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A Probabilistic Representation for Solutions of Nonlinear Equations with Catalyst and Large Deviations for Scaled Catalytic Processes

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1. Introduction

We consider the nonlinear differential equation with catalytic noise[4,12,16]. This is a rather new type of equation where the coefficient function of the nonlinear term is given by the so-called catalyst process. The purpose of this paper is to discuss the existence of the catalytic superprocess associated with the equation and establish the exponential moment formula (cf. (5.1) in Theorem 5.1, Sec.5). Based on this, we give a probabilistic interpretation of solutions for the nonlinear equation with catalytic noise. Furthermore, as its application, we derive the large deviation principle for superprocesses.

More precisely, we consider the following nonlinear reaction diffusion equation with catalytic noise:

$$\frac{\partial}{\partial s}u(s,y)+\frac{\kappa}{2}\Delta u(s,y)+\psi(s,y)=X_t(\omega)u^2(s,y). \tag{1.1}$$

Here we mean by the catalytic noise X_t a super-Brownian motion (or Dawson-Watanabe superprocess)[3,28] with a simple branching rate functional which is given by the Lebesgue measure multiplied by a constant (cf. Remark 2.1 in Sec.2). So that, it turns out to be that the system governed by the above equation describes its time evolution in a catalytic medium[12]. In Section 3 we treat the nonlinear differential equation with catalytic noise and discuss the existence and uniqueness of its solutions (see Theorem 3.2 and Theorem 3.3). Next we construct the Brownian collision local time (BCLT) $L=L_{[W,\rho]}$ (cf. Proposition 4.3 in Sec.4) and study some basic properties that the BCLT should satisfy (cf. Proposition 4.4 in Sec.4). The catalytic superprocess is constructed in Section 5 that is associated with the nonlinear equation with catalytic noise (see Theorem 5.1). In other words, the integral of the nonlinear term of the equation relative to the BCLT L provides with a rigorous expression of the corresponding log-Laplace equation (cf. (5.2) in Sec.5). This enables us to establish the Laplace functional formalism for the catalytic superprocess. On this account, we can derive the probabilistic representation of solutions for the nonlinear equation with catalytic noise (cf. Theorem 5.2). As its application, we discuss in Section 6 the large deviation and prove the large deviation principle (LDP) for the catalytic superprocess.

This is the main result in this paper (cf. Theorem 6.7). The proof of weak large deviation principle, which is the key result for LDP, will be given in the succeeding section (see Sec.7). For other related results on LDP, see Refs. 7, 9, 22, 24 and 27 (see also Ref. 6 for large deviation techniques). Furthermore, we study in Section 8 some scaling property for catalytic processes in question and derive a certain type of limit theorem for those processes.

Let p be a positive number such that $p > d$ where d is the space dimension. Define the reference function φ_p by

$$\varphi_p(y) := \left(1 + |y|^2\right)^{-p/2} \quad \text{for} \quad y \in R^d. \tag{1.2}$$

We denote by $\mathcal{B}^p \equiv \mathcal{B}^p(R^d)$ the space of real-valued Borel measurable functions f on R^d such that

$$|f(x)| \leq C(f) \cdot \varphi_p(x)$$

holds for every x in R^d, for some positive constant $C(f)$ depending on the function f. The space $\mathcal{B}^p_+$ consists of all positive elements in $\mathcal{B}^p$. For a time interval I in R_+, $\mathcal{B}^{p,I}$ denotes the space of the functions $f = f(s,x)$ in $\mathcal{B}(I \times R^d)$ such that there is a positive constant $C(f)$ depending on f satisfying

$$|f(s,\cdot)| \leq C(f) \cdot \varphi_p \quad \text{for} \quad s \in I.$$

$\mathcal{C}^p \equiv \mathcal{C}^p(R^d)$ is the space of real-valued continuous functions f on R^d such that $|f(x)| \leq C(f) \cdot \varphi_p(x)$, $\forall x \in R^d$ for some positive constant $C(f)$ depending on f. Both $\mathcal{B}^p$ and $\mathcal{C}^p$ are equipped with the norm $\|f\| := \|f/\varphi_p\|_\infty$, where $\|\cdot\|_\infty$ is the uniform norm. Likewise, $\mathcal{C}^p_+$ (resp. $\mathcal{C}^{p,I}$) is the counterpart of $\mathcal{B}^p_+$ (resp. $\mathcal{B}^{p,I}$) for continuous functions.

Let $\mathcal{M}_p \equiv \mathcal{M}_p(R^d)$ be the set of all locally finite non-negative measures μ on R^d, such that

$$\|\mu\|_p := \langle \mu, \varphi_p \rangle = \int_{R^d} \varphi_p(y)\mu(dy) < \infty. \tag{1.3}$$

$\mathcal{M}_p$ is the set of tempered measures on R^d with the p-vague topology. While, $M_F = M_F(R^d)$ is the set of all finite measures on R^d. We denote by

$$W^\kappa := \{W^\kappa_s, \Pi^\kappa_{s,a}, s \geq 0, a \in R^d\} \tag{1.4}$$

the d-dimensional Brownian motion with generator $\kappa\Delta/2$ and the canonical path space $\Omega = C(R_+; R^d)$. We write $W^\kappa_s(\omega) = \omega(s)$, $\omega \in \Omega$ for its canonical realization. Especially when $\kappa = 1$, we call it the standard Brownian motion and suppress the parameter $\kappa = 1$ in its notation. For convention in the theory of measure-valued processes, we would rather use the notation $P[X]$ for the mathematical expectation than the usual $E[X] = E^P[X] = \int_\Omega X dP$.

2. Preliminaries

According to e.g. Ref. 3, we may use the Brownian motion (BM) with generator $\frac{1}{2}\Delta$ (as an underlyong process) to define the super-Brownian motion (SBM) (or Dawson-Watanabe superprocess) in terms of the martingale problem formulation (see also Ref. 10). That is to say, for each initial measure μ in M_F, there exists a probability measure P_μ on $(\Omega', \mathcal{F}')$ with $\Omega' = C(R_+; M_F)$ such that $X_0 = \mu$, P_μ-a.s.,

$$M_t(\psi) := \langle X_t, \psi\rangle - \langle \mu, \psi\rangle - \int_0^t \langle X_s, \left(\frac{1}{2}\Delta\right)\psi\rangle ds, \quad \forall t > 0, \quad \psi \in \mathrm{Dom}(\Delta/2),$$

is a continuous $(\mathcal{F}'_t)$-martingale under P_μ, where the quadratic variation process $\langle M(\psi)\rangle_t$ is given by

$$\langle M_.(\psi)\rangle_t = 2\gamma \int_0^t \int_\Omega \psi(\eta)^2 X_s(d\eta)ds, \quad \forall t > 0, \quad P_\mu - a.s.$$

In connection with application to the succeeding section, we shall introduce an alternative formulation of SBM. We begin with the nonlinear parabolic equation and determine the superprocess by making use of its solution, via the Laplace functional. In fact, we consider the following nonlinear reaction diffusion equation in backward formulation (of convenience for later discussion) with the terminal condition:

$$-\frac{\partial v}{\partial s} = \frac{1}{2}\Delta v - \gamma \cdot v^2, \qquad v|_{s=t} = \varphi \in C_+^p, \tag{2.1}$$

where γ is a positive constant. It is well known (e.g. Ref. 20) that the solution $v \equiv v(\cdot, t, \cdot)$ of the log-Laplace equation

$$v(s,t,a) = \Pi_{s,a}\left[\varphi(W_t) - \int_s^t v^2(r,t,W_r)\gamma dr\right], \quad 0 \le s \le t, \ \ a \in R^d \tag{2.2}$$

uniquely solves (2.1) in the backward formulation.

Remark 2.1. It is interesting to note that the second term at the right-hand side in (2.2) can be regarded as an integral with respect to the branching rate functional K (cf. §2.2 in Ref. 4 and §4 in Ref. 16) of the special case, namely, $K(dr) = \gamma dr$. Heuristically, this simply corresponds to the event that each corresponding X-particle branches with the constant rate $\gamma > 0$, on a phenomenal basis.

Now we introduce the super-Brownian motion (SBM) as a catalyst process, which is used to describe the catalytic medium in the next section. According to Dynkin's approach,[20] there exists an $\mathcal{M}_p$-valued critical SBM (or Dawson-Watanabe superprocess) $X = X^K$ with branching rate functional $K(dr) = \gamma dr$ with the Laplace transition functional

$$P_{s,\mu} \exp\langle X_t^K, -\varphi\rangle = \exp\langle \mu, -v^{[\varphi]}(s,t,\cdot)\rangle \qquad 0 \le s \le t, \tag{2.3}$$

for μ in $\mathcal{M}_p$ and $\varphi \in \mathcal{B}_+^p$, where $v^{[\varphi]}(s,t,\cdot)$ is a solution of the nonlinear reaction diffusion equation (2.1). In fact, $X = X^K = X^{\gamma dr}$ is a time-homogeneous Markov process. From now onward, for the super-Brownian motion as a catalytic noise, we would rather use the notation ρ^γ than X^K with $K(dr) = \gamma dr$. We call it the catalyst process, as the naming is originally due to Dawson-Fleischmann.[4]

Consequently, the solution $v(s,t,\cdot) \equiv v^{[\varphi]}(s,t,\cdot)$ of (1) may be expressed by

$$v(s,t,a) = -\log P_{s,\delta_a}\left[\exp\langle \rho_t^\gamma, -\varphi\rangle\right], \qquad t \geq s, \;\; a \in R^d, \;\; \varphi \in C_+^p. \tag{2.4}$$

This is nothing but a probabilistic representation of solutions of the nonlinear reaction diffusion equation. This probabilistic interpretation is, for instance, due to E.B. Dynkin[17] (see also Ref. 11).

3. Nonlinear Differential Equation with Catalytic Noise

Let $\rho^\gamma = \{\rho_t^\gamma; t \geq s\}$ be the catalyst process defined in the previous section. The principal object of this section is the following nonlinear reaction diffusion equation with catalytic noise:

$$\mathcal{L}u \equiv \frac{\partial u}{\partial s} + \frac{\kappa}{2}\Delta u + \psi - \rho_s^\gamma u^2 = 0, \quad u|_{s=t} = \varphi. \tag{3.1}$$

Here we can regard the noise term ρ_s^γ as a $\mathcal{M}_p$-valued continuous path since we have a modification $\tilde{X}^K$ of the SBM X^K with continuous paths (e.g. Refs. 4, 16). The existence of solutions to (3.1) can be attributed to the problem for generalized cumulant equation, which is actually associated with a more general equation than (3.1).

Before discussing the cumulant equation, we need to introduce some notations. Let $\{T_t^\kappa; t \geq 0\}$ be the semigroup with generator $\kappa\Delta/2$, and set

$$(U^\kappa f)(s,\cdot) := \int_s^t T_{r-s}^\kappa f(r,\cdot)dr, \qquad \text{for} \;\; f \in C^{p,I}, \;\; s \in I = [L,T].$$

Definition 3.1. We say that a continuous additive functional (CAF) $A = A_{[W^\kappa]}$ of the Brownian motion W^κ belongs to the class $\mathcal{K}$ if A is locally admissible, i.e., if

$$\sup_{a \in R^d} \Pi_{s,a}^\kappa \int_s^t \varphi_p(W_r^\kappa) A_{[W^\kappa]}(dr)$$

vanishes as s,t tends to r_0 for some positive r_0.

For $f \in C^{p,I}$ and $s \in I = [L,T]$, define

$$(Z^\kappa[A]f)(s,\cdot) := \Pi_{s,a}^\kappa \int_s^t f(r, W_r^\kappa) A_{[W^\kappa]}(dr), \qquad A \in \mathcal{K}.$$

For convention, we write it formally as $\int_{[s,t]} T_{r-s}^\kappa (f(r,\cdot)) A_{[W^\kappa]}(dr)$. We introduce the functional

$$F(\kappa, \varphi, \psi, \rho^\gamma, u) := u - T_{t-\cdot}^\kappa \varphi - U^\kappa \psi + Z^\kappa[A](u^2) \tag{3.2}$$

defined for $\{\kappa, \varphi, \psi, \rho^\gamma, u\} \in R_+ \times C^p \times C^{p,I} \times C(R_+; \mathcal{M}_p) \times C^{p,I}$. We will study the following generalized cumulant equation

$$F(\kappa, \varphi, \psi, \rho^\gamma, u) = 0 \tag{3.3}$$

which covers (3.1). The purpose of this section is to solve (3.3). A routine work for the functional equation theory allows us to obtain the uniqueness of solutions to (3.3).

Theorem 3.2. (Uniqueness) *Let $A \in \mathcal{K}$ be locally bounded characteristic. For each initial data $\{\kappa, \varphi, \psi, \rho^\gamma\}$ in $R_+ \times C^p \times C^{p,I} \times C(R_+; \mathcal{M}_p)$, there exists at most one element $u \in C^{p,I}$ in the sense of $\|\cdot\|$ which solves the generalized cumulant equation $F(\kappa, \varphi, \psi, \rho^\gamma, u) = 0$.*

Proof. Assume that there are two elements $u, v \in C^{p,I}$ such that $F(\kappa, \varphi, \psi, \rho^\gamma, u) = F(\kappa, \varphi, \psi, \rho^\gamma, v) = 0$ for $\{\kappa, \varphi, \psi, \rho^\gamma\} \in R_+ \times C^p \times C^{p,I} \times C(R_+; \mathcal{M}_p)$. From (3.2) we readily obtain

$$\|u(s) - v(s)\| \leq \left\| Z^\kappa[A](u^2)(s) - Z^\kappa[A](v^2)(s) \right\|_{C^p}. \tag{3.4}$$

Recall that C^p and $C^{p,I}$ are Banach algebra with respect to the pointwise product of functions. Moreover, (3.4) may be estimated majorantly by

$$\sup_{s \in I} \|u(s) + v(s)\| \cdot \left\| \int_s^t T^\kappa_{r-s}(u(r) - v(r))\varphi_p(\cdot) A_{[W^\kappa]}(dr) \right\|$$

$$\leq C_I \left\| \int_s^t T^\kappa_{r-s} \varphi_p A_{[W^\kappa]}(dr) \right\| \tag{3.5}$$

because linear operators $\{T^\kappa_t\}$ acting in C^p are uniformly bounded over the bounded region of t, κ, and we have only to pay attention to finiteness of the upper bound $\leq C_{\kappa,I} \|\varphi_p(\cdot)\|_\infty$ of (3.4). By local admissibility we can choose small $\varepsilon > 0$ which gives the upper estimate of (3.5), and the required result yields from reductio ad absurdum together with the above estimate. Q.E.D.

Moreover, resorting to functional analysis, we can prove the existence of solutions to (3.3) by emplying the implicit function theorem and the standard iteration scheme. Now we state the assertion with the proof divided into two parts, which will be given separately below.

Theorem 3.3. (Existence) *Suppose the same assumptions on A as in Theorem 3.2. For each initial data $\{\kappa, \varphi, \psi, \rho^\gamma\}$ in $R_+ \times C^p \times C^{p,I} \times C(R_+; \mathcal{M}_p)$, there exists a solution u in $C^{p,I}$ of (3.3).*

Proof. Note that we can choose an approximating sequence $\{A^{(n)}\}_n$ of $A \in \mathcal{K}$ such that with probability one $A^{(n)}(J) \nearrow A(J)$ as $n \to \infty$ for all open intervals J of R_+ (cf. Remark 3.6 below). Assume that there exists a unique nonnegative bounded solution u_n of (3.3) with $A_{[W^\kappa]}$ replaced by $A^{(n)}_{[W^\kappa]}$ for each n. Then we readily obtain

$$0 \leq u_n(s, a) \leq C_{\kappa,I} \cdot \varphi_p(a), \quad s \in I, \quad a \in R^d,$$

because both $T^\kappa\varphi$ and $U^\kappa\psi$ satisfy the locally bounded characteristic property. So that, the solution $\{u_n\}$ are uniformly dominated. Then the pointwise limit $u(\geq 0)$ of u_n as $n \to \infty$ is also dominated. The passage to the limit $n \to \infty$ of $F(\kappa, \varphi, \psi, \rho^\gamma, u_n) = 0$ leads to the required result, if we can show that

$$Z^\kappa[A^{(n)}](u_n^2)(s,a) \to Z^\kappa[A](u^2)(s,a) \qquad (\text{as} \quad n \to \infty) \tag{3.6}$$

for each $(s,a) \in I \times R^d$. In fact, we get

$$\text{(i)} \quad \left|Z^\kappa[A^{(n)}](u_n^2) - Z^\kappa[A^{(n)}](u^2)\right| \leq \sup_r \|u_n(r) + u(r)\| \cdot \|u_n - u\| C_{\kappa,I}\varphi_p(a) \to 0$$

(as $n \to \infty$) and also

$$\text{(ii)} \quad \left|Z^\kappa[A^{(n)}](u^2) - Z^\kappa[A](u^2)\right| \leq \Pi^\kappa_{s,a} \left|\langle (A^{(n)} - A)_{[W^\kappa]}, u^2(\cdot, W^\kappa_r)\rangle\right| \to 0$$

(as $n \to \infty$). To get (3.6) we have only to combine the above estimates (i), (ii). Therefore, it remains to show the existence of solutions u_n of $F(\kappa, \varphi, \psi, \rho^\gamma, u)[A^{(n)}] = 0$. This will be proved below shortly after introducing some preliminary results.

As to Proof of Existence (Theorem 3.3). To show the existence of solutions u_n of $F(\kappa, \varphi, \psi, \rho^\gamma, u)[A^{(n)}] = 0$. Let $\mathcal{H}$ be the set of all those $\{\kappa, \varphi, \psi\} \in R_+ \times C^p \times C^{p,I}$ such that there exists an element $u := u^{[\kappa,\varphi,\psi,\rho^\gamma]} \in C^{p,I}$, with $P_{s,\nu}$-probability one, for which

$$F(\kappa, \varphi, \psi, \rho^\gamma, u)[A^{(n)}] = 0, \quad P_{s,\nu} - a.s., \quad A^{(n)} \in \mathcal{K}_0,$$

where $\mathcal{K}_0$ is Dynkin's Admissible Class, cf. Proof of Theorem 5.1 in Sec.5 (see also Remark 3.6). To assert the existence of solutions is equivalent to show that the set $\mathcal{H}$ is open. Note that the mapping: $R_+ \times C^p \ni (\kappa, \varphi) \mapsto T^\kappa_{T-(\cdot)}\varphi \in C^{p,I}$ is continuous. By the domination property, $\{T^\kappa_t;\, t > 0\}$ as $T^\kappa_t \in \mathcal{L}(C^p; C^p)$ are uniformly bounded over a bounded region of t, κ. Hence, by the dominated convergence theorem we conclude that the mapping: $R_+ \times C^{p,I} \ni (\kappa, \psi) \mapsto U^\kappa f \in C^{p,I}$ is continuous. On this account, it is easy to see that

Lemma 3.4. *For $A \in \mathcal{K}_0$ given, $F(\cdot, \rho^\gamma, \cdot)[A]$ maps $R_+ \times C^p \times C^{p,I} \times C^{p,I}$ continuously into $C^{p,I}$, $P_{s,\nu}$-a.s.*

Furthermore, the Fréchet derivative of F with respect to u at each point is given by

$$D_u F(\kappa, \varphi, \psi, \rho^\gamma, u)(v) = v + 2 \cdot Z^\kappa[A^{(n)}](uv), \quad v \in C^{p,I}. \tag{3.7}$$

Proposition 3.5. *For each $\{\kappa, \varphi, \psi, u\} \in R_+ \times C^p \times C^{p,I} \times C^{p,I}$, $P_{s,\nu}$-a.s.,*
(a) *$D_u F(v)$ is linear in u;*

(b) $D_u F(v)$ *is continuous in* $\{\kappa, \varphi, \psi, u\}$;
(c) *the operator* $D_u F(\cdot) : \mathcal{C}^{p,I} \to \mathcal{C}^{p,I}$ *is bounded linear and bijective.*

Proof. (a), (b) are trivial. A similar estimate in the proof of Theorem 3.2 leads to that $D_u F$ is an injective operator. To complete the proof, we need to find $v \in \mathcal{C}^{p,I}$ with

$$v(s) + 2\int_s^T T^{\kappa}_{r-s}(u(r)v(r))A^{(n)}_{[W^\kappa]}(dr) = w(s) \in \mathcal{C}^{p,I}, \quad s \in I, \quad P_{s,\nu} - a.s.$$

However, the usual principal theorem for linear operator equations in Banach spaces may take care of this problem with an iteration scheme and induction argument (cf. Ref. 29). Q.E.D.

To go back to the proof of Theorem 3.3, assume that $F(\kappa_0, \varphi_0, \psi_0, \rho^\gamma, u_0) = 0$, $P_{s,\nu}$-a.s. for a fixed point $\{\kappa_0, \varphi_0, \psi_0, u_0\}$. An application of the implicit function theorem (cf. Theorem 4.B in Ref. 29) with Theorem 3.2 and (3.7) provides the existence of an open neighborhood U_0 of $\{\kappa_0, \varphi_0, \psi_0\}$ in $R_+ \times\ \mathcal{C}^p \times\ \mathcal{C}^{p,I}$ such that there exists a unique map : $\{\kappa, \varphi, \psi\} \mapsto u^{[\kappa,\varphi,\psi,\rho^\gamma]}$ defined on U_0 with

$$F(\kappa, \varphi, \psi, \rho^\gamma, u^{[\kappa,\varphi,\psi,\rho^\gamma]}) = 0, \qquad P_{s,\nu} - a.s., \tag{3.8}$$

because we employed Proposition 3.5. Thus we attain that the non-empty set $\mathcal{H}$ previously defined is open. This concludes the required assertion. Q.E.D.

Remark 3.6. For example, we can take an approximate $A^{(n)}$ as

$$A^{(n)}(\cdot) = \int_{(\cdot)} (n\varphi_p \wedge 1)(W^\kappa_r)A_{[W^\kappa]}(dr), \quad n \geq 1. \tag{3.9}$$

It is interesting to note that $\{A^{(n)}\}$ belongs to the Dynkin AF Class $\mathcal{K}_0$ in Section 3.3.3, Ref. 20. In fact, $\mathcal{K}_0$ is dense in $\mathcal{K}$ (cf. Remark 1 in Ref. 4).

Remark 3.7. Dawson and Fleischmann have treated in Ref. 4 only the special case $\kappa = 1$ and $\psi \equiv 0$ of (3.1). So our result is an extension of their existence and uniqueness theorem. Moreover, the method we have adopted here is based upon nonlinear functional analysis and is a more general approach to nonlinear equations than what they used in Ref. 4.

4. Regular Path and Collision Local Time

First we introduce a certain class of measure-valued continuous paths, which is suitable for defining the corresponding collision local time. See Remark 4.8 below.

Definition 4.1. The path η is said to be an element of the Regular Path Class $\mathcal{R}_p$ if $\eta \in C(R_+; \mathcal{M}_p)$ and for all $N > 0$,

$$\sup_{\substack{0 \leq s \leq N \\ a \in R^d}} \int_s^{s+\varepsilon} \langle \eta_r, \varphi_p \cdot p^\kappa(r-s, a, \cdot)\rangle dr \to 0 \quad (\text{as } \varepsilon \to 0)$$

where p^κ is the transition density function of the BM with generator $\kappa\Delta/2$.

Roughly speaking, that η is regular means that ε-accumulated densities of the finite measure-valued path $\varphi_p \cdot \eta$ vanishes uniformly on $[0, N] \times R^d$ ($\forall N > 0$) as ε tends to zero.

Remark 4.2. The original idea for branching rate functional is due to Dynkin's additive functional approach.[20] However, the theory is not directly applicable to the catalytic reaction diffusion equations. So Dawson-Fleischmann[4] extended it to cover the catalytic case (see also Ref. 16). Of course, our definition of regular paths and the following results are extensions of their work.

Let $\varepsilon \in (0, 1]$. Suggested by Ref. 4, we define $L^\varepsilon = L^\varepsilon_{[W^\kappa, \rho^\gamma]}$ by

$$L^\varepsilon_{[W^\kappa,\rho^\gamma]}(dr) := \langle \rho^\gamma_r, p^\kappa(\varepsilon, W^\kappa_r, \cdot)\rangle dr. \tag{4.1}$$

Then, L^ε is a continuous additive functional (CAF) of the Brownian motion W^κ in the sense of Dynkin.[20] Here L^ε is also the collision local time (CLT) of $\rho^\gamma \equiv X^K$ with the ε-vicinity of the Brownian path W^κ in the sense of Barlow-Evans-Perkins.[1] It is not difficult to show the following proposition. All through this section we may consider only the special case $\kappa = 1$ for the proofs of the propositions without loss of generality as far as the existence of collision local time is concerned.

Proposition 4.3. *Let $d \le 3$. If $\eta \in \mathcal{R}_p$, then there exists an additive functional $L = L_{[W^\kappa,\eta]}$ of the Brownian path W^κ such that, for every $\psi \in C_+^{p,[0,N]}$ ($N > 0$),*

$$\sup_{\substack{0\le s\le N\\ a\in R^d}} \Pi^\kappa_{s,a} \sup_{s\le t\le N} \left| \int_s^t \psi(r, W^\kappa_r) L^\varepsilon_{[W^\kappa,\eta]}(dr) - \int_s^t \psi(r, W^\kappa_r) L_{[W^\kappa,\eta]}(dr) \right|^2 \tag{4.2}$$

vanishes as ε approaches to zero.

Proof. Set $A^\varepsilon(dr) = \langle \psi(r, \cdot)\eta_r, p(\varepsilon, W_r, \cdot)\rangle dr$. Then note that

$$\sup_{\substack{s\le N\\ a\in R^d}} \int_s^{s+\varepsilon} \int \psi(r,b) p(r-s,a,b)\eta_r(db)dr \to 0 \quad (\text{as } \varepsilon \downarrow 0), \quad N > 0.$$

By virtue of the regularity of η, the assertion immediately follows from a slight modification of the proof of Proposition 6.(a) in Ref. 4 and Theorem 4.1 (on the general convergence of CAF's) in Ref. 23, together with the aforementioned fact. Actually, L is given by the limit of A^ε divided by ψ. Q.E.D.

Since it is true that the path $\rho^\gamma(\omega) \equiv X^K(\omega)$ is contained in $\mathcal{R}_p$ with probability one, we can apply Proposition 4.3 for the case $\eta = \rho^\gamma$. Consequently, there is the so-called Brownian collision local time $L \equiv L_{[W^\kappa,\rho^\gamma]}$ of the catalyst process $\rho^\gamma = \rho(\gamma)$. It is easy to show the following properties that this limit L possesses.

Proposition 4.4. *Let $\delta > 0$, $d \le 3$ and $\xi \in (0, 1/4)$. The Brownian collision local time $L \equiv L_\delta = L_{[W^\kappa, \rho^\gamma_{\delta+(\cdot)}]}$ satisfies the following properties:*
(a) *L_δ is continuous, i.e., it does not carry mass at any single point;*
(b) *L_δ is locally admissible, i.e., as $s, t \to r_0$, ($r_0 \ge 0$)*

$$\sup_{a \in R^d} \Pi^\kappa_{s,a} \int_s^t \varphi_p(W^\kappa_r) L_{[W^\kappa, \rho^\gamma_{\delta+(\cdot)}]}(dr) \to 0; \tag{4.3}$$

(c) *for each $N > 0$, there exists a positive constant $C_{\kappa,N}$ such that*

$$\Pi^\kappa_{s,a} \int_s^t \varphi_p(W^\kappa_r)^2 L_{[W^\kappa, \rho^\gamma_{\delta+(\cdot)}]}(dr) \le C_{\kappa,N} |t-s|^\xi \varphi_p(a) \tag{4.4}$$

for $0 \le s \le t \le N$ and $a \in R^d$.

Proof. Only the property (c) does matter. The same discussion as in the second step of the proof of Theorem 4 in Ref. 4 together with the domination property of the heat flow (e.g. Eq.(2.10) in Ref. 4) provides with an estimate

$$\Pi_{s,a} \int_s^{s+\varepsilon} \varphi_p^2(W_r) L_{[W,\rho^\gamma]}(dr) \le C \varepsilon^\xi \varphi_p(a).$$

This concludes the assertion. Q.E.D.

From (a) in the above, we know that the BCLT L_δ lies in the class $\mathcal{K}$. We say that the continuous additive functional A belongs to the class $\mathcal{K}^\xi$ if A satisfies the conditions (b), (c) in Proposition 4.4 additionally. Notice that $\mathcal{K}^\xi \subset \mathcal{K}$.

Remark 4.5. We assume tacitly that the parameter δ is positive in the above proposition. However, if $\delta = 0$, then some proper additional condition is required to let it make sense.

Indeed, we need to pose, for instance, the following condition for the case $\delta = 0$. The mapping : $R_+ \times R^d \ni (r, z) \mapsto \rho^\gamma_0 * q^\kappa(0, r, z)$ is locally ξ-Hölder continuous with Hölder constant proportional to $\|\rho^\gamma_0\|_p$, where the convolution is given by

$$\rho^\gamma_0 * q^\kappa(0, r, z) = \int_0^r \int_{R^d} p^\kappa(s, a, z) \rho^\gamma_0(da) ds \tag{4.5}$$

with $P_{0,\mu}$-probability one.

Remark 4.6. Since ρ^γ is an $\mathcal{M}_p$-valued SBM, the quantity $\|\rho^\gamma_0\|_p := \langle \rho^\gamma_0, \varphi_p \rangle = \int \varphi_p \, \rho^\gamma_0(da)$ is finite in the above-mentioned condition.

Remark 4.7. Heuristically, the collision local time L between a Brownian particle with path $W^\kappa(\omega)$ and the catalytic medium $\rho^\gamma(\omega)$ is given by

$$L_{[W^\kappa, \rho^\gamma]}((s,t)) = \int_s^t \int \delta_b(W^\kappa_r) \rho^\gamma_r(db) dr \tag{4.6}$$

(see e.g. §1.2 in Ref. 4). Here $\rho_r^\gamma(b)$ can be considered as the amount of catalyst present at time r at b which is encountered by a reactant particle with path W^κ, when ρ_r^γ is a singular measure (cf. Ref. 12). Hence, the term $\delta_r(W_r^\kappa)\ \rho_r^\gamma(db)$ allows us to realize the intuitive description at the microscopic level that a tagged Brownian particle with path W^κ enjoys branching according to a clock given by the additive functional $L = L_{[W^\kappa,\rho^\gamma]}$ (cf. Ref. 16).

Remark 4.8. For the one-dimensional case with $\kappa = 1$, it is well-known[3,12] that the continuous SBM lies in the nicer space of absolutely continuous measures. Therefore, there exists the Radon-Nikodym derivative $\rho_t(b) \equiv \rho_t(db)/db$ with respect to the Lebesgue measure db which is even a jointly continuous density field $\{\rho_t(b);\, t > 0,\, b \in R\}$ (e.g. Ref. 26). So that,

$$L_{[W,\rho]}(dr) := \left\{\int \delta_b(W_r)db\right\}\rho_r(W_r)dr \tag{4.7}$$

defines a continuous additive functional of BM. For dimensions $d \geq 4$, the Brownian collision local time $L = L_{[W,\rho]}$ degenerates to 0 because the closure of the graph of the SBM ρ never intersect with the graph of W, by Barlow-Perkins (cf. Proposition 1.3 in Ref. 2). On the other hand, the BCLT L exists non-trivially in dimensions $d = 2$ and $d = 3$ (cf. Ref. 4), although the random measures $\rho_t(db)$ are singular[5] and we cannot use the above-mentioned expression for the definition of BCLT. By virtue of Theorem 4.1 in Ref. 21 together with Proposition 4.7 in Ref. 21, we can show the similar result also for our case $L = L_{[W^\kappa,\rho^\gamma]}$. Therefore, the restriction $d \leq 3$ is even essential for our theory.

5. Catalytic Superprocess and Exponential Moment Formula

In Section 2, according to Dynkin's approach,[20] we have introduced the Laplace functional formalism with the log-Laplace equation, whereby the Dawson-Watanabe superprocess (a finite measure-valued Markov process) can be determined. The purpose of this section consists in reconstruction of the Laplace functional formalism with the log-Laplace equation to be adjusted for the catalytic superprocess associated with the nonlinear equation (3.1) with catalytic noise. We now introduce some additional notation. In analogy to $\mathcal{M}_p$, let $\mathcal{M}_p^I$ be the set of all measures ν on $I\times\, R^d$ such that

$$\langle \nu, \psi\rangle_I := \int\int_{I\times R^d} \psi(r,b)\nu(d[r,b]) < +\infty, \qquad \forall\quad \psi \in \mathcal{B}^{p,I}$$

where we furnish $\mathcal{M}_p^I$ with the weakest topology such that the maps : $\nu \mapsto \langle\nu,\ \psi\rangle_I$ are continuous for all $\psi \in \mathcal{C}^{p,I}$. Here we use the same notation for extended functions even in $C(I \times R_c^d)$ by the one point compactification if necessary.

Due to Dynkin's AF approach with a slight modification of Theorem 1.1 in Ref. 17 and other several results in Ref. 17 with minor change, we can derive the following formulation. Here by the symbol $L^\rho = L(\rho) = L(\rho^\gamma)$ we quote the BCLT L from Section 4 instead of $L_{[W^\kappa,\rho^\gamma]}$. Now we are ready to introduce one of the principal results in this paper, which includes the exponential moment formula (5.1) of solutions for nonlinear differential equation with catalytic noise.

Theorem 5.1. (Ref. 14) *Let $d \leq 3$. If the branching rate functional K is given $P_{s,\nu}$-a.s. by the BCLT $L(\rho^\gamma)$ of the SBM ρ^γ, there exist a time inhomogeneous $\mathcal{M}_p$-valued Markov process*

$$X^L \equiv X^{L(\rho^\gamma)} = \{X_t^L = X_t^{L(\rho^\gamma)}, \mathbf{P}_{s,\mu}^{\kappa,\rho^\gamma},\ s>0,\ \kappa \in R_+,\ \mu \in \mathcal{M}_p\}$$

and an $\mathcal{M}_p^{[s,t]}$-valued weighted occupation measure process $Y^\rho \equiv \{Y_t^\rho\}$ defined by

$$\langle Y_t^\rho, \psi\rangle_{[s,t]} := \int_s^t \langle X_r^{L(\rho^\gamma)}, \psi(r,\cdot)\rangle dr, \qquad \forall \quad \psi \in C_+^I$$

with Laplace transition functional

$$\mathbf{P}_{s,\mu}^{\kappa,\rho^\gamma} \exp\left\{-\left(\langle X_t^{L(\rho^\gamma)}, \varphi\rangle + \langle Y_t^\rho, \psi\rangle_{[s,t]}\right)\right\} = \exp\{-\langle \mu, u(s,t,\cdot)\rangle\} \tag{5.1}$$

$P_{s,\nu}$-a.s. *for $0 \leq s \leq t$, $\mu \in \mathcal{M}_p, \varphi \in \mathcal{B}_t^p$* and $\psi \in \mathcal{B}_t^{p,I}$. *Here the function $u(\cdot,t,\cdot) \in C_+^{p,I}$ is the unique solution of the log-Laplace equation*

$$u(s,t,a) = \Pi_{s,a}^\kappa\left[\varphi(W_t^\kappa) + \int_s^t \psi(r,W_r^\kappa)dr - \int_s^t u^2(r,t,W_r^\kappa)L(\rho^\gamma)(dr)\right]. \tag{5.2}$$

Proof. For $\mathcal{K} \ni L(\rho^\gamma)$, there exists an approximating sequence $\{L^n\}_n$ belonging to Dynkin's Admissible Class $\mathcal{K}_0$ ($\subset \mathcal{K}$), cf. §3.3.3 in Ref. 20. In fact, we can take L^n ($\in \mathcal{K}_0$) as $L^n := \int_{[\cdot]} (1\wedge\ n\varphi_p)\ (W_r^\kappa)\ L(\rho^\gamma)\ (dr)$, $n \geq 1$. Then the existence of a time-inhomogeneous $M_F \times M_F$-valued Markov process $Z^n = \{X^{L^n},\ Y^{L^n}\}$ with $\mathbf{P}_{s,\mu}$, $s \geq 0$, $\mu \in M_F$ satisfying

$$\mathbf{P}_{s,\mu} \exp\left\{-\langle X_t^{L^n}, \theta\varphi\rangle - \langle Y_t^{L^n}, \zeta\psi\rangle_{[s,t]}\right\} = \exp\langle \mu, -v(s,t,\cdot)\rangle, \tag{5.3}$$

for $0 \leq s \leq t$, $\mu \in M_F$, $(\varphi,\psi) \in \mathcal{B}_+^p \times \mathcal{B}_+^{p,[0,t]}$, $\theta \geq 0, \zeta \geq 0$, and

$$v(s,t,a) = \theta \cdot f(s,a) + \zeta \cdot g(s,a) - \Pi_{s,a}^\kappa \int_s^t v^2(r,t,W_r^\kappa)L^n(dr) \tag{5.4}$$

for fixed $t > 0$, $f,g \in \mathcal{B}_+^{p,[0,t]}$ is guaranteed by Dynkin's additive functional theory (cf. Theorem 1.4 in Ref. 18). Indeed, (5.3), (5.4) are the particular cases of (1.36), (1.37) in Ref. 19. The domination property:

$$\sup_{s\in I=[S,L]} T_{L-s}^\kappa \varphi(a) \leq C_0\|\varphi\|\varphi_p(a), \qquad \varphi \in \mathcal{B}_+^p$$

allows us to extend it to $\mathcal{M}_p$-valued Markov process version $\tilde{Z}^n$. The fact $L^n \nearrow L(\rho^\gamma)$ and the monotone convergence $v_n \searrow v$ in (5.4) verify the convergence of the corresponding

(5.3) with v replaced by v_n, for fixed s, t, μ. Note that the limit functional is continuous in (φ, ψ) and also that $v(s,t,a)$ vanishes if (φ, ψ) tends to null functions. Hence, the limiting functional is the Laplace functional of a random measure $\tilde{Z} = \{X^{L(\rho)}, Y^{L(\rho)}\}$ given by (5.3). Moreover, we can construct the laws of vectors $\{Z_{t_1}, \cdots, Z_{t_k}\}$ by taking advantage of the semigroup structure of the solutions for (5.4). On this account, the Markov process $\tilde{Z}$ in question is determined by these compatible finite dimensional distributions. Note that the Markov process $\tilde{Z}$ is independent of the choice of $\{L^n\}$. On the other hand, it follows from Kolmogorov's moment method that $X^{L(\rho^\gamma)}$ has continuous paths, since the corresponding branching rate functional (= BCLT) $L(\rho^\gamma)$ is contained in $\mathcal{K}^\xi$. Thus we attain the pathwise definition of Y^ρ as a measure on $[s,t] \times R^d$ as indicated in the statement of Theorem 5.1. With the help of the expectation formula for Y^ρ, it is routine work to verify that $\tilde{Z} = \{X^{L(\rho)}, Y^\rho\}$ satisfies (5.1), (5.2), namely, $f = T^\kappa\varphi$ and $g = U^\kappa\psi$, see Theorem 3.1 in Ref. 19 (also see Ref. 17). Clearly v solves the cumulant equation $F[A] = 0$ with $A_{[W^\kappa]}$ replaced by $L(\rho)$. Q.E.D.

In the above theorem $\mathbf{P}^{\kappa,\rho^\gamma}_{s,\mu}, \kappa \in R_+, s \geq 0, \mu \in \mathcal{M}_p$ is the quenched distribution of X^L given ρ^γ. The existence and uniqueness of the solution to the log-Laplace equation (5.2) are guaranteed by Theorem 3.2 and Theorem 3.3 in Section 3. We call $X^L = X^{L(\rho^\gamma)}$ the catalytic superprocess in the catalytic medium ρ^γ. By the same argument in Ref. 17 we readily obtain the probabilistic representation of solutions for (3.1) which realizes a probabilistic interpretation of the nonlinear equation with catalytic noise in terms of catalytic superprocess.

Theorem 5.2. *Let u be the unique solution to* (3.1). *Then it may be expressed by*

$$u(s,y) \equiv u^{[\kappa,\varphi,\psi,\rho^\gamma]}(s,y) = V[\varphi,\psi]$$

$$:= -\log \mathbf{P}^{\kappa,\rho^\gamma}_{s,\delta_y} \exp\left\{-\left(\langle X^L_t, \varphi\rangle + \langle Y^\rho_t . \psi\rangle_{[s,t]}\right)\right\}, \quad P_{s,\nu} - a.s. \tag{5.5}$$

for $(s,y) \in I \times R^d$, $I = [L,T]$, $L < T$, and $(\varphi,\psi) \in \mathcal{C}^p_+ \times \mathcal{C}^{p,I}_+$. Here $P_{s,\nu}$-a.s. means almost all catalyst process realizations.

Proof. To get (5.5) from (5.2), it suffices to choose, in particular, $\mu = \delta_y$. Q.E.D.

Remark 5.3. The study to derive the probabilistic representation of the log type like (5.5) from the exponential moments is originally due to Dynkin.[17] For other similar expressions for nonlinear elliptic equations and distinct probabilistic representations in terms of path-valued process, see Ref. 11.

6. Large Deviation Principle for Catalytic Processes

Recall that $\mathcal{C}_p$ is a separable Banach space. Let $(\mathcal{C}^*_p, \|\cdot\|_*)$ denote the dual Banach space relative to $(\mathcal{C}_p, \|\cdot\|)$. Then note that we can regard $\mathcal{M}_p$ as a convex subset of $\mathcal{C}^*_p$ equipped with the weak-* topology. The p-vague topology in $\mathcal{M}_p$ is equivalent to the induced topology in $\mathcal{M}_p$ by the weak-* topology in C_p^*. We have

Lemma 6.1. *The duality pairing $\ll \cdot,\cdot \gg$ between $\mathcal{M}_p$ and C^p is continuous in both components, and in particular*

$$\|\mu\|_* = |\mu|_p (:= \langle \mu, \varphi_p \rangle), \qquad \mu \in \mathcal{M}_p. \tag{6.1}$$

Proof. It suffices to note that the estimate

$$| \ll \mu, \varphi \gg | \leq \|\varphi\| \langle \mu, \varphi_p \rangle, \quad \varphi \in C^p, \quad \mu \in \mathcal{M}_p \tag{6.2}$$

holds. From (6.2) the continuity follows immediately. The equality (6.1) yields from (6.2), too, by the definition of $\|\mu\|_*$. Q.E.D.

Now we introduce a metric d_p in $\mathcal{M}_p$. Denote by pC_K the space of all positive elements f in $C_K(R^d; R)$ with compact support, equipped with uniform convergence topology.

Definition 6.2. The meric d_p in $\mathcal{M}_p$ is defined by

$$d_p(\mu,\nu) := \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \left(1 - \exp\left\{ - \frac{| \ll \mu, f_n \gg - \ll \nu, f_n \gg |}{\|f_n\|} \right\}\right)$$

for $\mu, \nu \in \mathcal{M}_p$ and some sequence $\{f_n\}_n \subset C_+^p$.

Then the followings are well known results in connection with this metric.

Lemma 6.3. (Ref. 25) (a) *There exists a sequence $\{f_n\}_{n\geq 1}$ of functions in pC_K such that, with $f_0 = \varphi_p$, $d_p(\cdot,\cdot)$ is a translation-invariant metric on $\mathcal{M}_p$, which generates the p-vague topology.*
(b) *$\mathcal{M}_p$ is a separable metric space with respect to the metric d_p.*

We define the open ball $B(\nu, r)$ in $\mathcal{M}_p$ with center ν and radius r by

$$B^{d_p}(\nu; r) := \{\, \mu \in \mathcal{M}_p;\ d_p(\mu,\nu) < r \,\}, \quad \nu \in \mathcal{M}_p, \quad r > 0.$$

Lemma 6.4. (Ref. 22) *Each open ball $B^{d_p}(\nu; r)$, $\nu \in \mathcal{M}_p$, $r > 0$ is a convex subset of $\mathcal{M}_p$.*

Set $R_c^d := R^d \cup \{\infty\}$ with an isolated point ∞. φ_p^* denotes the extension of φ_p to R_c^d by setting $\varphi_p^*(\infty) := 1$. Define $\mathcal{M}_p^*$ as the set of all measures μ on R_c^d satisfying $\langle \mu, \varphi_p^* \rangle < \infty$.

So that, we can define the p-vague topology in $\mathcal{M}_p^*$ as in the case $\mathcal{M}_p$. Then the following gives a criterion for relative compactness in $\mathcal{M}_p^*$.

Lemma 6.5. (Ref. 23) *Let A be a subset of $\mathcal{M}_p^*$. Then A is relative compact if and only if there is some $k \in \mathrm{N}$ such that $A \subset \{\mu \in \mathcal{M}_p^*;\ \langle \mu, \varphi_p^* \rangle \leq k\ \}$.*

Let $\mathcal{U}$ be the system of all those non-empty subsets of $\mathcal{M}_p$ which are open and convex. Define

$$S_{\mu,t}(A) := -\lim_{\Lambda\to\infty} \frac{1}{\Lambda} \log \mathbf{P}_{0,\Lambda\mu}^{\kappa,\rho^\gamma}(X_t^{L(\rho^\gamma)}/\Lambda \in A), \quad P_{0,\nu} - a.s. \tag{6.3}$$

for all $A \in \mathcal{U}$, $\mu \in \mathcal{M}_p$, $t \geq 0$. We set

$$I_{\mu,t}(\nu) := \lim_{r\downarrow 0} S_{\mu,t}(B^{d_p}(\nu;r)), \quad (\nu \in \mathcal{M}_p) \tag{6.4}$$

for $\mu \in \mathcal{M}_p$ and $t \geq 0$. Now we are in a position to state one of the main results in this paper, which provides the weak large deviation principle (LDP) for catalytic superprocesses given in the previous section.

Theorem 6.6. (Weak LDP[13]) *The family $\{\frac{1}{\Lambda}\log \mathbf{P}_{0,\Lambda\mu}^{\kappa,\rho^\gamma}(X_t^L/\Lambda \in (\cdot));\ \Lambda > 0\}$ satisfies a weak large deviation principle with convex rate functional $I_{\mu,t}$. In other words, the following two types of estimates hold: for every $t \geq 0$ and $\mu \in \mathcal{M}_p$,*

$$\text{(I)} \quad \liminf_{\Lambda\to\infty} \frac{1}{\Lambda} \log \mathbf{P}_{0,\Lambda\mu}^{\kappa,\rho^\gamma}(X_t^{L(\rho^\gamma)}/\Lambda \in G) \geq -\inf_{\lambda\in G} I_{\mu,t}(\lambda), \quad P_{0,\nu} - a.s.,$$

for $\forall$ open $G \subset \mathcal{M}_p$; and

$$\text{(II)} \quad \limsup_{\Lambda\to\infty} \frac{1}{\Lambda} \log \mathbf{P}_{0,\Lambda\mu}^{\kappa,\rho^\gamma}(X_t^{L(\rho^\gamma)}/\Lambda \in C) \leq -\inf_{\lambda\in C} I_{\mu,t}(\lambda), \quad P_{0,\nu} - a.s.,$$

for $\forall$ compact $C \subset \mathcal{M}_p$.

This is the key result for derivation of the full large deviation principle (Theorem 6.7 below), which is one of the main results in this paper. The proof of Theorem 6.6 shall be given in the next section (see Sec.7). For $\mu \in \mathcal{M}_p$, $t > 0$, we denote by $\mathcal{C}_t^p[\mu]$ the largest open set of all those functions $\varphi \in \mathcal{C}^p$ such that $\Xi(\mu, t, \varphi) := \log\ \mathbf{P}_{0,\mu}^{\kappa,\rho^\gamma}\ \exp\ \langle X_t^{L(\rho)}, \varphi \rangle <$ $+\infty$.

Theorem 6.7.(Full LDP[15]) *The family $\{\frac{1}{\Lambda}\log \mathbf{P}_{0,\Lambda\mu}^{\kappa,\rho^\gamma}(X_t^L/\Lambda \in (\cdot));\ \Lambda > 0\}$ satisfies the full large deviation principle with good rate functional $I_{\mu,t}$.*

Proof. Take $\mu \in \mathcal{M}_p$, $t > 0$ and $\varphi \in \mathcal{C}_t^p[\mu]$ and fix them. Since $\mathcal{C}_t^p[\mu]$ is open, we can find $\theta > 0$ such that $(1+\theta)\varphi \in \mathcal{C}_t^p[\mu]$. Set the event $A := \{\langle X_t^{L(\rho)}/\Lambda, \varphi \rangle > N\} \subset \Omega$. A simple calculation reads

$$\frac{1}{\Lambda} \log \mathbf{P}_{0,\Lambda\mu}^{\kappa,\rho^\gamma}\left[\exp\langle X_t^{L(\rho)}, \varphi \rangle;\ A\right] \leq \Xi(\Lambda\mu, t, (1+\theta)\varphi)/\Lambda - \theta N, \tag{6.5}$$

$\Lambda, N > 0$. The branching property (see e.g. Ref. 20) implies that $\Xi(\Lambda\mu, t, \varphi) = \Lambda \cdot \Xi(\mu.t.\varphi)$, hence the right-hand side in (6.5) proves to be finite. Letting first $\Lambda \to \infty$ and then $N \to \infty$, we obtain

$$\lim_{N\to\infty} \lim_{\Lambda\to\infty} \log \mathbf{P}^{\kappa,\rho^\gamma}_{0,\Lambda\mu} \left[\exp\langle X^{L(\rho)}_t, \varphi\rangle;\ A \right] = -\infty. \tag{6.6}$$

For $\theta > 0$ small enough, we can deduce from (6.6) that

$$\lim_{N\to\infty} \lim_{\Lambda\to\infty} \frac{1}{\Lambda} \log \mathbf{P}^{\kappa,\rho^\gamma}_{0,\Lambda\mu} \left(\langle X^{L(\rho)}_t/\Lambda, \theta\varphi_p\rangle \geq N \right) = -\infty. \tag{6.7}$$

By taking (6.1) and Lemma 6.5 into consideration, for each $M > 0$ we can find a compact subset C_M of $\mathcal{M}_p$ such that

$$\limsup_{\Lambda\to\infty} \frac{1}{\Lambda} \log \mathbf{P}^{\kappa,\rho^\gamma}_{0,\Lambda\mu} \left(X^{L(\rho)}_t/\Lambda \in (C_M)^c \right) \leq -M, \tag{6.8}$$

as far as we reinterpret measure and distribution as those on R^d_c (or on $\mathcal{M}^*_p$) respectively. (6.8) implies that $\{X^{L(\rho)}_t/\Lambda;\ \Lambda > 0\} \subset \mathcal{M}^*_p$ is exponentially tight.[6] Hence the assertion immediately yields from Lemma 2.1.5 in Ref. 8 together with Weak LDP (Theorem 6.6). Q.E.D.

7. Proof of Weak Large Deviation Principle

For $t > 0$, $\mu \in \mathcal{M}_p$, and a convex subset $A \in \mathcal{B}(\mathcal{M}_p)$ (fixed), we define the function

$$F(\Lambda) := \mathbf{P}^{\kappa,\rho^\gamma}_{0,\Lambda\mu}(X^{L(\rho)}_t/\Lambda \in A), \qquad \Lambda > 0. \tag{7.1}$$

Then we have

Lemma 7.1. (Supermultiplicativity) *The inequality*

$$F(\Lambda + \Lambda') \geq F(\Lambda)F(\Lambda') \tag{7.2}$$

holds for any $\Lambda, \Lambda' > 0$.

Proof. Let $\Lambda, \Lambda' > 0$ and fixed. Suppose that $\{X^{L,1}_t,\ X^{L,2}_t\}$ be distributed according to the product measure $\mathbf{P}^{\kappa,\rho}_{0,\Lambda\mu} \times \mathbf{P}^{\kappa,\rho}_{0,\Lambda'\mu}$. Then it follows that

$$F(\Lambda)F(\Lambda') = (\mathbf{P}^{\kappa,\rho}_{0,\Lambda\mu} \times \mathbf{P}^{\kappa,\rho}_{0,\Lambda'\mu})(X^{L,1}_t/\Lambda \in A, X^{L,2}_t/\Lambda' \in A).$$

Recall that A is convex in $\mathcal{M}_p$, so that, the convex combination $(\Lambda + \Lambda')^{-1}(X^{L,1}_t + X^{L,2}_t)$ lies in A if both $X^{L,1}_t/\Lambda$ and $X^{L,2}_t/\Lambda'$ are contained in A. Therefore we get immediately

$$F(\Lambda)F(\Lambda') \leq (\mathbf{P}^{\kappa,\rho}_{0,\Lambda\mu} \times \mathbf{P}^{\kappa,\rho}_{0,\Lambda'\mu}) \left((\Lambda + \Lambda')^{-1}(X^{L,1}_t + X^{L,2}_t) \in A \right).$$

While, the exponential moment formula (5.1) in Theorem 5.1, Sec.5 yields the branching property directly. Hence the law of the sum $X_t^{L,1} + X_t^{L,2}$ is given by $\mathbf{P}^{\kappa,\rho}_{0(\Lambda+\Lambda')\mu}$. Thus we attain the required inequality. Q.E.D.

For $\mu \in \mathcal{M}_p$, $t > 0$, and $A \in \mathcal{U}$, we set

$$\Phi(\Lambda) := -\log \mathbf{P}^{\kappa,\rho}_{0,\Lambda\mu}(X_t^L/\Lambda \in A), \qquad R > 0. \tag{7.3}$$

An application of supermultiplicativity (Lemma 7.1) concludes that the function $\Phi(\Lambda) \in R_+ \cup \{\infty\}$ is subadditive. We need the following result.

Proposition 7.2. *Suppose that A is an open convex Borel subset of $\mathcal{M}_p$. If there is some positive number Λ such that $F(\Lambda) > 0$, then F is bounded away from 0 on some non-trivial open interval.*

Proof. The result follows from the branching property, the translation invariance of the metric d_p (Lemma 6.3) and the supermultiplicativity. In fact, the standard routine argument takes care of it with Lemma 6.4. The whole proof is quite longsome but easy, hence omitted. Q.E.D.

Then we can conclude from Proposition 7.2 that Φ is either bounded on some non-empty open interval, or identically $+\infty$. Consequently, we know that the subadditivity of Φ guarantees the existence of all limits $S_{\mu,t}(A) \in [0, +\infty]$, $A \in \mathcal{U}$, ($\mu \in \mathcal{M}_p$, $t \geq 0$), by repeating the discussion in Lemma 4.2.5 in Ref. 8. Moreover, by virtue of monotonicity we obtain

Lemma 7.3. *For $\pi \in \mathcal{M}_p$, the value $I_{\mu,t}(\pi)$ is equal to* $\sup\{S_{\mu,t}(A); \pi \in A \in \mathcal{U}\}$.

since all open balls $B^{d_p}(\nu; r)$, $\nu \in \mathcal{M}_p$, $r > 0$, belong to $\mathcal{U}$ by Lemma 6.4. Clearly, we have

Lemma 7.4. *$I_{\mu,t} : \mathcal{M}_p \to [0, +\infty]$ is a lower semi-continuous functional.*

The convexity of $I_{\mu,t}$ is obvious from the inequality

$$I_{\mu,t}(\pi_1) + I_{\mu,t}(\pi_2) \geq 2 I_{\mu,t}\left(\frac{\pi_1 + \pi_2}{2}\right), \qquad \pi_1, \pi_2 \in \mathcal{M}_p. \tag{7.4}$$

The above inequality follows from a direct computation together with the branching property and Lemma 7.3. On this account, the first type estimate (I) in Theorem 6.6 is derived immediately from (6.3), (6.4), and Lemma 7.3. On the other hand, we can deduce the second type estimate (II) in Theorem 6.6 from compactness, by employing the similar argument in §3.1 of Ref. 8. Summing up, we complete the proof of weak LDP.

8. Scaling Property and Limit Theorem

In this section we treat only the case $\psi \equiv 0$ in (3.1) of Section 3 for simplicity. We first consider some scaling property for the catalytic superprocess $X^L = \{X_t^{L(\rho^\gamma)}, t \geq 0\}$. In what follows we fix a measure $\mu \in \mathcal{M}_p$, a nonnegative constant c (≥ 0), and a time parameter $t > 0$. $\mathcal{L}(X)$ denotes the law of the random variable X. We use the same notation $\mathbf{P}_{s,\mu}^{\kappa,\rho^\gamma}$, $\kappa \in R_+$, $s \geq 0$, which was introduced in Section 5 as the quenched distribution of X^L ($P_{s,\nu}$-a.s.) given the catalyst process ρ^γ.

Proposition 8.1. *If X^L has the law $\mathcal{L}(X^L) = \mathbf{P}_{0,\mu}^{\kappa,\rho^\gamma}$, then cX^L is distributed according to the law $\mathcal{L}(cX^L) = \mathbf{P}_{0,c\mu}^{\kappa,c\rho^\gamma}$.*

Proof. Let $\mathcal{C}_-^p$ be the space of all continuous functions on R^d such that $-f \in \mathcal{C}_+^p$. Recall that we have the following Laplace transition functional

$$\mathbf{P}_{s,\mu}^{\kappa,\rho^\gamma} \exp\langle X_t^{L(\rho^\gamma)}, \varphi\rangle = \exp\langle \mu, u^{[\kappa,\varphi,0,\rho^\gamma]}(-(t-s))\rangle, \tag{8.1}$$

where $t \geq s$, $\varphi \in \mathcal{C}_-^p$ (cf. (5.1)). The arguments in Section 3 and Section 5 yields that $u \equiv u^{[\kappa,\varphi,0,\rho^\gamma]}$ solves the equation (cf. (3.2),(3.3) and (5.2))

$$\Pi_{0-,x}^{\kappa}\varphi(W_{-t}^{\kappa}) = u(0-,t,x) + \Pi_{0-,x}^{\kappa}\int_t^0 u^2(r,t,W_r^{\kappa})L(\rho^\gamma)(dr). \tag{8.2}$$

We claim that the identity

$$u^{[\kappa,c\varphi,0,\rho^\gamma]} = c \cdot u^{[\kappa,\varphi,0,c\rho^\gamma]} \tag{8.3}$$

for $\varphi \in \mathcal{C}_-^p$, by employing (8.1), (8.2) together with the Markov property. The assertion is a direct result from (8.3). Q.E.D.

Moreover, based upon the above result we can derive the following limit theorem (cf. Theorem 8.4). Actually it is nothing but a version of law of large numbers (LLN) for the scaled catalytic superprocesses. We derive by $U^{t,\kappa}(\mu)$ a neighborhood of $T_t^\kappa\mu$, $\mu \in \mathcal{M}_p$. The elememt $T_t^\kappa\mu$ is defined by the duality

$$\langle T_t^\kappa\mu, \varphi\rangle = \langle \mu, T_t^\kappa\varphi\rangle. \tag{8.4}$$

Lemma 8.2. *We have*

$$\langle \mu, u^{[\kappa,\varphi,0,0]}(t)\rangle = \langle T_t^\kappa\mu, \varphi\rangle, \qquad \varphi \in \mathcal{C}^p. \tag{8.5}$$

Proof. Since $u^{[\kappa,\varphi,0,0]}(t)$ is expressed as $T_t^\kappa\varphi$, (8.5) directly follows from the duality (8.4). Q.E.D.

Lemma 8.3. *For $\varphi \in C^p_-$, we have*

$$\mathbf{P}^{\kappa,\rho^\gamma}_{0,\Lambda\mu} \exp\langle X^{L(\rho^\gamma)}_t/\Lambda, \varphi\rangle = \exp\langle \mu, u^{[\kappa,\varphi,0,\rho^\gamma/\Lambda]}(-t)\rangle, \quad P_{0,\nu} - a.s. \tag{8.6}$$

Proof. By virtue of Proposition 8.1, we readily derive

$$\mathbf{P}^{\kappa,\rho^\gamma}_{0,\Lambda\mu} \exp\langle X^{L(\rho^\gamma)}_t/\Lambda, \varphi\rangle = \mathbf{P}^{\kappa,\rho^\gamma/\Lambda}_{0,\mu} \exp\langle X^{L(\rho^\gamma)}_t, \varphi\rangle, \quad P_{0,\nu} - a.s. \tag{8.7}$$

While, taking (8.1) into consideration, we obtain

$$\mathbf{P}^{\kappa,\rho^\gamma/\Lambda}_{0,\mu} \exp\langle X^{L(\rho^\gamma)}_t, \varphi\rangle = \exp\langle \mu, u^{[\kappa,\varphi,0,\rho^\gamma/\Lambda]}(-t)\rangle, \quad P_{0,\nu} - a.s. \tag{8.8}$$

for any element φ in C^p_-. Hence the assertion (8.6) follows immediately from (8.7) and (8.8). Q.E.D.

Lastly we introduce the main result in this section.

Theorem 8.4. (LLN) *For all neighborhoods $U^{t,\kappa}(\mu)$ of $T^\kappa_t\mu$,*

$$\lim_{\Lambda\to\infty} \mathbf{P}^{\kappa,\rho^\gamma}_{0,\Lambda\mu} \left\{X^{L(\rho^\gamma)}_t/\Lambda \in U^{t,\kappa}(\mu)\right\} = 1, \tag{8.9}$$

$P_{0,\nu}$-a.s *holds.*

Proof. By virtue of Lemma 8.2 and Lemma 8.3, the assertion of Theorem 8.4 directly yields from continuity property, because we have only to combine (8.5) to (8.6) with limiting procedure of $\Lambda \to \infty$. Q.E.D.

References

1. M. T. Barlow, S. N. Evans and E. A. Perkins, *Collision local times and measure-valued processes, Can. J. Math.* **43** (1991) 897-938.
2. M. T. Barlow and E. A. Perkins, *On the filtration of historical Brownian motion, Ann. Probab.* **22** (1994) 1273-1294.
3. D. A. Dawson, *Measure-valued Markov processes, Lecture Notes in Math.* **1541** (Springer-Verlag, 1993) 1-260.
4. D. A. Dawson and K. Fleischmann, *A continuous super-Brownian motion in a super-Brownian medium, J. Theor. Probab.* **10** (1997) 213-276.
5. D. A. Dawson and K. J. Hochberg, *The carrying dimension of a stochastic measure diffusion, Ann. Probab.* **7** (1979) 693-703.

6. A. Dembo and O. Zeitouni, **Large Deviations Techniques and Applications** (Jones and Bartlett, 1993).

7. J.-D. Deuschel and J. Rosen, *Occupation time large deviations for critical branching Brownian motion, super-Brownian motion and related processes, Ann.Probab.* **26** (1998) 602-643.

8. J.-D. Deuschel and D. W. Stroock, **Large Deviations** (Academic Press, 1989).

9. I. Dôku, *A lower bound estimate of large deviation type, J. Saitama Univ. Math. Nat. Sci.* **45** (1996) 1-5.

10. I. Dôku, *Nonlinear SPDE with a large parameter and martingale problem for the measure-valued random process with interaction, J. Saitama Univ. Math. Nat. Sci.* **46** (1997) 1-9.

11. I. Dôku, *A note on characterization of solutions for nonlinear equations via regular set analysis, J. Saitama Univ. Math. Nat. Sci.* **48**-1 (1999) 1-14.

12. I. Dôku, *An overview of the studies on catalytic stochastic processes, RIMS Kokyuroku* (*Kyoto Univ.*) **1089** (1999) 1-14.

13. I. Dôku, *Weak large deviation principle for superprocesses related to nonlinear differential equations with catalytic noise, J. Saitama Univ. Math. Nat. Sci.* **48**-2 (1999) 11-22.

14. I. Dôku, *Exponential moments of solutions for nonlinear equations with catalytic noise and large deviation*, preprint, (1999), 15p.

15. I. Dôku, *Measure valued processes associated with nonlinear equations in a catalytic medium*, to appear in *Proc. RIMS Workshop on New Development of Infinite-Dim. Anal. Quant. Probab.*,Sep.16-17,1999, RIMS, Kyoto Univ. (2000) 18p.

16. I. Dôku and N. Kojima, *An introduction to the super-Brownian motion with catalytic medium in Dawson-Fleischmann's work, RIMS Kokyuroku* (*Kyoto Univ.*) **1089** (1999) 15-26.

17. E. B. Dynkin, *A probabilistic approach to one class of nonlinear differential equations, Prob. Th. Rel. Fields* **89** (1991) 89-115.

18. E. B. Dynkin, *Path processes and historical superprocesses, Prob. Th. Rel. Fields* **90** (1991) 1-36.

19. E. B. Dynkin, *Superprocesses and partial differential equations, Ann. Probab.* **21** (1993) 1185-1262.

20. E. B. Dynkin, **An Introduction to Branching Measure-Valued Processes** (Amer. Math. Soc., 1994).

21. S. N. Evans and E. A. Perkins, *Measure-valued branching diffusions with singular interactions, Can. J. Math.* **46** (1994) 120-168.

22. K. Fleischmann and I. Kaj, *Large deviation probabilities for some rescaled superprocesses, Ann. Inst. Henri Poincaré* **30** (1994) 607-645.

23. I. Iscoe, *A weighted occupation time for a class of measure-valued branching processes, Prob. Th. Rel. Fields* **71** (1986) 85-116.

24. I. Iscoe and T.-Y. Lee, *Large deviations for occupation times of measure-valued branching Brownian motions, Stoch. Stochastics Rept.* **45** (1993) 177-209.

25. O. Kallenberg, **Random Measures**, 3rd Edition, (Akademie-Verlag, 1983).

26. N. Konno and T.Shiga, *Stochastic partial differential equations for some measure-valued diffusions, Prob. Th. Rel. Fields* **79** (1988) 201-225.

27. A. Schied, *Moderate deviations and functional LIL for super-Brownian motion, Stoch. Proc. Appl.* **72** (1997) 11-25.

28. S. Watanabe, *A limit theorem of branching processes and continuous state bran- ching processes, J. Math. Kyoto Univ.* **8** (1968) 141-176.

29. E. Zeidler, **Nonlinear Functional Analysis and its Applications**, Vol.I, (Springer-Verlag, 1986).