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Particle systems corresponding to Fleming-Viot processes with selection

板津 誠一 Seiichi Itatsu

静岡大学理学部

Department of Mathematics, Faculty of Science, Shizuoka University

1 Introduction

Let us denote the operator L of the infinitesimal generator in $C(R^{K})$ by the following:

$$L=\frac{1}{2}\sum_{i,j=1}^{K}x_{i}(\delta_{ij}-x_{j})\frac{\partial^{2}}{\partial x_{i}\partial x_{j}}+\sum_{i=1}^{K}b_{i}(x)\frac{\partial}{\partial x_{i}}$$

where $b_{i}(x)=\sum_{j=1}^{K}q_{ji}x_{j}+x_{i}(\sum_{j=1}^{K}\sigma_{ij}x_{j}-\sum_{k,l=1}^{K}\sigma_{kl}x_{k}x_{l})$, $q_{ij}\geq 0$ for $i\neq j$ and $\sum_{j}q_{ij}=0$ and $\sigma_{ij}=\sigma_{ji}$. This defines the infinitesimal generator of a Markov process on $\Delta_{K}=\{x=(x_{1},\cdots,x_{K}):x_{1}\geq 0,\cdots,x_{K}\geq 0,x_{1}+\cdots+x_{K}=1\}$, this process is called the Wright-Fisher diffusion model with selection according to Ethier and Kurtz [5]. Here x_{i} is a gene frequency of type i, q_{ij} is mutation intensity of $i\rightarrow j$, and σ_{ij} is selection intensity of (i,j)-type. In particular the haploid case,we assume that $\sigma_{ij}=\sigma_{i}+\sigma_{j}$.

This diffusion can be generalized as followings. Let E be a locally compact separable metric space and $\mathcal{P}(E)$ be the space of all probability measures on E. For $\mu\in \mathcal{P}(E)$ let us denote $\langle f,\mu\rangle=\int_{E}fd\mu$. For any $f_{1},\cdots,f_{m}\in \mathcal{D}(A)$ and $F\in C^{2}(R^{m})$ let $\varphi(\mu)=F(\langle f_{1},\mu\rangle,\cdots,\langle f_{m},\mu\rangle)=F(\langle f,\mu\rangle)$.

$$\mathcal{L}\varphi(\mu)=\frac{1}{2}\sum_{i,j=1}^{m}(\langle f_{i}f_{j},\mu\rangle-\langle f_{i},\mu\rangle\langle f_{j},\mu\rangle)F_{z_{i}z_{j}}(\langle f,\mu\rangle) \tag{1}$$
$$+\sum_{i=1}^{m}\{\langle Af_{i},\mu\rangle+\langle f_{i}h,\mu\rangle-\langle f_{i},\mu\rangle\langle h,\mu\rangle\}F_{z_{i}}(\langle f,\mu\rangle).$$

Here E is the space of genetic types and A is a mutation operator in $\bar{C}(E)(\equiv$ the space of bounded continuous functions on $E)$ which is the generator for a Feller semigroup $\{T(t)\}$ on $\hat{C}(E)(\equiv$ the space of continuous functions vanishing at infinity). Here we consider of the haploid case and that $h=h(x)\in\bar{C}(E)$ is a selection intensity for type $x\in E$. According to [5], this operator defines a generator corresponding to a Markov process $\{\mu_t\}$ on $\mathcal{P}(E)$ in the sense that the $C_{\mathcal{P}(E)}[0,\infty)$ martingale problem for $\mathcal{L}$ is well posed. This process is called the Fleming-Viot process. We denote μ^n the n-fold product of μ. The aim of this paper is to consider duality for this process in the form

$$E_\mu[\langle f, \mu_t\rangle] = \sum_{k=1}^{\infty}\langle f_k(t), \mu^k\rangle$$

for any $t \geq 0$, $n \in N$ and $f \in \bar{C}(E^n)$ with sup-norm $\|\cdot\|$. Here $f_k(t) \in \bar{C}(E^k)$ and satisfy $\sum_{k=1}^{\infty}\gamma^k\|f_k(t)\| < \infty$ for some $\gamma > 1$ and $f_n(0) = f$ and $f_k(0) = 0$ for $k \neq n$, and we construct the strongly continuous semigroup for this process.

2 Fleming-Viot processes with selection

According to Ethier and Kurtz [5], the operator (1) can be generalized as following formula.

$$\begin{aligned}
\mathcal{L}\varphi(\mu) &= \frac{1}{2}\sum_{i,j=1}^{m}(\langle f_i f_j, \mu\rangle - \langle f_i, \mu\rangle\langle f_j, \mu\rangle)F_{z_i z_j}(\langle f, \mu\rangle) \\
&+ \sum_{i=1}^{m}(\langle Af_i, \mu\rangle + \langle Bf_i, \mu^2\rangle)F_{z_i}(\langle f, \mu\rangle) \qquad (2) \\
&+ \sum_{i=1}^{m}\{\langle (f_i \circ \pi)\sigma, \mu^2\rangle - \langle f_i, \mu\rangle\langle \sigma, \mu^2\rangle\}F_{z_i}(\langle f, \mu\rangle).
\end{aligned}$$

Here B is a recombination operator defined by

$$Bf(x,y) = \alpha\int_E (f(x') - f(x))R((x,y), dx')$$

where $\alpha \geq 0$ and $R((x,y), dx')$ is a one step transition function on $E^2 \times \mathcal{B}(E)$, and $\sigma = \sigma(x,y)$ is a bounded symmetric function on $E \times E$ which is selection parameters for types $x, y \in E$ and $(f_i \circ \pi)(x,y) = f_i(x)$. According to [5], this operator defines a generator corresponding to a Markov process on $\mathcal{P}(E)$ in the sense that the $C_{\mathcal{P}(E)}[0,\infty)$ martingale problem for $\mathcal{L}$ is well posed. This process is called the Fleming-Viot process. In the case (1) $\sigma(x,y) = h(x) + h(y)$ and $B = 0$.

3 Construction of semigroups

We consider that E is a locally compact separable metric space, and treat the case of the formula (2) and assume $\{T(t)\}$ is a Feller semigroup on $\hat{C}(E)$ with the generator A. Denote the semigroup $T_k(t) = \overbrace{T(t)\otimes\cdots\otimes T(t)}^{k\ times}$ on $\bar{C}(E^k)$ and its generator $A^{(k)}$.

We now construct the strongly continuous contraction semigroup for the diffusion. In this section we consider the operator of the form

$$\mathcal{L}\varphi(\mu) = \frac{1}{2}\sum_{i,j=1}^{m}(\langle f_i f_j,\mu\rangle - \langle f_i,\mu\rangle\langle f_j,\mu\rangle)F_{z_i z_j}(\langle f,\mu\rangle) + \sum_{i=1}^{m}(\langle Af_i,\mu\rangle + \langle \tilde{B}f_i,\mu^{\infty}\rangle)F_{z_i}(\langle f,\mu\rangle). \tag{3}$$

Here $\tilde{B}$ is an operator from $\hat{C}(E)$ to $\bar{C}(E^{\infty})$ with $\tilde{B}f = \sum_{l=1}^{\infty} B_l f$ and B_l:$\hat{C}(E) \to \hat{C}(E^l)$ a bounded operator and $\sum_{l=1}^{\infty}\|B_l\|\gamma^{l-1} < \infty$ for some $\gamma > 1$ and $\langle \tilde{B}f_i,\mu^{\infty}\rangle = \sum_{k=1}^{\infty}\langle B_k f_i,\mu^k\rangle$. In the formula (2) we consider $\tilde{B}f(x) = Bf(x_1,x_2) + \sigma(x_1,x_2)f(x_1) - \sigma(x_2,x_3)f(x_1)$ and in this case $\mathcal{L}$ is well defined. Let us define the space $S = \{f = (f_1,f_2,\cdots) \in \sum_{k=1}^{\infty}\hat{C}(E^k) : \|f\|_{\gamma} \equiv \sum_{k=1}^{\infty}\gamma^k\|f_k\| < \infty\}$. Let $\mathcal{C} = \{\varphi_f(\mu) = \sum_{k=1}^{\infty}\langle f_k,\mu^k\rangle, f_k \in \hat{C}(E^k), \|f\|_{\gamma} < \infty\}$, and $\mathcal{D} = \{\varphi_f(\mu) = \sum_{k=1}^{\infty}\langle f_k,\mu^k\rangle \in \mathcal{C}, f_k \in \mathcal{D}(A^{(k)})\}$.

Theorem 1. *Assume* above and $\mathcal{L}$ *of* (3) *defined on* $\mathcal{D}$ *is well defined, closable, and dissipative, then* $\mathcal{L}$ *with the domain* $\mathcal{D}$ *generates a strongly continuous contraction semigroup* $\mathcal{T}(t)$ on $\mathcal{C}(\mathcal{P}(E))$.

Proof. For $\varphi_f(\mu) = \sum_{k=1}^{\infty}\langle f_k,\mu^k\rangle \in \mathcal{D}$ and $\varphi_g(\mu) = \sum_{k=1}^{\infty}\langle g_k,\mu^k\rangle \in \mathcal{C}$, the equation $\mathcal{L}\varphi_f(\mu) = \varphi_g(\mu)$ follows from the formula

$$g_k = (\hat{\mathcal{L}}f)_k \equiv \sum_{1\le i<j\le k+1}\Phi_{ij}^{(k+1)}f_{k+1} + (A^{(k)} - \binom{k}{2})f_k + \sum_{l=1}^{k}B_l^{(k-l+1)}f_{k-l+1}$$

for $k \geq 1$, and $B_l^{(k)} : \hat{C}(E^k) \to \hat{C}(E^{k+l-1})$ defined by

$$B_l^{(k)}f(x_1,\cdots,x_{k+l-1}) = \sum_{i=1}^{k}B_l f(x_1,\cdots,x_{i-1},\cdot,x_i,\cdots,x_{k-1})(x_k,\cdots,x_{k+l-1})$$

for $f \in \bar{C}(E^k)$, and for $i < j$

$$\Phi_{ij}^{(k)}f_k(x_1,\cdots,x_{k-1}) = f_k(x_1,\cdots,x_{j-1},x_i,x_{j+1},\cdots,x_{k-1})$$

for $f \in \bar{C}(E^k)$.

For given $g = (g_1,g_2,\cdots) \in S$ let us consider the equation on S

$$\lambda f_k - (\hat{\mathcal{L}}f)_k = g_k \qquad ,k \geq 1. \tag{4}$$

Then

$$(\lambda + \binom{k}{2} - A^{(k)})f_k = g_k + \sum_{1\le i<j\le k+1}\Phi_{ij}^{(k+1)}f_{k+1} + \sum_{l=1}^{k}B_l^{(k-l+1)}f_{k-l+1}$$

holds for $k \geq 1$. This is equivalent to the equation

$$f_k = (\lambda + \binom{k}{2} - A^{(k)})^{-1}\{g_k + \sum_{1\leq i<j\leq k+1} \Phi_{ij}^{(k+1)} f_{k+1} + \sum_{l=1}^{k} B_l^{(k-l+1)} f_{k-l+1}\}$$

for $k \geq 1$. Put $u = (u_1, u_2, \cdots)$ by

$$u_k = (\lambda + \binom{k}{2} - A^{(k)})^{-1}\{ \sum_{1\leq i<j\leq k+1} \Phi_{ij}^{(k+1)} f_{k+1} + \sum_{l=1}^{k} B_l^{(k-l+1)} f_{k-l+1}\}.$$

Then we have

$$\|u\|_\gamma \leq \sup_k(\frac{\binom{k}{2}\gamma^{k-1}}{(\lambda + \binom{k-1}{2})\gamma^k} + \sum_{l=1}^{\infty} \frac{\|B_l^{(k)}\|\gamma^{k+l-1}}{(\lambda + \binom{k+l-1}{2})\gamma^k})\|f\|_\gamma.$$

Because $\|B_l^{(k)}\| \leq k\|B_l\|$, for any $\delta > 0$ let a positive constant be $L = L(\delta) = \frac{9\delta^2-10\delta+4}{8\delta}$ such that $k \leq L + \delta\binom{k-1}{2}$,then

$$\frac{\binom{k}{2}\gamma^{k-1}}{(\lambda + \binom{k-1}{2})\gamma^k} + \sum_{l=1}^{\infty} \frac{\|B_l^{(k)}\|\gamma^{k+l-1}}{(\lambda + \binom{k+l-1}{2})\gamma^k} \leq \frac{(L + (1+\delta)\binom{k-1}{2})\gamma^{k-1}}{(\lambda + \binom{k-1}{2})\gamma^k} + \frac{(L + \delta\binom{k-1}{2}) \sum_{l=1}^{\infty} \|B_l\|\gamma^{k+l-1}}{(\lambda + \binom{k-1}{2})\gamma^k} \qquad (5)$$

for any k. Let

$$d(\gamma) = \sum_{l=1}^{\infty} \|B_l\|\gamma^{l-1},$$

and put $\delta > 0$ so that $\rho = (1+\delta)/\gamma + \delta d(\gamma) < 1$, then we have that

$$\|u\|_\gamma \leq \rho\|f\|_\gamma$$

for $\lambda \geq L(\gamma^{-1}+d(\gamma))/\rho$. For this λ we conclude that the equation (4) have a unique solution $f \in \mathcal{D}$ satisfying that $\|f\|_\gamma \leq \frac{1}{(1-\rho)\lambda}\|g\|_\gamma$. The equation (4) implies $(\lambda - \mathcal{L})\varphi_f(\mu) = \varphi_g(\mu)$. Because $\mathcal{D}$ is dense in $C(\mathcal{P}(E))$, this implies that the operator $\mathcal{L}$ with the domain $\mathcal{D}$ generates a strongly continuous semigroup by Hille-Yoshida theory. Q.E.D.

Next we will construct a strongly continuous semigroup $\{U(t)\}$ corresponding to $\hat{\mathcal{L}}$ on Banach space S with the norm $\|\cdot\|_\gamma$. For given $h \in S$ we consider $f(t) = (f_1(t), f_2(t), \ldots)$ with $f_k(t) \in \bar{C}(E^k)$ and $f(0) = h$ such that

$$\begin{aligned} \frac{d}{dt}f_k(t) &= (\hat{\mathcal{L}}f(t))_k \qquad (6) \\ &= \sum_{1\leq i<j\leq k+1} \Phi_{ij}^{(k+1)} f_{k+1}(t) \\ &\quad +(A^{(k)} - \binom{k}{2})f_k(t) + \sum_{l=1}^{k} B_l^{(k-l+1)} f_{k-l+1}(t) \end{aligned}$$

for $k \geq 1$. This is equivalent to

$$\begin{aligned} f_k(t) &= e^{-\binom{k}{2}(t-u)} T_k(t-u) f_k(u) \\ &+ \int_u^t e^{-\binom{k}{2}(t-s)} T_k(t-s)\{ \sum_{1\leq i<j\leq k+1} \Phi_{ij}^{(k+1)} f_{k+1}(s) \\ &+ \sum_{l=1}^{k} B_l^{(k-l+1)} f_{k-l+1}(s)\} ds \end{aligned} \tag{7}$$

for $k \geq 1$ and $t > u$, and we have that

$$\begin{aligned} \|f(t)\|_\gamma &\leq \|f(u)\|_\gamma + \int_u^t \sup_k (\binom{k}{2} \gamma^{k-1} e^{-\binom{k-1}{2}(t-s)}/\gamma^k \\ &+ \sum_{l=1}^{\infty} \|B_l^{(k)}\| \gamma^{k+l-1} e^{-\binom{k+l-1}{2}(t-s)}/\gamma^k) \|f(s)\|_\gamma ds, \end{aligned}$$

then

$$\begin{aligned} \|f(t)\|_\gamma &\leq \|f(u)\|_\gamma \\ &+ \int_u^t \sup_k [(L + (1+\delta)\binom{k-1}{2}) \gamma^{k-1} e^{-\binom{k-1}{2}(t-s)}/\gamma^k \\ &+ (L + \delta\binom{k-1}{2})) \sum_{l=1}^{\infty} \|B_l\| \gamma^{k+l-1} e^{-\binom{k+l-1}{2}(t-s)}/\gamma^k] \|f(s)\|_\gamma ds. \end{aligned}$$

Let $r(t) = \sup_{0\leq s\leq t} \|f(s)\|_\gamma$, then $r(t) \leq r(u) + (L(1+d(\gamma))(t-u) + \rho)r(t)$. Therefore by $\rho < 1$, we have

$$r(t) \leq (1 - L(1+d(\gamma))(t-u) - \rho)^{-1} r(u).$$

Therefore

$$r(t) \leq e^{Mt} r(0) \qquad \text{for} \qquad t > 0 \tag{8}$$

where $M = \frac{L(1+d(\gamma))}{1-\rho}$. By this equation $r(0) = 0$ implies $r(t) = 0$. So the equation (6) has a unique solution for $f(0) = h \in \mathcal{C}$ and implies

$$\frac{d}{dt} \varphi_{f(t)}(\mu) = \mathcal{L} \varphi_{f(t)}(\mu).$$

Therefore $f(t)$ satisfies

$$\mathcal{T}(t) \varphi_h(\mu) = \langle f(t), \mu^\infty \rangle.$$

So we have

$$E_\mu[\langle h, \mu_t^\infty \rangle] = \sum_{k=1}^{\infty} \langle f_k(t), \mu^k \rangle.$$

By the inequality (8) there exists a strongly continuous semigroup $\{U(t)\}$ on S corresponding to $\hat{\mathcal{L}}$ such that

$$\|U(t)\| \leq e^{Mt}.$$

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