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DERIVATION OF STOCHASTIC OSCILLATOR OF THE DUFFING TYPE FROM LORENZ EQUATION AND IDENTIFICATION OF THE LIMIT PROCESS*

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Abstract Lorenz(1963) obtained numerical results by integrating a simple third order system of ordinary differential equations on a computer. These equations were derived from a simple model of the weather and Lorenz was trying to show that the solutions of ordinary differential equations could be (in practice) unpredictable despite being deterministic. Here we consider the system of stochastic differential equations of the Lorenz type where the drift and diffusion coefficients are allowed to be increasing functions in a small parameter $\varepsilon > 0$. Then, by a space-time transformation, we derive a new stochastic system from the Lorenz-like equations and show that this new system tends to the stochastic Duffing oscillator as $\varepsilon \to 0$. This research is motivated by a singular perturbation problem for stochastic oscillators, such as the Duffing model, the Liénard model.

1 Lorenz Model and Duffing Oscillator

The *Lorenz equations* first arose in 1963 from a drastically over-simplified model of thermal convection in a layer of fluid. In their 'usual' form they are

$$\begin{aligned} \frac{dx}{dt} &= -\sigma(x-y), \\ \frac{dy}{dt} &= rx - y - xz, \\ \frac{dz}{dt} &= -bz + xy, \end{aligned}$$

where the parameters σ, b and r are real and positive. The values chosen by Lorenz are

$$\sigma = 10, \quad b = \frac{8}{3} \quad \text{and} \quad r = 28,$$

but in the context of bifurcation theory it is usual to treat σ and b as fixed and allow r to vary. In the original context, r acted as a measure of the

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imposed temperature difference between the bottom of the fluid layer and the top, which is what was driving the convective motion. In the same vein, x measured the flow speed, while y and z denoted certain broad features of the temperature distribution (see Addison (1997, pp.125-126)).

The Lorenz equations are symmetric under the operation

$$(x, y, z) \longrightarrow (-x, -y, z),$$

a fact that will be useful later and has stationary points at the origin $(0,0,0)$ and at solutions of $x = y$ (from $dx/dt = 0$) and $bz = x^2$ (from $dz/dt = 0$) and so

$$b(r-1)x - x^3 = 0 \qquad \text{(from} \quad dy/dt = 0).$$

Hence there are two other stationary points (see Glendinning (1994, p.359) and Rasband (1990, p.96)),

$$C_{\pm} = \left(\pm\sqrt{b(r-1)},\, \pm\sqrt{b(r-1)},\, r-1\right)$$

provided $r > 1$. A little linear analysis shows that the origin is stable if $0 < r < 1$ and loses stability in a pitchfork bifurcation at $r = 1$, creating the two non-trivial stationary points which are (initially) stable. To determine the stability of these points we look at the following Jacobian matrix:

$$\begin{pmatrix} -\sigma & \sigma & 0 \\ r-z & -1 & -x \\ y & x & -b \end{pmatrix}.$$

This has eigenvalues given by the roots of

$$\lambda^3 + (\sigma + b + 1)\lambda^2 + b(\sigma + r)\lambda + 2\sigma b(r-1) = 0$$

when evaluated at either C_+ or C_-. Note that since C_- is the image of C_+ under the symmetry the stability properties of the two stationary points must be the same. We can look for bifurcations of the stationary points, i.e. values of the parameters for which either $\lambda = 0$ or $\lambda = i\omega$ are solutions of the eigenvalue equation. The Jacobian matrix at the origin is

$$\begin{pmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{pmatrix}$$

and so there is one eigenvalue of $-b$ with the z-axis as its associated eigenvector and the other eigenvalues $\lambda_{\pm}$ are the solutions of $\lambda^2 + (\sigma+1)\lambda - \sigma(r-1) = 0$, i.e.

$$\lambda_{\pm} \equiv -(\sigma+1)\left[\frac{1}{2} \pm \frac{1}{2}\sqrt{1 - 4\sigma(1-r)/(\sigma+1)^2}\right],$$

which gives $\lambda_- < 0$ and $\lambda_+ > 0$. At $r = 1$ the eigenvalue $\lambda_- = 0$. So as r passes through 1, there is a bifurcation with a change of stability.

Let us consider a schematic diagram of Rayleigh-Benard convection between two horizontal plates. The bottom plate is at a temperature T_b which is greater than that of the top plate, T_t. For small differences between the two temperatures, heat is conducted through the stationary fluid between the plates. However, when $T_b - T_t$ becomes large enough, buoyancy forces within the heated fluid overcome internal fluid viscosity and a pattern of counter-rotating, steady recirculating vortices is set up between the plates. Lorenz noticed that, in his simplified mathematical model of Rayleigh-Benard convection, very small differences in the initial conditions blew up and quickly led to enormous differences in the final behaviour. He reasoned that if this type of behaviour could occur in such a simple dynamical system, then it may also be possible in much more complex physical system involving convection: the weather system. Thus, a very small perturbation, caused for instance by a butterfly flapping its wings, would lead rapidly to complete change in future weather patterns. The system of the Lorenz equations has two nonlinearities, the xz term and the xy term, and exhibits both periodic and chaotic motion depending upon the values of the control parameters σ, r and b. The parameter σ is the Prandtl number which relates the energy losses within the fluid due to viscosity to those due to thermal conduction; r corresponds to the dimensionless measure of the temperature difference between the plates known as the Rayleigh number; and b is related to the ratio of the vertical height of the fluid layer to the horizontal extent of the convective rolls within it. Note also that the variables x, y and z are not spatial co-ordinates but rather represent the convective overturning, horizontal temperature variation, and vertical temperature variation respectively.

Now, consider the numerical case where

$$\sigma = 10 \qquad \text{and} \qquad b = \frac{8}{3}.$$

The origin

$$x = 0, \quad y = 0, \quad z = 0$$

is clearly an stationary point for all r, but it turns out to be stable according to linear theory only for $r < 1$. If we increase r beyond 1 we find two 'new' stationary points

$$x = y = \pm\sqrt{\frac{8}{3}(r-1)}, \qquad z = r - 1.$$

These exist for all $r > 1$ but turn out to be linearly stable only for $1 < r < 24.74$. No other stationary points exist. Some typical numerical solutions in the case where $r = 28$ can be obtained by using the program C. The chaotic nature of these solutions is evident, not just because of their irregularity, but because of their extreme sensitivity to initial conditions. With an initial difference of 1 part in 1000 the oscillation sequences are seen diverging as t becomes greater than about 13. Moreover, even if we reduce the discrepancy in the initial conditions by a factor of 100, to just 1 part in 100 000, we only manage to keep the solutions together for a little longer, till t is about 16, after which they once again go their separate ways (see Acheson (1997, pp.158-161)).

Lorenz saw this behaviour to be a general property of irregular oscillations in nonlinear systems; indeed, he realized that this extreme sensitivity to the initial conditions was essentially the *cause* of irregularity. He realized, too, the practical implications, remarking in his 1963 paper that

... When our results ... are applied to atmosphere, ... they indicate that prediction of the sufficiently distant future is impossible by any method, unless the present conditions are known exactly. In view of the inevitable inaccuracy and incompleteness of weather observations, precise very-long-range forecasting would seem to be non-existent.

If we view the Lorenz equations as a fluid flow in phase space, writing $u = -\sigma(x - y)$ etc., the divergence of the flow is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = -\sigma - 1 - b, \qquad \cdots (\sharp)$$

which is negative; this value is estimated at $-13\frac{2}{3}$ in the particular case where $\sigma = 10$ and $b = \frac{8}{3}$. Consider, then, a small blob of phase fluid initially centred on $(5, 5, 5)$. Because this represents a whole set of slightly different initial conditions, and because we know the outcome to be sensitive to these, we know that the blob will become greately deformed and spread about all over the attractor in quite a short time. Yet because of the result $(\sharp)$ the blob must manage to do this while *decreasing in volume* all the time. Moreover, a divergence of -13.667 corresponds to a quite spectacular rate of shrinking.

A good introduction to the deterministic Lorenz system is given in Guckenheimer and Holms (1983), Sparrow (1983) and Lichtenberg and Lieberman (1982) who discuss the derivation from fluid dynamics. Noise-driven oscillator analogue to the deterministic Lorenz equations are investigated by various authors, who are, for example, cited in Schaffer and Truty (1989).

A simple forced, damped nonlinear oscillator is known as *the Duffing* *oscillator*, and has the equation of motion

$$m\,\frac{d^2x}{dt^2} + k\,\frac{dx}{dt} + \beta\,x^3 = A_f\,\cos\,\omega t.$$

Withoutloss of generality, we may simplify the equation of motion of the Duffing oscilator by setting the mass, m, spring stiffness, β, and the angular frequency, ω, to unity, to get

$$\frac{d^2x}{dt^2} + k\,\frac{dx}{dt} + x^3 = A_f\,\cos\,t.$$

We now have only two *control parameters*: a damping coefficient k, and the amplitude of forcing, A_f. By varying these two parameters we can locate regimes of periodic and chaotic oscillations (see Addison (1997, pp.120-122)).

The above equation is a special case of the following equation:

$$\frac{d^2x}{dt^2} + k\,\frac{dx}{dt} + \alpha\,x + \beta\,x^3 = A_f\,\cos\,t.$$

Let us consider the point mass m in a spring oscillator, which moves to and fro under the action of a spring. Let $x = 0$ denote the equilibrium point, at which the spring is neither extended nor compressed, and let the force exerted by the spring be $F(x)$ in the negative x-direction. In general, this will be a complicated function of x, determined by the detailed elastic properties of the spring, but we do know that $F(0) = 0$, because the spring force must be zero at the equilibrium point$x = 0$. The above-cited equation arises quite naturally from the forced oscillator problem of

$$m\,\frac{d^2x}{dt^2} = -\,F(x),$$

with $m = 1$, if the spring behaves the same way in compression as it does in extension, so that $F(-x) = -F(x)$, i.e. $F(x)$ is an odd function of x. If we confine attention to *small* values of $|x|$, so that the particle is close to the equilibrium point, we may approximate $F(x)$ by the first two terms of its Taylor series about $x = 0$ and then obtain $F(x) \approx \alpha\,x + \beta\,x^3$, because there can be no x^2 term. The coefficient $\beta = \frac{1}{6}\,F'''(0)$ may be positive or negative, depending on the nature of the spring (see Acheson (1997, p.164)).

Our research is motivated by a singular perturbation problem for stochastic oscillators as described in Narita (1993, 1994).

2 Derivation of Stochastic Duffing Oscillator from Lorenz System

Let $(\Omega, \mathbf{F}, P)$ be a probability space with an increasing family $\{\mathbf{F}_t, t \geq 0\}$ of sub-σ-algebras of $\mathbf{F}$, and let $W(t) = (w_0(t), w_1(t), w_2(t))$ be a three-dimensional Brownian motion process adapted to $\mathbf{F}_t$. Then we consider the following system of stochastic differential equations of the Lorenz type:

$$
\begin{aligned}
dx(t) &= -\sigma\,[\,x(t) - y(t)\,]\,dt, \\
dy(t) &= [\,r\,x(t) - y(t) - x(t)z(t)\,]\,dt + \delta_1\,dw_1(t), \\
dz(t) &= [\,-\,b\,z(t) + x(t)y(t)\,]\,dt + \delta_2\,dw_2(t)
\end{aligned}
\tag{2.1}
$$

with the family $\{\sigma, r, b, \delta_1, \delta_2\}$ of positive constants.

Definition 2.1 Let ε be a small parameter such that $0 < \varepsilon \ll 1$. For the solution $(x(t), y(t), z(t))$ of (2.1), define $x^\varepsilon(t)$, $y^\varepsilon(t)$ and $z^\varepsilon(t)$ by

$$
x^\varepsilon(t) = x\left(\frac{\varepsilon\,t}{\sqrt{\sigma}}\right), \qquad y^\varepsilon(t) = y\left(\frac{\varepsilon\,t}{\sqrt{\sigma}}\right), \qquad \text{and} \quad z^\varepsilon(t) = z\left(\frac{\varepsilon\,t}{\sqrt{\sigma}}\right).
$$

Then it is easy to see that $(x^\varepsilon(t), y^\varepsilon(t), z^\varepsilon(t))$ satisfies the following system of stochastic differential equations:

$$
\begin{aligned}
dx^\varepsilon(t) &= -\sqrt{\sigma}\,\varepsilon\,[\,x^\varepsilon(t) - y^\varepsilon(t)\,]\,dt, \\
dy^\varepsilon(t) &= \frac{1}{\sqrt{\sigma}}\,\varepsilon\,[\,r\,x^\varepsilon(t) - y^\varepsilon(t) - x^\varepsilon(t)z^\varepsilon(t)\,]\,dt + \left(\delta_1\,\sigma^{-1/4}\right)\sqrt{\varepsilon}\,d\widetilde{w}_1(t), \\
dz^\varepsilon(t) &= \frac{1}{\sqrt{\sigma}}\,\varepsilon\,[\,-\,b\,z^\varepsilon(t) + x^\varepsilon(t)y^\varepsilon(t)\,]\,dt + \left(\delta_2\,\sigma^{-1/4}\right)\sqrt{\varepsilon}\,d\widetilde{w}_2(t).
\end{aligned}
\tag{2.2}
$$

Here and hereafter, $\widetilde{w}_1(t)$ and $\widetilde{w}_2(t)$ are (new) one-dimensional Brownian motion processes which are defined by

$$
\widetilde{w}_1(t) = \sigma^{1/4}\,\frac{1}{\sqrt{\varepsilon}}\,w_1\left(\frac{\varepsilon\,t}{\sqrt{\sigma}}\right) \qquad \text{and} \qquad \widetilde{w}_2(t) = \sigma^{1/4}\,\frac{1}{\sqrt{\varepsilon}}\,w_2\left(\frac{\varepsilon\,t}{\sqrt{\sigma}}\right),
$$

so that they are adapted to $\mathbf{F}_t$ and independent each other.

Definition 2.2. For the solution $(x^\varepsilon(t), y^\varepsilon(t), z^\varepsilon(t))$ of (2.2), define $\xi^\varepsilon(t), \eta^\varepsilon(t)$ and $q^\varepsilon(t)$ by

$$
\xi^\varepsilon(t) = \frac{\varepsilon}{\sqrt{2\,\sigma}}\,x^\varepsilon(t), \qquad \eta^\varepsilon(t) = -\,\frac{\varepsilon^2}{\sqrt{2}}\,(\,x^\varepsilon(t) - y^\varepsilon(t)\,)
$$

$$\text{and} \qquad q^{\varepsilon}(t) = \frac{\varepsilon^2}{\sigma}\left(\sigma\, z^{\varepsilon}(t) - \frac{1}{2}\, x^{\varepsilon}(t)^2\right).$$

Moreover, for the positive parameters σ and b, define a, β and h by

$$a = \frac{b}{\sqrt{\sigma}}, \qquad \beta = \frac{2\,\sigma - b}{\sqrt{\sigma}} \qquad \text{and} \qquad h = \frac{\sigma + 1}{\sqrt{\sigma}}.$$

Then $(\xi^{\varepsilon}(t), \eta^{\varepsilon}(t), q^{\varepsilon}(t))$ satisfies the following system of stochastic differential equations:

$$(2.3)\qquad \begin{aligned} d\xi^{\varepsilon}(t) &= \eta^{\varepsilon}(t)\, dt, \\ d\eta^{\varepsilon}(t) &= -\left[(\varepsilon\, h)\eta^{\varepsilon}(t) + \xi^{\varepsilon}(t)^3 + \{q^{\varepsilon}(t) - \varepsilon^2(r-1)\}\, \xi^{\varepsilon}(t)\right] dt \\ &\qquad + \left(\tfrac{1}{\sqrt{2}}\, \delta_1\, \sigma^{-1/4}\right) \varepsilon^{5/2}\, d\tilde{w}_1(t), \\ dq^{\varepsilon}(t) &= \left[-(\varepsilon\, a) q^{\varepsilon}(t) + (\varepsilon\, \beta)\, \xi^{\varepsilon}(t)^2\right] dt + \left(\delta_2\, \sigma^{-1/4}\right) \varepsilon^{5/2}\, d\tilde{w}_2(t). \end{aligned}$$

Remark 2.1. In the system (2.3), the solutions $\xi^{\varepsilon}(t)$ and $q^{\varepsilon}(t)$ can be regarded as the response to the stochastic Duffing oscillator and the Ornstein-Uhlenbeck type process, respectively, as follows:

$$\frac{d^2\xi}{dt^2} + (\varepsilon\, h)\frac{d\xi}{dt} + \xi^3 + \{q - \varepsilon^2\,(r-1)\}\,\xi = \left(\frac{\delta_1}{\sqrt{2}}\,\sigma^{-1/4}\right) \varepsilon^{5/2}\, \frac{d\tilde{w}_1}{dt},$$

and

$$\frac{dq}{dt} = -(\varepsilon\, a)\, q + (\varepsilon\, \beta)\, \xi^2 + \left(\delta_2\, \sigma^{-1/4}\right) \varepsilon^{5/2}\, \frac{d\tilde{w}_2}{dt},$$

where $\frac{d\tilde{w}_1}{dt}$ and $\frac{d\tilde{w}_2}{dt}$ are the formal white noises.

For the analysis we treat the case when $r > 1$ and allow the coefficients δ_1 and δ_2 of the intensity of the fluctuation to blow up.

Assumption 2.1 In the original system (2.1), the coefficients r, δ_1 and δ_2 depend on a small parameter $0 < \varepsilon \ll 1$ as follows:

(i) $r = 1 + \varepsilon^{-2}$, that is $\varepsilon\sqrt{r-1} = 1$.

(ii) $\delta_1 = \delta_1(\varepsilon) = c_1\, \varepsilon^{k-5/2}$ with constants $c_1 > 0$ and $k \geq 0$.

(iii) $\delta_2 = \delta_2(\varepsilon) = c_2\, \varepsilon^{l-5/2}$ with constants $c_2 > 0$ and $l \geq 0$.

Assumption 2.1 implies the following circumstances:

- $\delta_1(\varepsilon) \uparrow \infty$ and $\delta_2(\varepsilon) \uparrow \infty$ as $\varepsilon \downarrow 0$ provided that $0 \leq k < \frac{5}{2}$ and $0 \leq l < \frac{5}{2}$.

- $\delta_1(\varepsilon) \downarrow 0$ and $\delta_2(\varepsilon) \downarrow 0$ as $\varepsilon \downarrow 0$ provided that $k > \frac{5}{2}$ and $l > \frac{5}{2}$.
- $\delta_1(\varepsilon) \equiv c_1$ and $\delta_2(\varepsilon) \equiv c_2$ provided that $k = l = \frac{5}{2}$.

Under Assumption 2.1, the system (2.3) can be written by the following form:

$$(2.4)\qquad \begin{aligned} d\xi^\varepsilon(t) &= \eta^\varepsilon(t)\,dt, \\ d\eta^\varepsilon(t) &= -\left[(\varepsilon\, h)\eta^\varepsilon(t) + \xi^\varepsilon(t)^3 + \{q^\varepsilon(t) - 1\}\, \xi^\varepsilon(t)\right] dt \\ &\qquad + \varepsilon^k \left(\tfrac{1}{\sqrt{2}}\, c_1\, \sigma^{-1/4}\right) d\tilde{w}_1(t), \\ dq^\varepsilon(t) &= \varepsilon\left[-a\, q^\varepsilon(t) + \beta\, \xi^\varepsilon(t)^2\right] dt + \varepsilon^l \left(c_2\, \sigma^{-1/4}\right) d\tilde{w}_2(t). \end{aligned}$$

Our goal is to obtain the limit processes of the above system (2.4) as $\varepsilon \to 0$. For this purpose, taking $\varepsilon = 0$ in (2.4), we can derive the following system of *reduced equations*:

$$(2.5)\qquad \begin{aligned} &d\xi(t) = \eta(t)\,dt, \\ &d\eta(t) = -\left[\xi(t)^3 + \{q(t) - 1\}\, \xi(t)\right] dt + \varphi(k) \left(\frac{1}{\sqrt{2}}\, c_1\right) \sigma^{-1/4}\, d\tilde{w}_1(t), \\ &dq(t) = \psi(l)\left(c_2\, \sigma^{-1/4}\right) d\tilde{w}_2(t), \end{aligned}$$

where

$$(2.6)\qquad \varphi(k) = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{if } k > 0 \end{cases} \quad \text{and} \quad \psi(l) = \begin{cases} 1 & \text{if } l = 0, \\ 0 & \text{if } l > 0. \end{cases}$$

Remark 2.2 The solutions $\xi(t)$ and $q(t)$ of the above system (2.5) can be regarded as the response to the stochastic Duffing oscillator and the Brownian motion-like process, respectively, as follows:

$$\frac{d^2\xi}{dt^2} + \xi^3 + (q-1)\xi = \varphi(k)\left(\frac{1}{\sqrt{2}}\, c_1\, \sigma^{-1/4}\right)\frac{d\tilde{w}_1}{dt}$$

and

$$\frac{dq}{dt} = \psi(l)\left(c_2\, \sigma^{-1/4}\right)\frac{d\tilde{w}_2}{dt}$$

with the formal white noises $\frac{d\tilde{w}_1}{dt}$ and $\frac{d\tilde{w}_2}{dt}$.

The system (2.1) with $\delta_1 = \delta_2 = 0$ is the deterministic Lorenz model, in which the Brownian motion processes do not appear; moreover, the expression of the limit system (2.5) with $c_1 = c_2 = 0$ is equivalent to that with

$\varphi(k) = 0$ and $\psi(l) = 0$. The deterministic case when $\varphi(k) = 0$ and $\psi(l) = 0$ is formally obtained by Andreychikov and Yadovich (1981), that is introduced by Neimark and Landa (1992, pp.15-16) without the proof. Our main theorem corresponds to an extension of the above deterministic case to the stochastic one with mathematically rigorous proof. In particular, Theorem 4.3 shows that the system (2.4) with $\beta = 0$ converges to the stochastic system (2.5) in the sense of mean square as $\varepsilon \to 0$. This involves the asymptotic behaviour of the system (2.1) with large parameters r, δ_1 and δ_2 as described in Assumption 2.1.

In Remark 2.2, the *second-order* stochastic differential equation for $\xi(t)$ can be interpreted as the equation of *forced motion* (without damping) of a particle on a spring which provides a *restoring force*:

$$\begin{aligned} \text{forcing term} = F(t) &\equiv \varphi(k)\left(\frac{1}{\sqrt{2}}c_1\,\sigma^{-1/4}\right)\frac{d\tilde{w}_1}{dt}, \\ \text{coefficient in restoring force} &= q(t) - 1. \end{aligned}$$

$\varphi(k)$	$\psi(l)$	Duffing oscillator $\xi = \xi(t)$	coefficient $q = q(t)$
0	0	deterministic ($F(t) \equiv 0$)	$q(t) \equiv q(0) =$ constant
0	1	stochastic ($F(t) \equiv 0$)	$q(t) =$ Brownian motion-like
1	0	stochastic ($F(t) =$ white noise-like)	$q(t) \equiv q(0) =$ constant
1	1	stochastic ($F(t) =$ white noise-like)	$q(t) =$ Brownian motion-like

3 Stochastic Lorenz System

For the solutions $x(t), y(t)$ and $z(t)$ of SDE(2.1), define $X(t)$ by

$$X(t) = (x(t), y(t), z(t)).$$

Assumption 3.1. Let $X_0 = (x_0, y_0, z_0)$ be any three-dimensional random vector independent of the Brownian motion process $W(t) = (w_0(t), w_1(t), w_2(t))$ for $t \geq 0$, such that

$$E\left[\,|\,X_0\,|^{2m}\,\right] < \infty \qquad \text{for an integer} \quad m \geq 1.$$

Then we have the following theorems.

Theorem 3.1. Suppose that Assumption 3.1 holds for X_0. Then there exists a pathwise unique solution $X(t)$ of SDE(2.1) with the initial state $X(0) = X_0$. Moreover, for $(x, y, z) \in R^3$, define $V(x, y, z)$ by $V(x, y, z) = \frac{1}{2}\,(x^2 + y^2 + z^2)$. Then

$$E[\,(\,1 + V(X(t))\,)^m\,] \leq E[\,(\,1 + V(X_0)\,)^m\,]\exp[\,C(m)\,t\,] \quad \text{for all} \quad t \geq 0,$$

where $C(m) = m\{ (\sigma + r + \delta^2) + 2\,\delta^2(m-1) \}$, $\delta^2 = \frac{1}{2}(\, {\delta_1}^2 + {\delta_2}^2 \,)$ and m is the same integer as that in Assumption 3.1.

In the following we give the outline of the proof.

Let $V = V(x, y, z)$ be the function as given in the hypothesis. Denote by L the differential generator associated with SDE(2.1). Then it is easy to see that

$$LV \leq (\,\sigma + r\,)\,V + \delta^2 \qquad \text{for all} \quad (x, y, z) \in R^3,$$

where $\delta^2 = \frac{1}{2}(\, {\delta_1}^2 + {\delta_2}^2 \,)$. Note that the function V is radially unbounded, namely

$$V(x, y, z) \longrightarrow \infty \qquad \text{as} \quad (x^2 + y^2 + z^2)^{1/2} \longrightarrow \infty.$$

Then, according to nonexplosion criteria for solutions of SDEs by the Lyapunov function method, as follows from Hasminskii (1980), McKean (1969) and Narita (1982a,b), any solution of SDE(2.1) with the initial state $X(0) = X_0$ cannot explode. Hence SDE(2.1) has a pathwise unique solution. Apply Ito's formula concerning stochastic differentials to the Lyapunov function $(1+V)^m$ and take mathematical expectations. Then Gronwall's lemma yields the estimate of the moments of the solution.

Theorem 3.2. Let $X(t)$ be the solution of SDE(2.1) with the initial state $X(0) = X_0$. For any $k > 0$, set

$$D(k) = \left\{ (x, y, z) : r x^2 + y^2 + b\,(z - r)^2 \leq c + \frac{k}{\sigma} \right\},$$

where $c = b\,r^2 + \delta^2$ and $\delta^2 = \frac{1}{2}(\, {\delta_1}^2 + {\delta_2}^2 \,)$. Denote by $D(k)^c$ the complement of $D(k)$ in R^3, that is $D(k)^c = R^3 \setminus D(k)$.
Suppose that the initial state $X_0 = (x_0, y_0, z_0)$ is deterministic and that $X_0 \in D(k)^c$. For such solution $X(t)$, define the random time $\tau(\, D(k)^c \,)$ by

$$\tau(\, D(k)^c \,) = \inf \{\, t \,:\, X(t) \notin D(k)^c \,\}.$$

Then

$$E\,[\,\tau(\, D(k)^c \,)\,] \leq \frac{1}{k}\,U(X_0),$$

where $U(X) = \frac{1}{2}(\, r\,x^2 + \sigma\,y^2 + \sigma\,(z - 2r)^2 \,)$ for $X = (x, y, z) \in R^3$.

In the following we give the outline of the proof.

Let $U = U(x, y, z)$ be the function as given in the hypothesis. Denote by L the differential generator associated with SDE(2.1). Then U satisfies

$$LU < -k \qquad \text{for all} \quad (x, y, z) \in D(k)^c.$$

Set $\tau = \tau(\,D(k)^c\,)$. Apply Ito's formula to the function U and take mathematical expectations. Then, the above-cited inequality yields

$$\begin{aligned} 0 \le E[\,U(\,X(\tau\wedge t))\,] &\le U(X_0) - k\cdot E[\,\tau\wedge t\,], \\ \text{namely,}\quad E[\,\tau\wedge t\,] &\le \frac{1}{k}U(X_0) \qquad \text{for any}\quad t>0, \end{aligned}$$

where $\tau\wedge t = \min\{\,\tau,\,t\,\}$. Since t is arbitrary, the assertion of the theorem holds.

Let $x^\varepsilon(t), y^\varepsilon(t)$ and $z^\varepsilon(t)$ of SDE(2.2) be the same processes as those in Definition 2.1, so that they satisfy SDE(2.2). Set

$$X^\varepsilon(t) = (x^\varepsilon(t), y^\varepsilon(t), z^\varepsilon(t)).$$

Then we have the following theorem.

Theorem 3.3. Let $X^\varepsilon(t)$ be the solution of SDE(2.2) with the initial state $X^\varepsilon(0) = X_0$ and suppose that X_0 satisfies Assumption 3.1. Then $X^\varepsilon(t)$ is the pathwise unique solution of SDE(2.2). Moreover, for $(x,y,z)\in R^3$, define the function $V(x,y,z)$ by $V(x,y,z) = \frac{1}{2}\,(x^2+y^2+z^2)$. Then

$$E[\,(\,1+V(X^\varepsilon(t))\,)^m\,] \;\le\; E[\,(\,1+V(X_0)\,)^m\,]\exp\left[\frac{1}{\sqrt{\sigma}}\varepsilon\cdot C(m)\,t\right] \qquad \text{for all}\quad t\ge 0,$$

where $C(m) = m\,\{\,(\sigma+r+\delta^2) + 2\,\delta^2\,(m-1)\,\}$, $\delta^2 = \frac{1}{2}\,(\,{\delta_1}^2+{\delta_2}^2\,)$ and m appears in Assumption 3.1.

In the following we give the outline of the proof.

Denote by L^ε the differential generator associated with $X^\varepsilon(t)$. Then, the function $V(x,y,z)$ as given in the hypothesis of the theorem satisfies

$$L^\varepsilon\,V = \left(\frac{1}{\sqrt{\sigma}}\varepsilon\right)\,LV \;\le\; \left(\frac{1}{\sqrt{\sigma}}\varepsilon\right)\{(\sigma+r)\,V+\delta^2\} \qquad \text{for all}\quad (x,y,z)\in R^3,$$

where L is the differential generator associated with the solution $X(t)$ of SDE(2.1). Apply Ito's formula to the function $(\,1+V\,)^m$ and take mathematical expectations. Then Gronwall's lemma yields the estimate of the moments of the solution.

Let $\xi^\varepsilon(t), \eta^\varepsilon(t)$ and $q^\varepsilon(t)$ be the same processes as those defined in Definition 2.2. Set

$$\chi^\varepsilon(t) = (\xi^\varepsilon(t), \eta^\varepsilon(t), q^\varepsilon(t)).$$

Then, by a straightforward calculation of stochastic differengtials we can obtain the following theorem.

Theorem 3.4. The process $\chi^{\varepsilon}(t)$ satisfies SDE(2.3). In particular, suppose that r, δ_1 and δ_2 depend on a small parameter ε such that $0 < \varepsilon \ll 1$, satisfying Assumption 2.1. Then $\chi^{\varepsilon}(t)$ satisfies SDE(2.4).

For the solutions $\xi(t), \eta(t)$ and $q(t)$ of SDE(2.5), set

$$\chi(t) = (\xi(t), \eta(t), q(t)).$$

Assumption 3.2 Let $\chi_0 = (\xi_0, \eta_0, q_0)$ be any three-dimensional random vector independent of the Brownian motion process, such that

$$E\left[\,|\,\chi_0\,|^{4m}\,\right] < \infty \qquad \text{for an integer} \quad m \geq 1.$$

Then we have the following theorem.

Theorem 3.5. Suppose that Assumption 3.2 holds for χ_0. Then there exists a pathwise unique solution $\chi(t)$ of SDE(2.5) with the initial state $\chi(0) = \chi_0$. Moreover, for $(\xi, \eta, q) \in R^3$, define $U(\xi, \eta, q)$ by $U(\xi, \eta, q) = \frac{1}{4}\xi^4 + \frac{1}{2}\eta^2 + \frac{1}{4}q^4$. Then

$$E[\,(\,1 + U(\,\chi(t)\,)\,)\,] \leq (\,1 + E[\,U(\chi_0)\,]\,)\,\exp[\,C(k,l)\,t\,] \quad \text{for all} \quad t \geq 0,$$

where

$$\begin{aligned} C(k,l) &= (3 + 6\,\psi(l)\,\sigma^{-1/2}\cdot c_2^2) \\ &\qquad + \frac{1}{2}\left[\sigma^{-1/2}\left\{\varphi(k) + \frac{1}{2}\,c_1^2\,\psi(l)\cdot 3\,c_2^2\right\} + 1\right]. \end{aligned}$$

In the following we give the outline of the proof.

Let $\overline{L}^0$ be the differential generator associated with the solution $\chi(t)$, and let $U(\xi, \eta, q)$ be the function as given in the hypothesis of the theorem. Then U satisfies

$$\overline{L}^0\,U \leq A(l)U + B(k,l) \qquad \text{for all} \quad (\xi, \eta, q) \in R^3$$

with constants $A(l) > 0$ depending on l and $B(k,l) > 0$ depending on k and l. Namely, the radially unbounded function U satisfies the criterion of nonoccurrence of an explosion, and hence the pathwise uniqueness holds for the solution of SDE(2.5). Apply Ito's formula to the function $1 + U$ and take mathematical expectations. Then Gronwall's lemma yields the estimate of the moments of the solution.

4 Identification of Limit Process

According to Theorem 3.4, the process $\chi^\varepsilon(t) = (\xi^\varepsilon(t), \eta^\varepsilon(t), q^\varepsilon(t))$ satisfies SDE(2.4) under Assumption 2.1. The following theorem guarantees the pathwise uniqueness for solutions to SDE(2.4).

Theorem 4.1. Let $\chi_0 = (\xi_0, \eta_0, q_0)$ be the random vector satisfying Assumption 3.2. Suppose that the parameters σ and b satisfy the relation

$$2\,\sigma = b, \quad \text{namely,} \quad \beta = 0.$$

Then there exists a pathwise unique solution $\chi^\varepsilon(t)$ of SDE(2.4). Moreover, for $(\xi, \eta, q) \in R^3$, set $U(\xi, \eta, q) = \frac{1}{4}\xi^4 + \frac{1}{2}\eta^2 + \frac{1}{4}q^4$. Then

$$E[\,(1 + U(\,\chi^\varepsilon(t)\,)\,)\,] \leq E[\,(1 + U(\,\chi_0\,)\,)\,]\exp\,[K(\varepsilon)\,t] \quad \text{for all} \quad t \geq 0,$$

where

$$\begin{aligned} K(\varepsilon) &= K_1(\varepsilon) + K_2(\varepsilon), \\ K_1(\varepsilon) &= \frac{1}{2} + \frac{1}{2}\left(\varepsilon^{2k} \cdot \frac{1}{2}\,c_1^2 + \varepsilon^{2l} \cdot 3\,c_2^2\right)\,\sigma^{-1/2} \\ \text{and} \quad K_2(\varepsilon) &= 3 + 6\,\varepsilon^{2l} \cdot c_2^2 \cdot \sigma^{-1/2}. \end{aligned}$$

In the following we give the outline of the proof.

By $\overline{L}^\varepsilon$ denote the differential generator associated with SDE(2.4), and let $U = U(\xi, eta, q)$ be the same function as that in the hypothesis. Then it is easy to see that

$$\overline{L}^\varepsilon\,U \leq K_1(\varepsilon) + K_2(\varepsilon)\,U \qquad \text{for all} \quad (\xi, \eta, q) \in R^3$$

with the constants $K_1(\varepsilon)$ and $K_2(\varepsilon)$ as given in the preceding. Namely, U is a radially unbounded Lyapunov function that satisfies the sufficient condition for nonoccurrence of an explosion, which guarantees the pathwise uniqueness for the solution of SDE(2.4). Moreover, apply Ito's formula to the function $1 + U$ and take mathematical expectations. Then Gronwall's lemma yields the estimate of the moment of the solution.

Theorem 4.2. Let $\chi_0 = (\xi_0, \eta_0, q_0)$ be the random vector independent of the Brownian motion process, such that

$$E\left[\frac{1}{4}\,\xi_0^4 + \frac{1}{2}\,\eta_0^2 + \frac{1}{4}\,q_0^4\right] < \infty.$$

Let $\chi^\varepsilon(t)$ and $\chi(t)$ be the solutions of SDE(2.4) with $\beta = 0$ and SDE(2.5), respectively, such that $\chi^\varepsilon(0) = \chi(0) = \chi_0$. For any $M > 0$, define τ_M^ε and τ_M by

$$\tau_M^\varepsilon = \inf\,\{\,t : |\,\chi^\varepsilon(t)\,| \geq M\,\} \qquad \text{and} \qquad \tau_M = \{\,t : |\,\chi(t)\,| \geq M\,\}\,.$$

Further, for any $t \geq 0$, set $t_M^\varepsilon = \min\{t,\, \tau_M^\varepsilon,\, \tau_M\}$. Let $T < \infty$ be arbitrary and be fixed. Then

$$E\left[\,|\,\chi^\varepsilon(t_M^\varepsilon) - \chi(t_M^\varepsilon)\,|^2\,\right] \;\leq\; H(\varepsilon, T, M)\,\exp[\,I(T, M)\,t\,] \qquad \text{for all} \quad 0 \leq t \leq T,$$

where $H(\varepsilon, M, T)$ is a positive constant which depends on ε, T and M, satisfying

$$H(\varepsilon, T, M) \longrightarrow 0 \qquad \text{as} \qquad \varepsilon \to 0,$$

and $I(T, M)$ is a positive constant which depens only on T and M.

The above-mentioned theorem is obtained by an application of the truncation procedure, Schwarz's inequality and Gronwall's lemma.

Theorem 4.3. Let $\chi_0 = (\xi_0, \eta_0, q_0)$ be any three-dimensional random vector satisfying Assumption 3.2. Suppose that the parameters σ and b satisfy the relation

$$2\,\sigma = b, \qquad \text{namely,} \qquad \beta = 0.$$

Let $\chi^\varepsilon(t)$ and $\chi(t)$ be the solutions of SDE(2.4) and SDE(2.5), respetively, such that $\chi^\varepsilon(0) = \chi(0) = \chi_0$. Let $T < \infty$ be arbitrary and be fixed. Then

$$E\left[\,|\,\chi^\varepsilon(t) - \chi(t)\,|^2\,\right] \longrightarrow 0 \qquad \text{as} \qquad \varepsilon \to 0$$

for every $t \leq T$.

In the following we give the outline of the proof.

Consider t in the interval $[0,\, T]$. Then

$$\chi^\varepsilon(t) - \chi(t) = (\,\chi^\varepsilon(t) - \chi^\varepsilon(t_M^\varepsilon)\,) + (\,\chi^\varepsilon(t_M^\varepsilon) - \chi(t_M^\varepsilon)\,) + (\,\chi(t_M^\varepsilon) - \chi(t)\,),$$

and so

$$E\left[\,|\,\chi^\varepsilon(t) - \chi(t)\,|^2\right] \;\leq\; 3\,\left[\,E\left[\,|\,\chi^\varepsilon(t) - \chi^\varepsilon(t_M^\varepsilon)\,|^2\,\right] + E\left[\,|\,\chi^\varepsilon(t_M^\varepsilon) - \chi(t_M^\varepsilon)\,|^2\,\right] + E\left[\,|\,\chi(t_M^\varepsilon) - \chi(t)\,|^2\,\right]\,\right],$$

where $t_M^\varepsilon = \min\{t,\, \tau_M^\varepsilon,\, \tau_M\}$, and τ_M^ε and τ_M are the same random times as those in Theorem 4.2. By Theorems 4.1 and 3.5, note that $\chi^\varepsilon(t)$ and $\chi(t)$ cannot explode, so that

$$t_M^\varepsilon \longrightarrow t \qquad \text{as} \qquad M \longrightarrow \infty \qquad \text{with probability 1,}$$

and hence

$$\chi^{\varepsilon}(t_M^{\varepsilon}) \longrightarrow \chi^{\varepsilon}(t) \quad \text{as} \quad M \longrightarrow \infty \quad \text{and} \quad \chi(t_M^{\varepsilon}) \longrightarrow \chi(t) \quad \text{as} \quad M \longrightarrow \infty$$

with probability 1. Moreover, Theorem 4.1 and Theorem 3.5 imply that

$$E\left[\mid \chi^{\varepsilon}(t_M^{\varepsilon}) \mid^2 \right] < \infty \quad \text{and} \quad E\left[\mid \chi(t_M^{\varepsilon}) \mid^2 \right] < \infty \quad \text{uniformly in } M \text{ and } \varepsilon.$$

Therefore, by Theorem 4.2 we can obtain the conclusion of the theorem.

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