{s,w},\ s \in I, w \in \mathbf{C}^s]$ the Brownian path process on I. The semigroup of $\tilde{W}$ is given by

$$\tilde{S}_{s,t}\varphi(w) = \tilde{\Pi}_{s,w}\varphi(\tilde{W}_t), \qquad 0 \le s \le t \le T, \quad w \in \mathbf{C}^t, \quad \varphi \in b\mathcal{B}(\mathbf{C}), \tag{1}$$

and by $\{\tilde{A}_s; s \in I\}$ the corresponding generator. $M_F^t(\mathbf{C}^t)$ denotes the totality of nonnegative finite measures μ on $\mathbf{C} = C(I, \mathbf{R})$ such that $\mu(\mathbf{C} \setminus \mathbf{C}^t) = 0$, equipped with topology of weak convergence.

According to [4], for each $t \in I$ there exists an $M_F^t(\mathbf{C}^t)$-valued time inhomogeneous right Markov process

$$\tilde{X} = \left\{\tilde{X}^{L(\rho)}, \tilde{\mathbf{P}}_{s,\mu},\ s \in I, \mu \in M_F^s(\mathbf{C}^s)\right\}.$$

We call $\tilde{X}$ a historical catalytic process (HCP).

Proposition 1. *The Laplace transition functional of HCP $\tilde{X}$ is given by*

$$\tilde{\mathbf{P}}_{s,\mu} \exp\langle \tilde{X}_t^{L(\rho)}, -\varphi\rangle = \exp\langle \mu, -u_\varphi(s, \cdot, t)\rangle \tag{2}$$

for $0 \le s \le t \le T$, $\mu \in M_F^s(\mathbf{C}^s)$ and $\varphi \in pb\mathcal{B}(\mathbf{C}^t)$. Here the function $u_\varphi(\cdot, \cdot, t)$ is the unique $\mathcal{B}([0, t] \hat{\times} \mathbf{C}^{\cdot})$ -measurable bounded nonnegative solution of the equation :

$$\begin{cases} -\dfrac{\partial}{\partial s} u_\varphi(s, w, t) = & \tilde{A}_s u_\varphi(s, w, t) - \rho_s^\gamma u_\varphi^2(s, w, t), \qquad 0 \le s \le t, \quad w \in \mathbf{C}^s \\ \quad u_\varphi(t, \cdot, t) = & \varphi. \end{cases} \tag{3}$$

Proposition 2. *The corresponding log-Laplace equation to* (3) *is given by*

$$\tilde{\Pi}_{s,w}\varphi(\tilde{W}_t) = u_\varphi(s, w, t) + \tilde{\Pi}_{s,w} \int_s^t u_\varphi^2(r, \tilde{W}_r, t) L_{[W,\rho^\gamma]}(dr). \tag{4}$$

Moreover, an easy computation leads to expectation formula.

Corollary 3. *The following expectation formula holds :*

$$\tilde{\mathbf{P}}_{s,\mu}\langle \tilde{X}_t^{L(\rho)}, \varphi\rangle = \int \tilde{\Pi}_{s,w}\varphi(\tilde{W}_t)\mu(dw). \tag{5}$$

2. Historical Catalytic Occupation Time Measure (HCOTM)

Suggested by Dawson et al. [4] (cf. (2.4.1), p.43), from (3.23) [3, p.240] we can certainly define pathwise the historical catalytic occupation time measure (HCOTM) $\tilde{Y}$ related to $\tilde{X}^{L(\rho)}$ by

$$\langle \tilde{Y}_{[s,t]}, \psi \rangle := \int_s^t dr \langle \tilde{X}_r^{L(\rho)}, \psi(r, \cdot) \rangle \qquad \text{for} \quad \psi \in pb\mathcal{B}([0,t]\hat{\times}\mathbf{C}^{\cdot}), \tag{6}$$

for fixed $0 \le s \le t \le T$. Since $\tilde{X}^{L(\rho)}$ is right continuous from definition, the integral (6) makes sense, and actually HCOTM $\tilde{Y}_{[s,t]}$ is a finite random measure defined on $[s,t]\hat{\times}\mathbf{C}^{\cdot}$. By the standard argument of Dynkin (1991) [16], we can obtain the following characterization theorem of HCOTM $\{\tilde{Y}_{[s,t]}\}$ via Laplace transition functional.

Theorem 4. *The Laplace transition functional of* $\{\tilde{X}, \tilde{Y}\}$ *is given by*

$$\tilde{\mathbf{P}}_{s,\mu} \exp\left\{-\langle \tilde{X}_t^{L(\rho)}, \varphi \rangle - \langle \tilde{Y}_{[s,t]}, \psi \rangle\right\} = \exp\langle \mu, -u_{\varphi,\psi}(s, \cdot, t)\rangle \tag{7}$$

for $\mu \in M_F^t(\mathbf{C}^t)$, *for fixed* $0 \le s \le t \le T$ *and for* $\varphi \in pb\mathcal{B}(\mathbf{C}^t)$ *and* $\psi \in pb\mathcal{B}([0,t]\hat{\times}\mathbf{C}^{\cdot})$. *Here the function* $u_{\varphi,\psi}(\cdot,\cdot,t)$ *is the unique* $\mathcal{B}([0,t]\hat{\times}\mathbf{C}^{\cdot})$ *-measurable bounded nonnegative solution of the nonlinear integral equation :*

$$\begin{aligned} u_{\varphi,\psi}(s,w,t) &+ \tilde{\Pi}_{s,w} \int_s^t u_{\varphi,\psi}^2(r, \tilde{W}_r, t) L_{[W,\rho^\gamma]}(dr) \\ &= \tilde{\Pi}_{s,w}\varphi(\tilde{W}_t) + \tilde{\Pi}_{s,w} \int_s^t dr \psi(r, \tilde{W}_r). \end{aligned} \tag{8}$$

3. Historical Catalytic Occupation Density Measure (HCODM)

For the occupation time process there is a sufficient L^2-criterion for the existence of absolutely continuous states. In particular, it is well known [3] that the L^2-density $y_{[s,t]}(z) \equiv y_{[s,t]}^{L(\rho)}$ of catalytic occupation time measure $Y \equiv Y^{L(\rho)} \equiv Y^{L(\rho)}[X^{L(\rho)}]$ exists.

Lemma 5. (cf. [3], Proposition 5 (a), p.240) *Let* $p(r,a,b)$ *be the transition density of the standard Brownian motion* $B = (B_r)$. *Then the* $L^2(\mathbf{P}_{s,\mu})$ *-limit* $y_{[s,t]}^{L(\rho)}(z)$ *of*

$$y_{[s,t]}^{\varepsilon}(z) := \langle Y_{[s,t]}, p(\varepsilon, \cdot, z)\rangle, \qquad \varepsilon > 0$$

exists as $\varepsilon \downarrow 0$ *for each* $z \in \mathbf{R}^d$, $d \le 3$.

Lemma 6. (cf. [3], Proposition 5 (b), p.241) *With respect to* $\mathbf{P}_{s,\mu}$, *the random measure* $Y_{[s,t]}$ *on* $\mathbf{R}^d$ $(d \le 3)$ *is absolutely continuous with density function* $y_{[s,t]}$, *namely,*

$$\mathbf{P}_{s,\mu}\left\{Y_{[s,t]}(dz) = y_{[s,t]}(z)dz\right\} = 1 \qquad \textit{holds.}$$

In what follows we assume that $d = 1$. Note that when $d = 1$, we have the sharpened existence result of Brownian collision local time $L_{[W,\rho^\gamma]}$ of the catalyst process ρ^γ. Indeed, it belongs to the class K^ξ of branching rate functionals with parameter $\xi = 1/2$. That is to say,

$$\Pi_{s,a}\int_s^t \varphi_p^2(W_r)L_{[W,\rho^\gamma]}(dr) \le C_N|t-s|^{1/2}\varphi_p(a), \quad 0\le s\le t\le N, \quad a\in\mathbf{R}.$$

When the occupation time process

$$Y_t := \int_0^t drX_r(\cdot), \qquad t\ge 0$$

possesses a jointly continuous occupation density field $y = \{y_t(z);\ t\ge 0, z\in\mathbf{R}\}$ with probability one, we can define the occupation density measure (or the so-called super-Brownian local time measure)

$$\lambda^z(dr) := dy_r(z), \qquad z\in\mathbf{R}$$

on $\mathbf{R}_+$. Analogously, we can consider the historical version of the catalytic occupation density measure. In fact, the historical catalytic occupation density measure (HCODM) $\tilde\lambda^z_{[s,t]}(d[r,w])$ is defined by

$$\langle\tilde\lambda[s,t]^z,\psi\rangle := \int \tilde Y^{L(\rho)}_{[s,t]}(d[r,w])\psi(r,w)\delta_z(W_r) \tag{9}$$

for $z\in\mathbf{R}$, $0\le s\le t\le T$, and $\psi\in pb\mathcal{B}([s,t]\ \hat{\times}\mathbf{C}^\cdot)$. The main assertion of this paper is as follows.

Theorem 7. *The HCODM* $\tilde\lambda^z_{[s,t]}$ *is* $\tilde{\mathbf{P}}_{s,\mu}$ -a.s. *diffuse as a measure on* $[s,t]\hat{\times}\mathbf{C}^\cdot$, i.e., *it does not carry mass at any single point set in* $[s,t]\hat{\times}\mathbf{C}^\cdot$.

Proof. Let $x_r(z)$ be the density field of $X_r^{L(\rho)}(\cdot)$. From the definition of occupation time measure, it follows that

$$y_r(z) = \int_0^r dsx_s(z), \qquad t>0, \quad z\in\mathbf{R},$$

implying that $y_r(z)$ provides almost surely with an absolutely continuous measure on the time parameter set $\mathbf{R}_+$. Hence, it is obvious that the original catalytic occupation density measures $\{\lambda^z(\cdot);\ z\in\mathbf{R}\}$ are almost surely diffuse (cf. [2]). While, the almost sure diffuseness of HCODM $\tilde\lambda^z_{[s,t]}$ yields from identical law property. It suffices to note that the law $\mathcal{L}(\lambda^z_{[s,t]}(F);\ \mathbf{P}_{s,\mu})$ of $\lambda^z_{[s,t]}$ is equivalent to the law $\mathcal{L}(\tilde\lambda^z_{[s,t]}(F\times\mathbf{C});\ \tilde{\mathbf{P}}_{s,\mu})$ of $\tilde\lambda^z_{[s,t]}$ for any $z\in\mathbf{R}$, any Borel set $F\in\mathcal{B}(I_T)$, and $0\le s\le t\le T$. This completes the proof.

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